Permutations, Strings and Feynman Graphs.

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Based on papers
From Matrix Models and quantum fields to Hurwitz space and the absolute Galois group
arXiv:1002.1634[hep-th] ; Robert de Mello Koch, Sanjaye Ramgoolam
Permutations, Strings and Feynman Graphs
Robert de Mello Koch, Sanjaye Ramgoolam ; to appear : arXiv:1109.****
Introduction

There is a very fundamental connection between permutations and strings.
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Example \( \sigma_1, \sigma_2, \sigma_3 \) are 3 permutations among the 6 in \( S_3 \).

\[
\begin{align*}
\sigma_1 & : \{1, 2, 3\} \rightarrow \{1, 2, 3\} \\
\sigma_2 & : \{1, 2, 3\} \rightarrow \{2, 1, 3\} \\
\sigma_3 & : \{1, 2, 3\} \rightarrow \{2, 3, 1\}
\end{align*}
\]
Other ways of describing $\sigma_2$:

\[
\sigma_2(1) = 2, \sigma_2(2) = 1, \sigma_2(3) = 3
\]
\[
\sigma_2 = 213
\]
\[
\sigma_2 = (12)(3)
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In cycle notation

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In cycle notation

$$\sigma_1 = (1)(2)(3)$$
$$\sigma_2 = (12)(3)$$
$$\sigma_3 = (123)$$

Two permutations $\sigma, \sigma'$ having the same cycle structure are related by conjugation.

$$\sigma = \gamma \sigma \gamma^{-1}$$

e.g. $(12)(3)$ and $(13)(2)$ are related by $\gamma = (1)(23)$. 
$S_3$ has 3 cycle structures, equivalently 3 conjugacy classes. Consider the possible states of 3-strings winding around a circle.

\[
\begin{align*}
X(\sigma) &= 0 & \text{Singly Wound} \\
X(\sigma) &= 2\pi & \text{Doubly Wound} \\
X(\sigma) &= 3\pi & \text{Triply Wound}
\end{align*}
\]

$X \sim X + (2\pi)$

$0 \sim 0 + (2\pi)$
Starting from a configuration of such strings we can label the points above a fixed spacetime point and obtain a permutation.
The number of winding states, or cycle structures in $S_n$, is the number of partitions of $n$, called $p(n)$: a well-studied number in Mathematics. Its asymptotics is relevant to Hagedorn transition.

\[
\begin{align*}
  3 &= 1 + 1 + 1 \\
  2 &= 2 + 1 \\
  3 &= 3
\end{align*}
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\end{align*}
\]

When we consider string interactions, the permutations themselves matter, not just the cycle structure they belong to.
The connection between strings and permutations plays a central role in gauge-string duality. I will explain 3 connections to illustrate this.

- Large N expansion of 2 dimensional Yang Mills partition function.

- Large N expansion of Hermitian Matrix model correlators.

- Feynman Graph counting in scalar field theory.

The third example suggests that Large $N$ is not crucial to strings emerging from Quantum Field theory.
Gauge-String Duality.
Some dynamics of quantum field theory with matrix fields has a
dual description in terms of String theory. Maldacena’s
AdS/CFT duality between strings on $AdS_5 \times S^5$ and $N = 4$
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$U(N)$ gauge group, on Riemann surface $\Sigma_G$ of genus $G$ and
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In lower dimensions, we have 2D pure Yang Mills theory with $U(N)$ gauge group, on Riemann surface $\Sigma_G$ of genus $G$ and area $A$.

This theory is quasi-topological and the partition function depends only on $G, A$. In this case, the string theory has 2D target $\Sigma_G$, as discovered by Gross and Taylor in mid-nineties.
Permutations are key to organising the gauge invariant operators. For 2dYM on a cylinder, defining the partition function requires specifying the boundary condition, which is a group element $U$ in $U(N)$ at each boundary.

\[ U_1 = e^{i \oint_{c_1} A \cdot ds} \]
\[ U_2 = e^{i \oint_{c_2} A \cdot ds} \]
The gauge-invariant functions are traces.

\[ tr(U^3), trU^2 trU, (trU)^3 \]

Some are linear in traces, some non-linear. Permutations give a unified linear way of thinking about all of them.
\[ trU^2 = U_{i_2}^{i_1} U_{i_1}^{i_2} \]
\[ = U_{i_{\sigma(1)}}^{i_1} U_{i_{\sigma(2)}}^{i_2} \]

with \( \sigma = (12) \)

\[ (trU)^2 = U_{i_1}^{i_1} U_{i_2}^{i_2} \]
\[ = U_{i_{\sigma(1)}}^{i_1} U_{i_{\sigma(2)}}^{i_2} \]

Multi-traces are constructed by using different permutations.

\[ tr_{V \otimes n}(\sigma U \otimes n) \]

Different permutations with the same cycle structure give the same trace. Replacing \( \sigma \rightarrow \gamma \sigma \gamma^{-1} \) leaves the trace invariant.
In 2dYM, the partition function $Z(U_1, U_2)$ on a cylinder (and any Riemann surface) can be written exactly in terms of representations of $U(N)$.

We can transform to a permutation basis

$$Z(\sigma_1, \sigma_2) = \int dU_1 dU_2 Z(U_1, U_2) tr_n(\sigma_1 U_1^\dagger) tr_n(\sigma_2 U_2^\dagger)$$

$$Z(\sigma_1, \sigma_2) = \sum_{\gamma \in S_n} \delta(\sigma_1 \gamma \sigma_2 \gamma^{-1})$$
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$$Z(\sigma_1, \sigma_2) = \sum_{\gamma \in S_n} \delta(\sigma_1 \gamma \sigma_2 \gamma^{-1})$$

This is the answer in the zero area limit.
The $\delta$ function is defined like a Kronecker delta, except over the symmetric group:

$$\delta(\sigma) = \begin{cases} 1 & \text{if } \sigma = \text{identity} \\ 0 & \text{otherwise} \end{cases}$$

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$$\delta(\sigma) = 1 \text{ if } \sigma = \text{identity}$$

$$= 0 \text{ otherwise}$$

Can be manipulated like Kronecker Delta.

The permutation $\gamma$ is the re-lebelling of sheets of the cover in going from one boundary to another. The delta function ensures that the two cycle structures are the same.
Figure: Paths and permutations on cylinder
At non-zero area the sum is modified to include additional permutations which can be interpreted as a counting of branched covers (holomorphic maps) where $\partial_z f$ of the map is allowed to vanish at certain points on the worldsheet.
Let us leave 2dYM aside, to give a simple illustration of this point in the Hermitian matrix model.

The physics of the Gaussian measure for the eigenvalue distribution is encoded in correlators of traces:

$$\int dX \ e^{-\frac{1}{2}trX^2} \mathcal{O}(X)$$

The observables are general traces:

$$trX, \ trX^2, \ trXtrX, \ trX^3, \ trX^2trX, \ trXtrXtrX$$
One finds that

$$
< \mathcal{O}_p > = \frac{1}{n!} \sum_{\sigma_1 \in p \in S_n} \sum_{\sigma_2 \in [2^{n/2}]} \sum_{\sigma_3} \delta(\sigma_1 \sigma_2 \sigma_3) N^{C_{\sigma_3}}
$$

Here \( n \) is even. The permutation \( \sigma_3 \) is arbitrary, but \( \sigma_2 \) has the cycle structure \([2^{n/2}]\), i.e. of type

$$
(12)(34) \cdots (n-1 \, n)
$$

This comes from the fact that the computation of the correlators can be done by Wick contractions, which are pairings of the \( n \) matrices in the observable.
One can use some **classical mathematics of Riemann** to relate this formula directly to the geometry of branched covers, i.e. holomorphic maps a worldsheet $\Sigma$ to a sphere.

**Holomorphy**: 

$$\partial \bar{z} f = 0$$

**The powers of $N$** keep track of the genus of the worldsheet.

**Three permutations**: If $\partial_z f(P) = 0$ then $f(P) \in \{0, 1, \infty\}$.

Details in arXiv:1010.1634
A third connection involves a QFT without large $N$. Just real scalar field theory, for concreteness, take vacuum diagrams in $\phi^4$ theory.

Calculations in QFT are simplified by organizing the large number of Wick contractions, into graphs, each of which comes with a symmetry factor.
For $v = 1$ there is one graph. For $v = 2$, there are 3 graphs, etc.

Figure: One vertex vacuum diagram in $\phi^4$ theory

Figure: Two vertex vacuum diagrams in $\phi^4$ theory
This sequence of vacuum diagrams

\[1, 3, 7, 20, 56, 187, 654, 2705, 12587, 67902, 417065, \ldots\]

has an expression in terms of string amplitudes, of the kind that appears in 2dYM.
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\[
\frac{1}{|H_1||H_2|} \sum_{\sigma_1 \in H_1} \sum_{\sigma_2 \in H_2} \sum_{\gamma \in S_{4v}} \delta(\sigma_1 \gamma \sigma_2 \gamma^{-1})
\]

Number of diagrams with \(v\) vertices
$H_1$ is a subgroup of $S_{4^v}$:

$$(S_4 \times S_4 \cdots \times S_4) \rtimes S_v \equiv S_v[S_4]$$

There are $v$ copies of $S_4$ and $S_v$ acts as an automorphism of this product group.
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$H_2$ is a subgroup of $S_{4^v}$:

$$(S_2 \times S_2 \cdots \times S_2) \rtimes S_{2^v} \equiv S_{2^v}[S_2]$$
$H_1$ is a subgroup of $S_{4v}$:

$$(S_4 \times S_4 \cdots \times S_4) \rtimes S_v \equiv S_v[S_4]$$

There are $v$ copies of $S_4$ and $S_v$ acts as an automorphism of this product group.

$H_2$ is a subgroup of $S_{4v}$:

$$(S_2 \times S_2 \cdots \times S_2) \rtimes S_{2v} \equiv S_{2v}[S_2]$$

$H_1$ is the symmetry of the $v$ 4-valent vertices. $H_2$ is the subgroup of permutations which commute with

$$(12)(34) \cdots (4v - 1 \ 4v)$$

which has to do with the pairing-property of Wick contractions.
The key step in deriving this expression is to describe the graph in terms of a pair of data $\Sigma_0, \Sigma_1$, where $\Sigma_0$ is associated with vertices and $\Sigma_1$ with Wick contractions.

Figure: Two vertex vacuum diagrams in $\phi^4$ theory
Figure: Numbering the half-edges
(a) \[ \Sigma_0 = <1, 2, 3, 4> <5, 6, 7, 8> \]
\[ \Sigma_1 = (12)(34)(56)(78) \]

(b) \[ \Sigma_0 = <1, 2, 3, 4> <5, 6, 7, 8> \]
\[ \Sigma_1 = (23)(16)(47)(58) \]

(c) \[ \Sigma_0 = <1, 2, 3, 4> <5, 6, 7, 8> \]
\[ \Sigma_1 = (15)(26)(37)(48) \]

This also leads to neat symmetric group expressions for symmetry factors which have a string interpretation.
Conclusions
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Are there physical versions of such dualities involving non-trivial dependence on space-time and momenta?