Feynman diagrams in Matrix Models and the absolute Galois group of rationals

Sanjaye Ramgoolam

Queen Mary - London

B(A)MC, April 7, 2010

Based on paper arXiv:1002.1634[hep-th]
with Robert de Mello Koch.

One Matrix Model

$$\mathcal{Z} = \int dX \;\; e^{-\frac{1}{2} trX^2}$$

X : $N \times N$ Hermitian matrix

$$dX \equiv \prod_{i<j} dRe(X_{ij}) dIm(X_{ij}) \prod_i dRe(X_{ii})$$

$$\mathcal{Z}(g) = \int dX e^{-\frac{1}{2} trX^2 + V(X,g)} = \int dX e^{-\frac{1}{2} trX^2 + g_3 trX^3 + g_4 trX^4 + \cdots}$$

One Matrix Model : Obserbvables

The Observables of interest : Trace moments of the matrix variables.

$$\langle \mathcal{O}(X) \rangle = \int dX e^{-\frac{1}{2} trX^2} \mathcal{O}(X) \cdots$$

The $\mathcal{O}(X)$ is a function of traces, e.g $\mathcal{O}(X) = (trX)^{p_1} (trX^2)^{p_2} \cdots$.

Fixing the total number of X to be n, the number of these observables is $p(n)$. The number of partitions of n.

$$n = p_1 + 2p_2 + 3p_3 + \cdots$$

Partitions of n correspond to conjugacy classes of the symmetric group S_n, of all permutations of n objects.

It is possible to associate observables to permutations

$$\mathcal{O}_\sigma(X)$$

which only depend of the conjugacy class.

$$\mathcal{O}_{\alpha\sigma\alpha^{-1}}(X) = \mathcal{O}_\sigma(X)$$

Conjugacy classes in S_n are characterized by the cycle decomposition of the permutations. e.g a permutation (123)(45) in S_5 cyclically permutes $1, 2, 3$ and swops $4, 5$.

The conjugacy class of such a permutation corresponds to $trX^3 trX^2$, i.e if $\sigma = (123)(45)$,

$$\mathcal{O}_\sigma(X) \sim trX^3 trX^2$$

We will choose a normalization of observables as

$$\begin{aligned}\mathcal{O}_\sigma(X) &= N^{-n+p_1(\sigma)+p_2(\sigma)+\cdots+p_n(\sigma)}(trX)^{p_1}(trX^2)^{p_2}\cdots(trX^n)^{p_n} \\ &= N^{C_\sigma-n}(trX)^{p_1}(trX^2)^{p_2}\cdots(trX^n)^{p_n}\end{aligned}$$

We will define a delta function over the symmetric group

$$\begin{aligned}\delta(\sigma) &= 1 \text{ if } \sigma = 1 \\ &= 0 \text{ otherwise}\end{aligned}$$

Theorem 1 :

$$\langle \mathcal{O}_\sigma \rangle = \frac{1}{(2n)!} \sum_{\sigma \in [\sigma]} \sum_{\gamma \in [2^n]} \sum_{\tau \in S_{2n}} \delta(\sigma\gamma\tau) N^{C_\sigma + C_\tau - n}$$

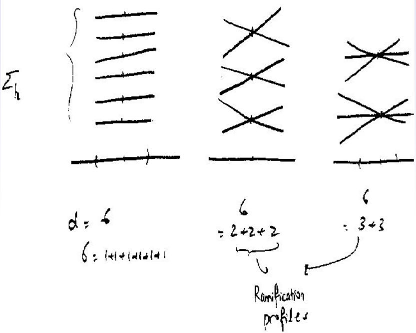
Follows using Wick's theorem. The sum over γ, which is in $[2^n]$ is the sum over Wick contractions.

Equivalently, this is the sum over Feynman diagrams of the Gaussian matrix model.

Use a classic theorem : The Riemann Existence theorem, which relates the counting of such strings of permutations to the counting of equivalence classes of holomorphic maps $f : \Sigma_h \to \mathbb{P}^1$, from Riemann surface Σ_h of genus h to target $\mathbb{P}^1$.

Holomorphic maps between Riemann surfaces are branched covers.

An interval through a generic point on the target Riemann surface : inverse image has d intervals, where d is the degree of the map. A branch point has fewer inverse images.



Σ_h
d = 6
6 = 1+1+1+1+1+1
6
= 2+2+2
6
= 3+3
Ramification profiles

Each branch point has a ramification profile which is a partition of the degree d. The ramification data determines the genus h of Σ_h by the Riemann Hurwitz formula.

$$\langle \mathcal{O}_\sigma \rangle = \sum_{f:\Sigma_h \to \mathbb{P}^1} \frac{1}{|Aut\ f|} N^{2-2h}$$

The Gaussian Matrix model correlator is a sum over equivalence classes of holomorphic maps to $\mathbb{P}^1$, branched at 3 points, with ramification profiles $[\sigma]$, $[\gamma] = [2^n]$ and $[\tau]$ which is general.

Weighted by g_{st}^{2h-2} where $g_{st} = \frac{1}{N}$

Physics Interpretation

The Gaussian Matrix model is equivalent to a topological string theory, with target space $\mathbb{P}^1$ which localizes to holomorphic maps with three branch points.

A perturbed Gaussian model also has such an interpretation with e^V treated as an observable.

MEANING OF THREE ?

Belyi theorem : A Riemann surface is defined over algebraic numbers iff it admits a map to $\mathbb{P}^1$ with three branch points.

Riemann surface can be described by algebraic equations, e.g an elliptic curve

$$y^2 = x^3 + ax^2 + bx$$

If a, b are algebraic numbers, i.e solutions to polynomial equations with rational coefficients $\mathbb{Q}$, then the Riemann surface is defined over $\bar{\mathbb{Q}}$, i.e for $x \in \bar{\mathbb{Q}}$, $y \in \bar{\mathbb{Q}}$

The field $\bar{\mathbb{Q}}$ is the algebraic closure of $\mathbb{Q}$ and contains all solutions of polynomial equations with rational coefficients.

It contains finite extensions of $\mathbb{Q}$ such as $\mathbb{Q}(\sqrt{2})$.

This is numbers of the form $a + b\sqrt{2}$, where a, b are rational. They form a field, closed under addition, multiplication, division.

An important group associated to this extension is the group of automorphisms which preserves the rationals. In this case, the only non-trivial element of the group is $\sqrt{2} \rightarrow -\sqrt{2}$. We say $Gal(\mathbb{Q}(\sqrt{2})/\mathbb{Q}) = \mathbb{Z}_2$

The absolute Galois group $Gal(\bar{\mathbb{Q}}/\mathbb{Q})$ contains as subgroups, all the finite Galois groups of finite dimensional extensions.

It acts on the algebraic numbers coefficients of the defining equations of the curve Σ_h and of the map f.

Hence the Galois group acts on the (equivalence classes of) permutation triples, equivalently the Feynman graphs of the 1-matrix model.

Grothedieck associated Dessins to the permutation triples, which are essentially the Feynman graphs of the 1-matrix model.

The multiplicity of Feynman graphs can be organised into orbits of the Galois group action.

Elements in the same orbit contribute with equal weight, since *Autf* is a Galois invariant.

$$\langle \mathcal{O}_{[\sigma]} \rangle = F_1 + F_2 + F_3 + F_4 + F_5$$

$F_1 + F_2 + F_3$: Galois orbit

$F_4 + F_5$: Galois orbit

F_1 → Equivalence class of σ, γ, τ ;

$$\frac{1}{\mathrm{Aut}\, f} = \frac{1}{\mathrm{Aut}(\sigma, \gamma, \tau)}$$

→ This is a Galois invariant

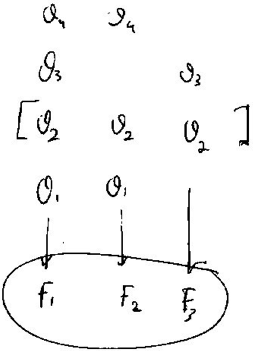
⇓

Feynman diagrams in a Galois orbit contribute with same weight

An obvious generalization to consider is multi-matrix models, where we have integrals over multiple matrix variables, e.g X, Y.

The edges of the Feynman graphs, which are propagators are now colored, i.e they can be X or Y propagators. So they correspond to colored-edge versions of Grothendieck Dessins.

- A given multi-matrix observable, e.g $trX^2Y^2trXtrY^3$ can receive zero contribution from one Dessin in a Galois orbit and non-zero from another.
- Colorings of the Dessins allow the definition of new invariants of the action of $Gal(\bar{\mathbb{Q}}/\mathbb{Q})$ on the Dessins : constructed from lists of multi-matrix observables which receive contributions from the Dessin.
- Known invariants can be described in terms of these lists.
- See paper



Intersection of lists of Multi-Matrix Observables is a Galois Invariant.

Two of Many questions

How are the Physics-inspired invariants constructed from coloured Dessins related to number theoretic invariants ?

Belyi theorem suggests that the string theory of 1-matrix model can be defined over $\bar{\mathbb{Q}}$. Is there an explicit construction of string amplitudes and path integrals over $\bar{\mathbb{Q}}$?