Lecture 3:

- Tensor Models

① Complex Matrix Model Correlators.

- Branched covers

- $\pi_1 \to S$.

- Emergent S^2.

② · Tensor Models :

- Paper with :
  Ben Geloun & S.R.
  (1307.6490)
  (1708.03524)

- around same time :
  - de Mello Koch & collabs,
  - Morozov & Mironov
  - Diaz & Rey.

fermions

– Counting ~~Matr~~ Invariants with Multi-Index tensors

– Counting fn & Belyi Maps

– Counting fn & Sums over perms

③ Correlators in Tensor Models

ⓐ – Gauss

ⓑ – S_n - TFT & Open Questions

ⓒ Orthogonality: → gens

④ Further Counts

– Finite N

– Color Symmetrized Counting

$$\left\langle \underbrace{\mathcal{O}^{R,S,T;\tau_1,\tau_2}} \qquad \underbrace{\mathcal{O}^{R'S'T';\tau_1'\tau_2'}} \right\rangle$$

$\sim \cdot \qquad \pi$.

Tensor Models:

(Introduction)

↦ ⊗ In previous lectures, we focused on 2 aspects of Matrix Models

- L1 ↦ 2d. Y.M. Theory

↦ ~~Dg~~ Physical variables are holonomies;

↦ $Z(G, F)$ can be calculated.

→ $Z(G; A, . B; U, \ldots U_B)$

gauge invariant function of boundary holonomies

- We talked about
  - covering spaces
  - Large N expansion

↳ counting at each order

↓

cov. spaces

↓

Perm :

(L2) ↦ Complex Matrix Model

$$\int e^{-\mathrm{tr}\, Z Z^{\dagger}} DZ .$$

$$-\int e^{-\mathrm{tr}\, ZZ^T - \mathrm{tr}\, YY^T} \, dZ\, dY$$

- We talked about

$$\langle \underbrace{f(Z)}\, \underbrace{f(Z^+)} \rangle$$

- complete set of gauge invts $\rightarrow \chi^{R}(Z)$

-

- Commutants :

- Use of permutations

- 2 - Matrix : $\rightarrow$ finite N

$$\chi^{R}_{R_1,R_2,\nu_1,\nu_2}(Z,Y)$$

- Use of permutations
- Schur-Weyl Duality
- Covering space.

→ all these things

- Algebras & geometry work equally well to yield counting fns for gauge invts,

- {Thermodynamics is different.} —

→ Put Perm-Algebra & Perm Geom.

① Gaussian Models :

$$\int e^{-t\,\Phi_{ijk}\bar{\Phi}_{ijk}}\; O(\Phi,\bar{\Phi})$$

1-point fn.

$$\int e^{-t\,\Phi\bar{\Phi}}\; \left\{ \underbrace{:O_1(\Phi,\bar{\Phi}):}_{\text{Normal ordering}}\; \underbrace{:O_2(\Phi,\bar{\Phi})}_{\text{Norm. ord.}} \right\}_0$$

⊘ Back to

$$\int DZ \; e^{-\mathrm{tr}(Z Z^\dagger)}$$

Σ : Multi-traced :

$$\langle \mathrm{tr}(\sigma_1 Z) \, \mathrm{tr}(\sigma_2 Z^\dagger) \rangle$$

$$= \sum_{\substack{\sigma_1, \sigma_2 \, \cancel{\in S_n} \\ \sigma_3 \in S_n}} \delta(\sigma_1 \gamma \sigma_2 \gamma^{-1} \sigma_3) N^{C_{\sigma_3}}$$

$$\cancel{\left\langle \left(\frac{\sum_{\sigma_1 \in T_1} \mathrm{tr}(\sigma_1 Z)}{|T_1|} \right) \left(\frac{\sum_{\sigma_2 \in T_2} \mathrm{tr}(\sigma_2 Z^\dagger)}{|T_2|} \right) \right\rangle}$$

$\exists$ $\quad O_{\sigma_1} = O_{\mu_1 \sigma_1 \mu_1^{-1}} \qquad O_{\sigma_2} = O_{\mu_2 \sigma_2 \mu_2^{-1}}$

$$= n! \sum_{\sigma_1 \in T_1} \sum_{\sigma_2 \in T_2} \sum_{\gamma, \sigma_3 \in S_n} \delta(\sigma_1 \sigma_2 \sigma_3) \, N^{C_{\sigma_3}}.$$

$$\langle \frac{\vartheta_{\sigma_1}}{\cancel{(N^{-C_{\sigma_1}+n})}} \frac{\vartheta_{\sigma_2}^{-1}}{\cancel{(N^{-C_{\sigma_2}+n})}} \rangle \underbrace{N^{C_{\sigma_1}-n}} \cdot \underbrace{N^{C_{\sigma_2}-n}} \cdot \frac{N^{-n}}{(n!)^2}.$$

$$= \frac{1}{n!} \sum_{\sigma_1 \in T_1} \sum_{\sigma_2 \in T_2} \sum_{\sigma_3 \in S_n}.$$

$$\delta(\sigma_1 \sigma_2 \sigma_3)$$

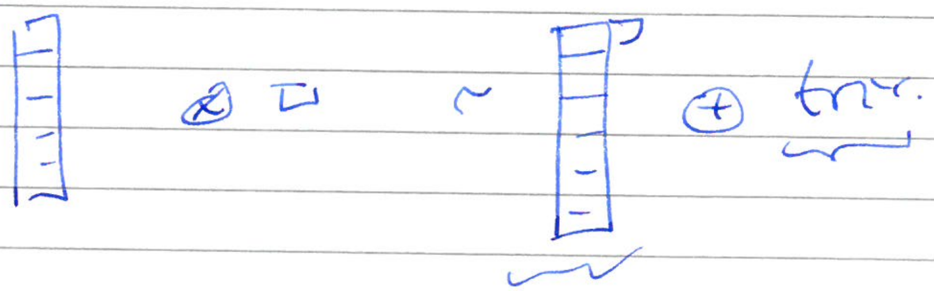
$$\underbrace{N^{(C_1-n)}} \cdot \underbrace{N^{(C_2-n)}} \cdot \underbrace{N^{(C_3-n)}}$$

↦ . What we have to do :

⊗ ⊗ Basic fact :

$$V \otimes \bar{V}$$

Decomposing under $U(N)$:



$$\rightarrow \quad N^2 = \underline{(N^2-1)} + \underline{1}.$$

~~Ad~~ adjoint

$$U \otimes U \; e_i \otimes \bar{e}_i$$

$$U_{ji} \, U^{+}_{ki} \qquad e_j \otimes e_k$$

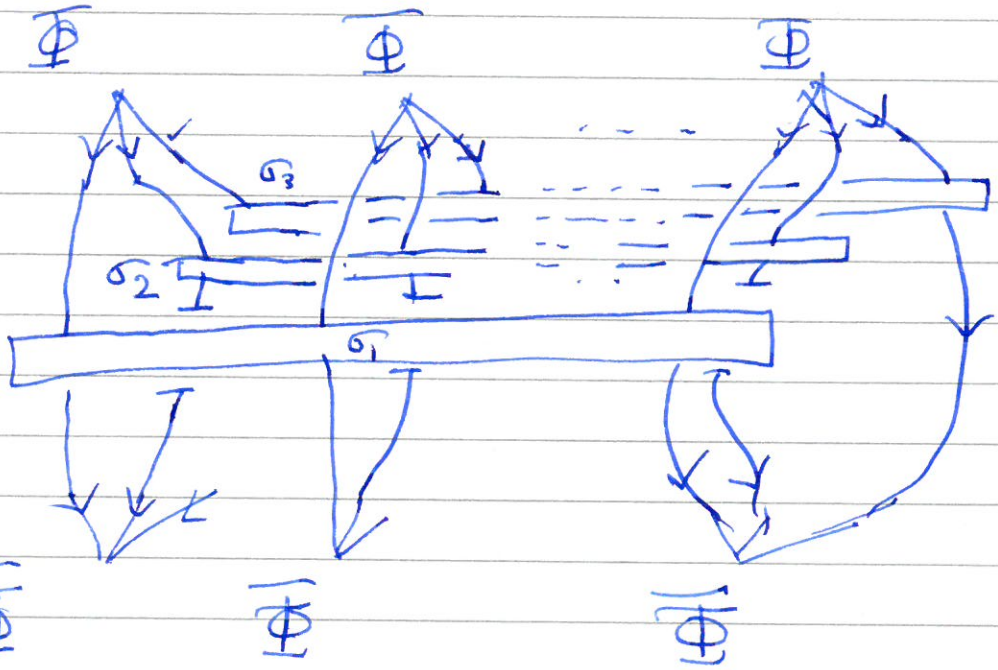
$$= (U_{ji} U^{+}_{ik}) \, e_j \otimes e_k$$

$$= \delta_{jk} \, e_j \otimes e_k = e_j \otimes e_j .$$

$$\nearrow \; \Phi_{i_1 j_1 k_1} \; \Phi_{i_2 j_2 k_2} \cdots \Phi_{i_n j_n k_n}$$

$$\overline{\Phi}_{i_{\sigma_1(1)} j_{\sigma_2(1)} k_{\sigma_3(1)}} \cdots \overline{\Phi}_{i_{\sigma_1(n)} j_{\sigma_2(n)} k_{\sigma_3(n)}}$$

$=$



$$\mapsto \; \mathcal{O}_{\sigma_1, \sigma_2, \sigma_3}(\Phi, \overline{\Phi})$$

$$= \; \mathcal{O}_{\gamma_1 \sigma_1 \gamma_2 , \; \gamma_1 \sigma_2 \gamma_2 , \; \gamma_1 \sigma_3 \gamma_2}(\Phi, \overline{\Phi})$$

Can do Matrix / Tensor Physics

$\mapsto (\sigma_1, \sigma_2, \sigma_3)$ without Matrices / Tensors

→ The information about gauge invariants is captured by equivalence classes of permutations:

$$(\sigma_1, \sigma_2, \sigma_3) \in S_n \times S_n \times S_n$$

$$(\sigma_1, \sigma_2, \sigma_3) \sim (\gamma_1 \sigma_1 \gamma_2, \gamma_1 \sigma_2 \gamma_2, \gamma_1 \sigma_3 \gamma_2)$$

⊗ How many invariant tensors are there :

$\frac{1}{n!}$ group $S_n \times S_n$

acting on $(S_n \times S_n \times S_N)$

→ Burnside lemma.

$$\rightarrow \frac{1}{(n!)^2} \sum_{\substack{\gamma_1, \gamma_2 \\ \sigma_1, \sigma_2, \sigma_3}} \delta(\gamma_1 \sigma_1 \gamma_2 \sigma_1^{-1}) \, \delta(\gamma_1 \sigma_2 \gamma_2 \sigma_2^{-1}) \, \delta(\gamma_1 \sigma_3 \gamma_2 \sigma_3^{-1})$$

Solve for γ_1 z: $\gamma_1 =$

$$\gamma_1 = (\sigma_1 \gamma_2^{-1} \sigma_1^{-1})$$

$$Z(n) = \frac{1}{(n!)^2} \sum_{\substack{\gamma_2, \\ \sigma_1 \sigma_2 \sigma_3}} \delta(\sigma_1 \gamma_2^{-1} \sigma_1^{-1} \cdot \sigma_2 \gamma_2 \sigma_2^{-1}) \, \delta(\sigma_1 \gamma_2^{-1} \sigma_1^{-1} \cdot \sigma_3 \gamma_2 \sigma_3^{-1})$$

$\tau_{12} = \sigma_1^{-1}\sigma_2$

$\tau_{13} = \sigma_1^{-1}\sigma_3$

$$= \frac{1}{(n!)^2} \sum_{\sigma_1} \sum_{\tau_{12},\tau_{13}} \delta\left(\tau_{12}\gamma_2\tau_{12}^{-1}\gamma_2^{-1}\right) \delta\left(\tau_{13}\gamma_2\tau_{13}^{-1}\gamma_2^{-1}\right)$$

$$= \left(\frac{1}{n!}\right) \sum_{\tau_{12},\tau_{13}} \delta\left(\tau_{12}\gamma_2\tau_{12}^{-1}\gamma_2^{-1}\right) \delta\left(\tau_{13}\gamma_2\tau_{13}^{-1}\gamma_2^{-1}\right)$$

Burnside Lemma:

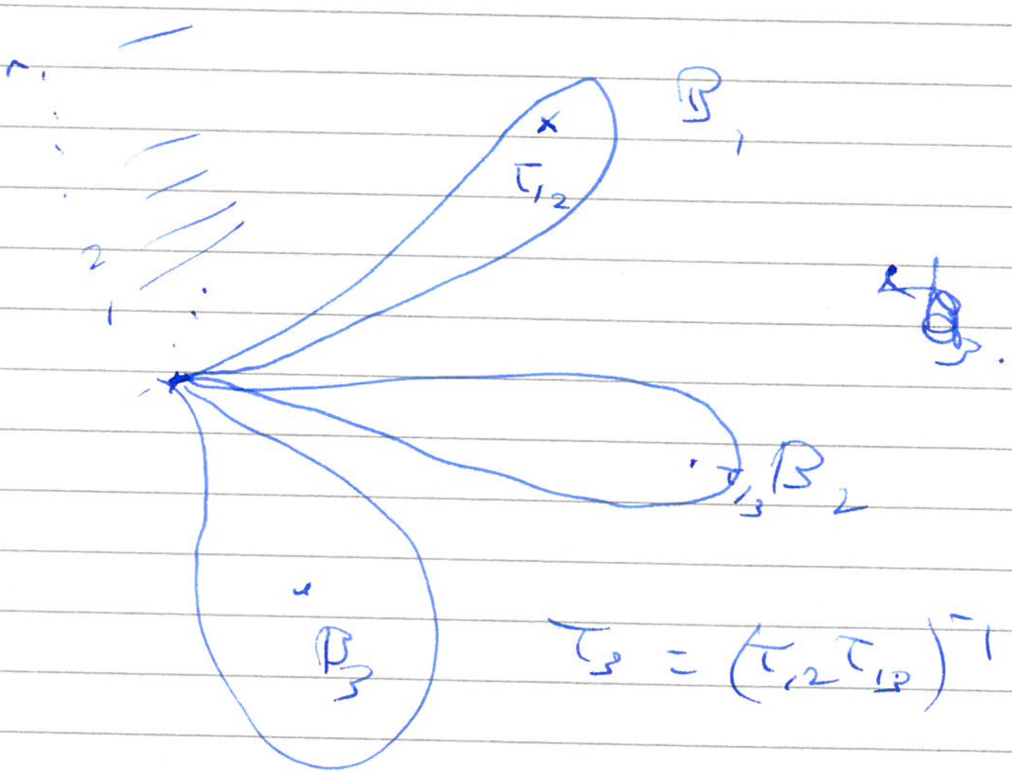
S_n acting on $\underbrace{S_n \times S_n}$

$$(\tau_{12}, \tau_{13}) \sim (\gamma\tau_{12}\gamma^{-1}, \gamma\tau_{13}\gamma^{-1})$$

Conjugacy classes of

τ_{12}, τ_{13}

geometry:



$\tau_3 = (\tau_{12}\tau_{13})^{-1}$

→ { Counting Branched Covers of Sphere with 3 branch points

$Z_3(n)$ = # of Branched Covers weighted with $\frac{1}{\text{Sym. factors}}$

↗ Branched covers with fixed branch points correspond to these pairs of perms :

$\left\{ \underline{Z_3(n)} \right.$ = # of Branched Covers with fixed Branch points

C.f. Complex Matrix Model

- 3-branch pts
- $\mathcal{O}_{\sigma_1}$, $\mathcal{O}^{+}_{\sigma_2}$ observables

: σ_3 – The power of N.

↳ Here No correlators yet.

$$\mapsto \sum_{\tau_1, \tau_2} \sum_{\gamma \in S_n} \delta(\tau_1 \gamma \tau_1^{-1} \gamma^{-1}) \, \delta(\tau_2 \gamma \tau_2^{-1} \gamma^{-1})$$

Sum over γ:

τ_1 commutes with γ.

$$\sum_\gamma \to \sum_\rho$$

$$\underbrace{\sum_{\rho \vdash n} \frac{1}{\operatorname{Sym} \rho'}}\, (\operatorname{Sym} \rho) \cdot (\operatorname{Sym} \rho')$$

$$= \underbrace{\sum_{\rho \vdash n} (\operatorname{Sym} \rho)}$$

$$= \underbrace{\sum_{\rho \vdash n} \left(\prod_{i=1} i^{\rho_i} \rho_i! \right)}$$

~~Work~~

— Due to the - - Sequence of Numbers

⊗ $\{1, 4, 11, 43, 161, 901 \ldots$

One Sequence:

$\mapsto$ (a) Rmk:

- Belyi Maps

- Any curve with $\{0, 1, \infty\}$ can be written in terms of ~~Curve Equat~~ $y = P(x)$

& $f : (y, x) \to f(y, x)$

- everything defined over

  $\mathbb{Q}$ or

  Extensions of $\mathbb{Q}$.

- Eg. with coeffs in $\mathbb{Q}$

- (i.e.) Curves & Maps are

  $\overline{\overline{\mathbb{Q}}}$

⊗ Talk about Color Sym. Cards

⊗ Give answer for Curves.

⊗ $K(1)$ & $K_{Y}(n)$

- Gauge-fixing:

Gauge fixing:

$(\sigma_1, \sigma_2, \sigma_3)$

Use σ_1^{-1} on left

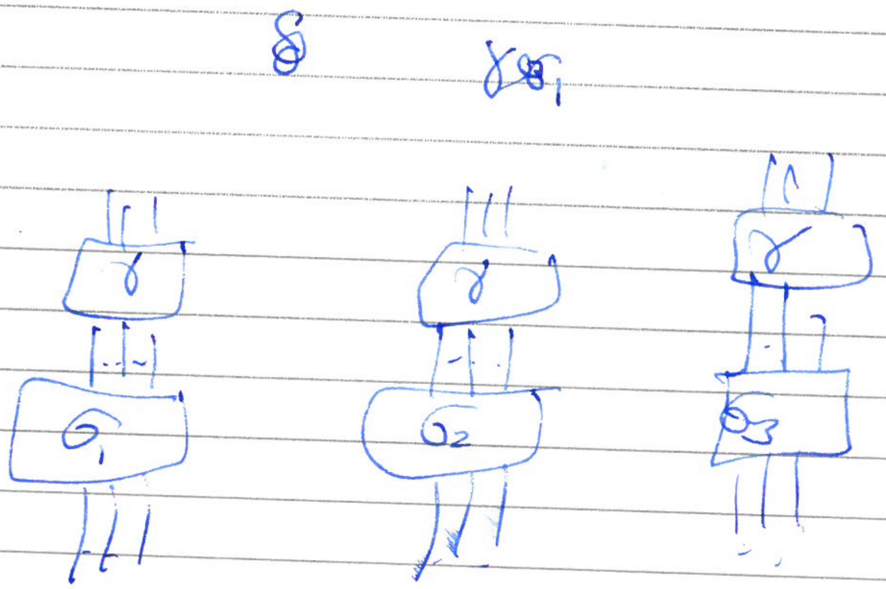
$(1, \sigma_1^{-1}\sigma_2, \sigma_1^{-1}\sigma_3)$

$$\begin{tikzcd}
\text{Use}\ (\sigma_1, \sigma_2, \sigma_3) \arrow[d, "\text{Use } \gamma_2"'] \arrow[r, phantom, "\sim"] & (\gamma_1\sigma_1, \gamma_1\sigma_2, \gamma_1\sigma_3) \arrow[d, Rightarrow, "\text{with } \gamma_1 = \sigma_1^{-1}"] \\
(\sigma_1\gamma_2, \sigma_2\gamma_2, \sigma_2\gamma_2) \arrow[dr, bend left, "\text{Use } \gamma_1 = \gamma_2^{-1}\sigma_1^{-1}"] & (1, \sigma_1^{-1}\sigma_2, \sigma_1^{-1}\sigma_3) \arrow[d, squiggly, "\gamma_2 \text{ by conj.}"] \\
 & (1, \gamma_2^{-1}(\sigma_1^{-1}\sigma_2)\gamma_2, \gamma_2^{-1}(\sigma_1^{-1}\sigma_3)\gamma_2)
\end{tikzcd}$$

~~$\gamma_2\sigma_1^{-1}$~~

Correlators:

$=$



$$\left\langle \theta_{\sigma_1, \sigma_2, \sigma_3} \right\rangle$$

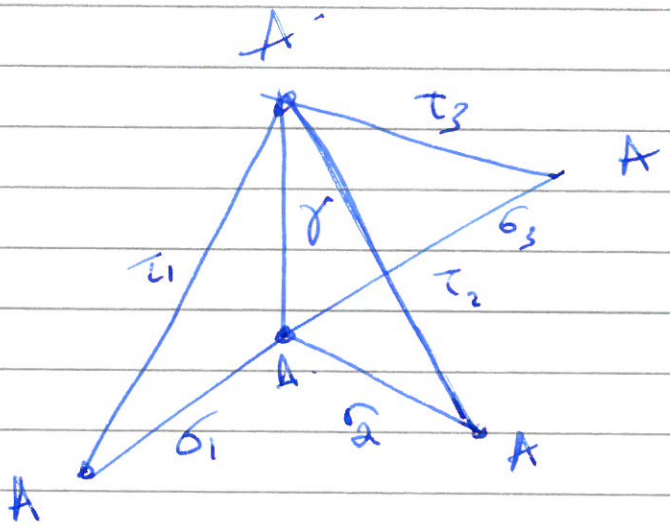
$$= \sum_{\gamma} N^{C_{\gamma\sigma_1} + C_{\gamma\sigma_2} + C_{\gamma\sigma_3}}$$

$$= \sum_{\gamma} \delta(\gamma\sigma_1 \tau_1)\, \delta(\gamma\sigma_2 \tau_2)$$

$$\delta(\gamma\sigma_3 \tau_3)$$

$$\underbrace{N^{C_{\tau_1} + C_{\tau_2} + C_{\tau_3}}}$$

$\mapsto$



- We get covering of a 2-complex which has the above shape.

- & specified.

$V+F-E = 1+1-\overset{2}{\cancel{3}} = \underline{\underline{-1}}$.

~~$3+1-3=1$~~.

$$\left\{\begin{array}{l} \cancel{=2-2G-B}\ \overset{1}{} \\ =1 \end{array}\right.$$

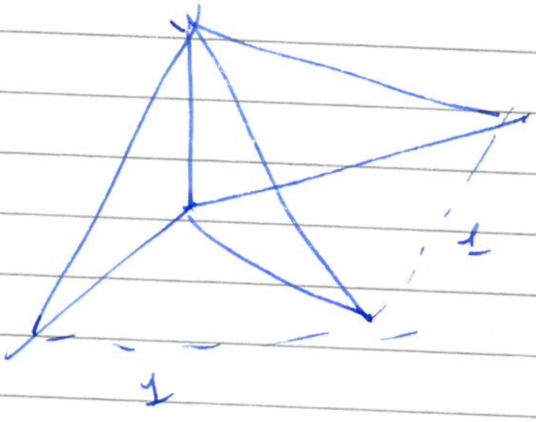
$2-2G-B = 2-3 = -1$ } — 3 holed sphere.

↦ This can be the
2-complex of a 3-manifold
(not of a smooth 2-manifold)

↦ Suggests top. strings with
a 3D-target:

↦ TFT2 – S_n with defects:

- γ – an internal line
  – sum as in TFT2
- $\sigma_1, \sigma_2, \sigma_3$: Perm. Constrained holonomies
- $\tau_i \to \underbrace{\Sigma_i N^{C\tau_i}} t_i$



- Unit defects :

Restricted diffeo. A

A e.g. Restrict the diffeos to avoid changing the intersections of these defect lines

Problem:

↳ Study the Restricted topological invariance in the presence of these defects.

→ O.P.F of these defects

⊘ Characterise the 3D TFT
of [illegible]. Underlying these
correlated

Finite N Counting:

⊘ β-index tensors : — Finite N

Counting

↳ $Z_3(n) = \sum_{p \vdash n} (\mathrm{Sym}\, p)^2 .$

$Z_d(n) = \sum_{p \vdash n} (\mathrm{Sym}\, p)^{d-2}$

- Tensors of rank d
- with $U(N)^{\times d}$ symmetry.

↳ Finite N Counting :

$$\rightarrow \sum_{\substack{R_1 \vdash n \\ R_2 \vdash n \\ R_3 \vdash n \\ l(R_i) \leq N}} \left(C(R_1, R_2, R_3) \right)^2$$

↦

$$\frac{1}{(n!)^2}\sum_{\substack{\gamma_1,\gamma_2\\ \sigma_1\,\sigma_2\,\sigma_3}} \delta(\gamma_1\,\sigma_1\,\gamma_2\,\sigma_1^{-1})\;\delta(\gamma_1\,\sigma_2\,\gamma_2\,\sigma_2^{-1})\;\delta(\gamma_1\,\sigma_3\,\gamma_2\,\sigma_3^{-1})$$

$$= \frac{1}{(n!)^2}\sum D^{R_1}_{i_1 j_1}(\gamma_1)\,D^{R_1}_{j_1 k_1}(\sigma_1)\,D^{R_1}_{k_1 l_1}(\gamma_2)\,D^{R_1}_{l_1 i_1}(\sigma_1^{-1})$$

$$D^{R_2}_{i_2 j_2}(\gamma_1)\,D^{R_2}_{j_2 k_2}(\sigma_2)\,D^{R_2}_{k_2 l_2}(\gamma_2)\,D^{R_2}_{l_2 i_2}(\sigma_2^{-1})$$

$$\cancel{\frac{1}{(n!)^4}}\frac{1}{(n!)^3}\times\begin{pmatrix} d_{R_1}\\ d_{R_2}\\ d_{R_3}\end{pmatrix}\times \underbrace{D^{R_3}_{i_3 j_3}(\gamma_1)}\,D^{R_3}_{j_3 k_3}(\sigma_3)\,\underbrace{D^{R}_{k_3 l_3}(\gamma_2)}\,D^{R_3}_{l_3 i_3}(\sigma_3^{-1})$$

$$= \frac{1}{(n!)^2}\left(n!\,S^{R_1\,R_2\,R_3\,;\,\tau_1}_{i_1\;i_2\;i_3}\;S^{R_1\,R_2\,R_3\,;\,\tau_1}_{j_1\;j_2\;j_3}\right)$$

$$\left(n!\,S^{R_1\,R_2\,R_3\,;\,\tau_2}_{k_1\;k_2\;k_3}\;S^{R_1\,R_2\,R_3\,;\,\tau_2}_{l_1\;l_2\;l_3}\right)$$

$$\frac{d_R\,d_{R_2}\,d_{R_3}}{(n!)^3}\;\frac{n!}{\cancel{d_{R_1}}}\,\frac{n!}{\cancel{d_{R_2}}}\,\frac{n!}{\cancel{d_{R_3}}}\;\delta_{j_1 i_1}\,\delta_{k_1 l_1}\;\delta_{j_2 i_2}\,\delta_{k_2 l_2}\;\delta_{j_3 i_3}\,\delta_{k_3 l_3}$$

$$= \sum_{R_1, R_2 R_3 \vdash n} \begin{matrix} C(R_1\ R_2\ R_3) \\ C(R_1, R_2, R_3) \end{matrix}$$

$$\underline{\underline{l(R_i) \leq H.}}$$

$$\sum_R d_R \chi_R$$

$$= n!\,\delta$$

$$\delta = \frac{1}{n!} \sum_R d_R \chi_R .$$

$\mapsto$. For Rank $\underline{\underline{d}}$.

$$\sum_{R_1 \cdots R_d \vdash n} \left(C(R_1, R_2, \cdots R_d) \right)^2$$

$$l(R_i) \leq H.$$

∘ Use group theory identities:

$$\delta(\sigma) = \sum_{R \vdash n} \frac{d_R \chi_R(\sigma)}{n!}$$

$$\sum_{\sigma} D^{R}_{ij}(\sigma) D^{S}_{kl}(\sigma) = \frac{n!}{d_R} \delta^{RS} \delta_{ik} \delta_{jl}$$

$$\sum_{\gamma} D^{R_1}_{i_1 j_1}(\gamma) D^{S}_{i_2 j_2}(\gamma) D^{T}_{i_3 j_3}(\gamma)$$

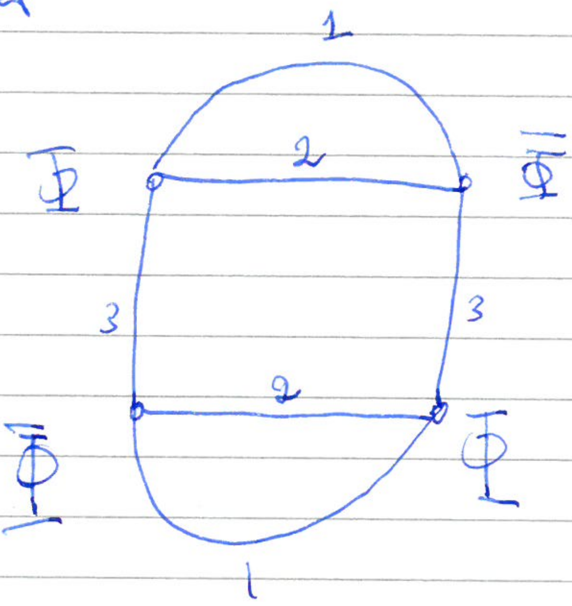
$$= \sum_{\tau_1} S^{R, S, T ; \tau_1}_{i_1 i_2 i_3} S^{R S T ; \tau_1}_{j_1 j_2 j_3}$$

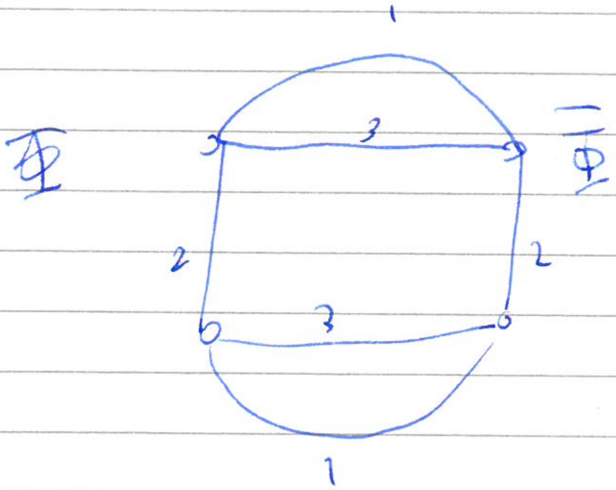
$$\boxed{1 \le \tau_1 \le \text{Inv.}(R_1 \otimes R_2 \otimes R_3)}$$

- Color - symmetrized Counting of Number sequences

- Color_Symm :

Consider





same up to color-symmetrization

→ $(\sigma_1, \sigma_2, \sigma_3) \sim (\sigma_2, \sigma_1, \sigma_3) \sim \cdots$

→ Equivalence classes gen. by swapping the 3 perms;
& left γ_1 –
Right γ_2 – mult.

→ Find color sym. subspace of

$$\mathbb{C}(S_n) \otimes \mathbb{C}(S_n) \otimes \mathbb{C}(S_n)$$

$$[\sigma_1 \otimes \sigma_2 \otimes \sigma_3]$$

$$= \sum_{\alpha \in S_3} \left(\frac{1}{3!}\right) \sigma_{\alpha(1)} \otimes \sigma_{\alpha(2)} \otimes \sigma_{\alpha(3)}$$

→ Then impose L_{γ_1}, R_{γ_2}.

$$\frac{1}{(n!)^2} \sum_{\sigma_1, \sigma_2, \gamma} \delta(\underbrace{\gamma \sigma_1 \gamma^{-1} \sigma_1^{-1}}) \; \delta(\gamma \sigma_2 \gamma^{-1} \sigma_2^{-1})$$

$$= \underbrace{\sum_{p} \left(\frac{1}{n!}\right)^2 \cdot \left(\frac{n!}{\text{Sym } p}\right)}_{\downarrow} \cdot \left(\text{Sym } p\right)^2$$

Converting $\sum_{\gamma}$: $\sum_{p \vdash n}$

$$= \frac{1}{n!} \sum_{p} \text{Sym } p$$

$\otimes$:

$$Z_{3;\, s.c.}(N)$$

$$= \frac{1}{6n!} \sum_{p \vdash n} \mathrm{Sym}\, p$$

$$+ \frac{1}{2n!} \sum_{\gamma \in S_n} \sum_{\sigma \in S_n} \delta\left(\gamma^{2} \sigma \gamma^{-2} \sigma^{-1}\right)$$

$$+ \frac{1}{3} \cdot \frac{1}{n!} \sum_{\gamma, \sigma \in S_n} \underbrace{\delta\left(\gamma^{3} . \sigma^{3}\right)}$$

$\otimes$: Find Nal.

Each term e.g.

$$\left(\frac{1}{3}\right)$$

$$\left(\frac{1}{6}\right) S^{(3)}_{[1^3]}(n) \quad ;$$

$$= \mathbb{Z}$$

$$S^{(3)}_{[1^3]}(n) = \sum_{R_1 R_2 R_3 \vdash n} \left(\hat{C}(R_1, R_2 R_3)\right)^2$$

$$= \frac{1}{n!} \sum_{p \vdash n} (\mathrm{Sym}\, p)$$

$$S^{(3)}_{[2,1]}(n) = \frac{1}{n!}\sum$$

$$= \left(\frac{1}{n!}\right) \sum_R \sum_\sigma \chi^R(\sigma^2)\chi^R(\sigma^2)$$

$$= \frac{1}{n!} \sum_{\gamma, \sigma} \delta(\gamma^2 \sigma \gamma^{-2} \sigma^{-1})$$

$$S^{(3)}_{[3]}(n) = \frac{1}{(n!)^2} \sum_R \sum_{\sigma_1, \sigma_2} \chi^R(\sigma_1^3)\chi^R(\sigma_2^3).$$

$$= \frac{1}{(n!)} \sum_{\gamma, \sigma} \delta(\gamma^3 \sigma^3)$$

$S^{(3)}_p(n)$: Each integral.

– The fractions not int.

– But combine to give integers

$\mathbb{Z}$ s. c

Ly in B.G.R 1708,

we showed that:

$$Z_{S.C.} = Z_{Y_0}$$

Consider also $\left\{ \begin{array}{l} Z_{Y=\begin{smallmatrix}\square\square\\ \square\end{smallmatrix}} \\ Z_{Y=\begin{smallmatrix}\square\\ \square\\ \square\end{smallmatrix}} \end{array} \right.$

i.e. $\left(\mathbb{C}(S_n) \otimes \mathbb{C}(S_n) \otimes \mathbb{C}(S_n) \right)$

$\left\{ \begin{array}{l} \text{is acted on by left } \mathbb{C}(A) \\ \qquad\qquad\quad\ \text{right } \mathbb{C}(A) \\ \text{\& } S_3 \text{ of swapping the 3 } S_n\text{'s} \end{array} \right.$

$$\underbrace{\left\{ \oplus\, \mathbb{C}(S_3) \otimes \mathbb{C}(S_n) \otimes \mathbb{C}(S_n) \right\}}_{\text{acts on it}}$$

- Can decompose into irreps of S_3 Y.

→ The multiplicities are integers

→ Using this integrality

~~we can~~ T

& integ. of $\chi^{\gamma}(\sigma)$

we can prove integrality of the

$$\underbrace{\int_p^{(n)}}$$

↙

This generalizes to any d.

i.e. any pLd,

◦ So how do we count the

color-symmetrized ~~graph~~ :

subspace :

— Dim. of color sym. subspace

— Counting without distinguishing graphs ~~then~~ if they are related by a re-coloring e.g. 2 ↔ 3 above

— $(\sigma_1, \sigma_2, \sigma_3)$

$Z(t, x_1, x_2, \cdots)$

$$= e^{\sum_{i=0}^{\infty} \left(\frac{x_i}{i} \right)}$$

↗ $Z^{(2)}(t, x_1, x_2, x_3, x_4 \ldots)$

$$= Z^{(0)}(t, x_1, x_2 = x_1^2, x_3, x_4 = x_2^2, \ldots)$$

↗ $Z^{(3)}(t, x_i)$

$$= Z\left(t, x_i \to (x_{\frac{i}{3}})^3 \Big|_{\text{if } i/3 \in \mathbb{Z}}\right)$$

↗. Similarly $\underbrace{Z^{(k)}}$ $\forall k$

show allow formulae.