Constructing N-body Simulations with General Relativistic Dynamics

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Motivation

On very large scales: fluctuations small
relativistic linear perturbation theory

On smaller scales: non-linear behaviour
Newtonian N-body simulations

We want to go beyond:

- because Newtonian theory is just an approximation
- to explore effects of backreaction
- observational surveys are becoming more and more accurate
- to explore alternatives to ΛCDM (Modified Gravity)
- to look for topological defects and how they effect formation of structure
- possible relativistic sources of perturbations
The formalism

The metric in longitudinal gauge:

\[ ds^2 = a^2(\tau)[-(1 + 2\Psi)\, d\tau^2 - 2B_i x^i \, d\tau + (1 - 2\Phi)\, \delta_{ij} \, dx^i \, dx^j + h_{ij} \, dx^i \, dx^j] \]

- gravitational potentials are \( \sim 10^{-5} \)
- gradients related to peculiar velocities and therefore \( \sim 10^{-3} \)
- second spatial derivatives: non perturbative

Approximation:
- every metric perturbation \( \sim \varepsilon \)
- every spatial derivative \( \sim \varepsilon^{-1/2} \)

Example: we keep terms:

\[ \sim \Phi \]
\[ \sim \Phi, i \Phi, j \quad \text{or} \quad \Phi \Phi, i j \]

“the weak-field limit”,
but with spatial derivatives
up to all orders

Green and Wald: arXiv:1011.4920, 1111.2997
Equations of motion:

\[(1 + 4\Phi)\Delta \Phi - 3\mathcal{H}\Phi' - 3\mathcal{H}^2\Psi + \frac{3}{2}\delta^{ij}\Phi_{,i}\Phi_{,j} = -4\pi Ga^2\delta T^0_0\]

(from the time-time part)

\[\frac{1}{2}h''_{ij} + \mathcal{H}h''_{ij} - \frac{1}{2}\Delta h_{ij} + B'_{(i,j)} + 2\mathcal{H}B_{(i,j)}\]

\[+ \left( \frac{\partial^2}{\partial x^i\partial x^j} - \frac{1}{3}\delta_{ij}\Delta \right) \left[ (\Phi - \Psi)(1 + \Phi - \Psi) + \Phi^2 \right] \]

\[+ 2\Psi\Phi_{,ij} - \frac{2}{3}\delta_{ij}\Psi\Delta \Phi - (\Phi - \Psi)_{,i}(\Phi - \Psi)_{,j} + \frac{1}{3}\delta_{ij}\delta_{kl}(\Phi - \Psi)_{,k}(\Phi - \Psi)_{,l} \]

\[= 8\pi Ga^2\Pi_{ij}\]

(from the transverse-traceless spatial part)

Where the energy-momentum tensor is given by:

\[T^\mu_\nu = \sum_n m_{(n)} \delta^{(3)}(x - x_{(n)}) \left( -g_{\alpha\beta}(x_{(n)}^\alpha \frac{dx^\alpha}{d\tau}(x_{(n)}^\beta \frac{dx^\beta}{d\tau}) \right)^{-1/2} \frac{dx^\mu_{(n)}}{d\tau} \frac{dx^\nu_{(n)}}{d\tau} \]

and:

\[\Pi_{ij} = \delta_{ik}T^k_j - \frac{1}{3}\delta_{ij}T^k_k\]
Post-Newtonian estimation has been used to probe relativistic effects: 

\((\Phi - \Psi)\)

\(B_i\)

\(h_{ij}\)

using the output of a 3D purely Newtonian simulation (Gadget 2)

\[
\begin{align*}
 ds^2 &= a^2(\tau)[-(1 + 2\Psi)d\tau^2 - 2B_i x^i d\tau + (1 - 2\Phi)\delta_{ij}dx^i dx^j + h_{ij}dx^i dx^j] \\
\end{align*}
\]
Toy model: spherical symmetry

Motivation: *

- *simplifies* the equations to solve
- much *faster* compared to the full 3D simulation
- you can run many simulations for different configurations
- easy to compare to the *analytical solution*
- in addition, useful to model e.g. expansion of a void

Spherical metric:

\[
 ds^2 = -a^2(\tau) \left[ (1 + 2\Psi(\tau, r)) \, d\tau^2 + (1 - 2\Phi(\tau, r)) \left( dr^2 + r^2 \, d\Omega^2 \right) \right]
\]

- *metric perturbations depend only on radial coordinate and comoving time*
- *particles are pressureless “spherically symmetric shells”*
The Einstein equations:

- From $G^0_0 = 8\pi G T^0_0$ we get:
  \[
  \Phi_{,rr} + \frac{2}{r} \Phi_{,r} - 3\mathcal{H} \Phi_{,\tau} - 3\mathcal{H}^2 (\Phi - \chi) + \frac{3}{2} (\Phi_{,r})^2 = -4\pi G a^2 (1 - 4\Phi) \delta T^0_0
  \]

- From $G^i_j - \frac{1}{3} \delta^i_j G^k_k = 8\pi G \left( T^i_j - \frac{1}{3} \delta^i_j T^k_k \right)$, the traceless part of the "space-space" component we get:
  \[
  \chi_{,rr} - \frac{1}{r} \chi_{,r} + \chi_{,r}^2 + 2\Phi_{,r}^2 + 2 \left( \Phi_{,rr} - \frac{1}{r} \Phi_{,r} \right) (2\Phi - \chi) = 12\pi G a^2 (1 - 2\chi) \Pi_{rr}
  \]
  \[
  \Pi_{ij} = \delta_{ik} T^k_j - \frac{1}{3} \delta_{ij} T^k_k
  \]
  where $\chi = \Phi - \Psi$

We define the covariant momentum:

\[
p = \frac{(1 - \Phi) \left( \frac{dr}{d\tau} \right)}{\sqrt{1 + 2\Psi - (1 - 2\Phi) \left( \frac{dr}{d\tau} \right)^2}}
\]

\[
\frac{dp}{d\tau} = - (\mathcal{H} - \Phi_{,\tau}) p - \Psi_{,r} \sqrt{1 + p^2}
\]

velocities don’t need to be small!
Discretisation

- In order to solve Einstein’s equations numerically we need to discretise them.
- The spatial domain is divided into \( n \) comoving “grid-cells”
- Fields live on the grid-cells
- Energy-momentum tensor lives in the grid cells
- Particles live in “continuous” space
- We need to establish a correspondence between the two
- Particle-to-mesh projection: cloud-in-cell, triangular …
- Interpolation of fields to propagate particles.
On each step of the simulation we:

1) project particles’ positions to the grid to determine the density and energy-momentum tensor at each grid cell using particle-to-mesh projection

2) discretise EOM and solve for potential $\Phi$ and $\Psi$

3) update particles’ momenta using geodesic equation:

$$\frac{dp}{d\tau} = -(\mathcal{H} - \Phi,_{\tau})p - \Psi,_{r}\sqrt{1 + p^2}$$

4) evolve particles’ positions using:

$$\frac{dr}{d\tau} = \frac{p}{\sqrt{1 + p^2}}(1 + \Phi + \Psi)$$

5) evolve the background:

$$\mathcal{H}^2 = \frac{8\pi G (\rho + \Lambda)}{3}$$

Steps 3-5 are done simultaneously, using Runge-Kutta 4.
Initial conditions

In the full 3D simulation:

* the Zel’dovich approximation
* 2LPT

To test our spherically symmetric case:
Compensated top-hat profiles.
Results: testing the code

An overdensity

\[ \rho / \bar{\rho} \]

\[ r \]

\[ \phi \]

\[ \chi = \phi - \psi \]

Phase-space portrait

Difference of the potentials
An underdensity (void)

Fields need to be weak, but density contrast doesn’t!
Comparison to the LTB solution

We want to compare our results to analytical Lemaitre-Tolman-Bondi (LTB) solution

The metric: \( ds^2 = -dt^2 + \frac{(\partial_r R(t, r))^2}{(1 + 2E(r))} dr^2 + R^2(t, r) d\Omega^2 \)

Solutions:

\[
R(t, r) = -\frac{M(r)(1 - \cos \eta)}{2E(r)} \quad (\eta - \sin \eta)^{2/3} = -\frac{2E(r)t^{2/3}}{M(r)^{2/3}}
\]

Where \( M(r) \) is just the mass within the radius \( r \).

Specifically, with our top-hat initial conditions, we get:

\[
E(r) \rightarrow -\frac{10a_{in}^2 \delta_1 r^2}{27t_{in}^2} \quad r < r_1
\]

\[
E(r) \rightarrow -\frac{10a_{in}^2 (\delta_2 r^3 + \delta_1 r_1^3 - \delta_2 r_1^3)}{27rt_{in}^2} \quad r_1 < r < r_2
\]

\* Initially, we can perform a linear gauge transformation between LTB and our metric

\* At later times linearity breaks down so we can no longer compare the two solutions.

\* Instead, we can compare gauge independent variables
1) Redshift of an in-falling source

- We track a ray of light, emitted at the boundary of top-hat overdensity through the simulation volume and “absorb” it at the other end.
- To propagate the ray, simulation update step is sub-divided into \( n \) smaller steps: this is defined by the courant factor which sets the resolution in time.

The redshift:

\[
1 + z = \frac{(g_{\mu\nu}k^{\mu}u^{\nu})|_{src}}{(g_{\mu\nu}k^{\mu}u^{\nu})|_{obs}}
\]

\[
g_{\mu\nu}k^{\mu}u^{\nu} = -k^0a\left[-(1 - \Phi - (\Psi + \Phi)p^2)p + (1 + \Psi - (\Psi + \Phi)p^2)\sqrt{1 + p^2}\right]
\]

From the null-shell condition \((ds^2=0)\) and geodesic equation you get:

\[
\frac{dr}{d\tau} = \pm (1 + \Psi + \Phi) \quad \Leftrightarrow \quad \frac{d\varphi}{d\tau} = 0
\]

\[
\frac{dk^0}{d\tau} + \left[\Psi,_{\tau} - \Phi,_{\tau} + 2\Psi,_{\tau}\left(\frac{dr}{d\tau}\right) + 2H\right]k^0 = 0
\]
2) Lensing of non-radial rays (on an overdensity)

* We solve the geodesic equation for number of rays incoming at different angles:
Comparison to the Schwarzschild solution

Motivation: independent test of our numerical scheme

Equations:

\[ ds^2 = -\left(\frac{1 - \frac{r_S}{4r}}{1 + \frac{r_S}{4r}}\right)^2 dt^2 + \left(1 + \frac{r_S}{4r}\right)^4 \left[ dr^2 + r^2 d\Omega^2 \right] \]

Schwarzschild metric in isotropic coordinates

vacuum stationary solution

Expansion for \( r >> r_s \):

\[ \left(\frac{1 - \frac{r_S}{4r}}{1 + \frac{r_S}{4r}}\right)^2 = 1 + 2\Psi(r) = 1 - \frac{r_S}{r} + \frac{r_S^2}{2r^2} + \ldots \]

\[ \left(1 + \frac{r_S}{4r}\right)^4 = 1 - 2\Phi(r) = 1 + \frac{r_S}{r} + \frac{3r_S^2}{8r^2} + \ldots \]

In post-Newtonian counting our numerical scheme is one order better than purely Newtonian.
Including the angular momentum

Motivation: to avoid spherical collapse and thus model stable, bound structures

- We imagine each “spherical particle” to be made up of infinitesimal point particles
- Each of these particles is given some initial angular momentum, but in such a way that once we average over all particles on the sphere, there is no net preferred direction
- The equation of motion for spheres is nevertheless affected by angular momentum

\[ p = \frac{(1 - \Phi) \left( \frac{dr}{d\tau} \right)}{\sqrt{1 + 2\Psi - (1 - 2\Phi) \left( \frac{dr}{d\tau} \right)^2}} \]

\[ p_r = \frac{(1 - \Phi) \left( \frac{dr}{d\tau} \right)}{\sqrt{1 + 2\Psi - (1 - 2\Phi) \left( \frac{dr}{d\tau} \right)^2 - (1 - 2\Phi)r^2 \left( \frac{d\varphi}{d\tau} \right)^2}} \]

\[ \frac{d\varphi}{d\tau} = \frac{L}{ar^2} \sqrt{\frac{1 + 4\Phi + 2\Psi - (1 + 2\Phi) \left( \frac{dr}{d\tau} \right)^2}{1 + \frac{L^2(1+2\Phi)}{a^2r^2}}} \]

\[ \frac{dp}{d\tau} = -(\mathcal{H} - \Phi, r)p - \Psi, r \sqrt{1 + p^2} \]

\[ \frac{dp_r}{d\tau} = -(\mathcal{H} - \Phi, r)p_r - \Psi, r \sqrt{1 + p_r^2 + p_{\varphi}^2} \]

\[ + \left( \frac{1}{r} - \Phi, r \right) \frac{p_{\varphi}^2(1 + \Phi + \Psi)}{\sqrt{1 + p_r^2 + p_{\varphi}^2}} \]

\[ \left( \text{where } p_{\varphi} = \frac{L(1 + \Phi)}{ar} \right) \]
Conclusions and future applications

- We implemented a general relativistic approximations that requires weak-field limit, but allows large gradients of the fields and arbitrarily large velocities of particles.
- Spherical symmetry makes numerical solving simple and fast.
- Comparing our results to the analytical LTB solution we find good agreement.
- Comparing to Schwarzschild solution we find the approximation to be at the post-Newtonian level of accuracy.
- In addition, our code can handle shell-crossing.
- Although spherical symmetry hides vector and tensor perturbations, one can still use our approach to explore relativistic effects due to the difference of the potentials.

In the future we want to:
- explore the effects of modified gravity or quintessence-type scalar field theory.
- statistical study of properties of environments that allow creation of primordial black holes in the early universe.
Thank you!