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Asen Ivanov

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School of Economics and Finance

Queen Mary University of London

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Asen Ivanov*

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Abstract

I consider an extension of the Borda count to the case when individuals can have weak preferences, and I show that it satisfies several normatively appealing axioms. The first axiom is an extension to the case of weak preferences of the Modified Independence of Irrelevant Alternatives axiom in Maskin (2020a). The second axiom is a new axiom which I call Up-Down Symmetry. The remaining axioms are more standard.

*School of Economics and Finance, 327 Mile End Road, London, United Kingdom; tel. +44 (0)20 7882 5886; a.ivanov@qmul.ac.uk

1 Introduction

I consider an extension of the Borda count (Borda (1781)) to the case when individuals can have weak preferences, and I show that it satisfies several normatively appealing axioms. The first axiom is an extension to the case of weak preferences of the Modified Independence of Irrelevant Alternatives (MIIA) axiom in Maskin (2020a). The second axiom is a new axiom which I call Up-Down Symmetry. The remaining axioms are more standard.

2 Preliminaries

Let X denote the set of social alternatives and let I be the set of individuals. Let $\succeq_i$ denote the preference of individual i over X. $\succ_i$ and $\sim_i$ are defined as usual. Let $\succeq$ denote the profile of individuals' preferences $\bigtimes_{i \in I} \succeq_i$, where "$\bigtimes$" is the Cartesian product. A social welfare function is a function $F : \bigtimes_{i \in I} \mathcal{R}_i \to \mathcal{R}$, where $\mathcal{R}_i$ is a set of transitive and complete preferences over X for individual i and $\mathcal{R}$ is a set of transitive and complete preferences over X for society.

3 MIIA*

Maskin (2020a and 2020b) argues that Arrow's Independence of Irrelevant Alternatives (IIA) axiom (Arrow (1951)) is too stringent because it rules out any sensitivity of the social welfare function to preference intensities. Maskin (2020a) proposes a weakening of IIA, MIIA, which does allow for some sensitivity to preference intensities. Maskin (2020a) defines MIIA for the case when individuals' preferences are strict.[1] I propose the following extension to the case of possibly weak preferences.[2]

Definition (MIIA*). *For all* $\succeq, \succeq' \in \bigtimes_{i \in I} \mathcal{R}_i$ *and all* $x, y \in X$*, if, for all* i*,*

a) $\succeq_i$ *ranks* x *and* y *in the same way that* $\succeq'_i$ *does and*

b) for all $z \in X - \{x, y\}$*, (i)* z *lies between* x *and* y *in* $\succ_i$ *if and only if* z *lies between* x *and* y *in* $\succ'_i$*, (ii)* $z \sim_i x$ *if and only if* $z \sim'_i x$*, and (iii)* $z \sim_i y$ *if and only if* $z \sim'_i y$*,*

then $F(\succeq)$ *ranks* x *and* y *in the same way that* $F(\succeq')$ *does.*[3]

The following examples illustrate the difference between IIA and MIIA*. In all three examples, $I = \{1, 2\}$, $X = \{x, y, z\}$, $y \succ_2 x \succ_2 z$, and $y \succ'_2 x \succ'_2 z$.

Example 1: $x \succ_1 z \succ_1 y$ and $x \succ'_1 y \succ'_1 z$.
Example 2: $x \succ_1 z \succ_1 y$ and $x \sim'_1 z \succ'_1 y$.
Example 3: $x \sim_1 z \succ_1 y$ and $x \succ'_1 y \succ'_1 z$.

In these examples, IIA insists that the social preferences, $F(\succeq)$ and $F(\succeq')$, rank x and y in the same way. MIIA* does not insist on this based on the argument that one can learn something about the intensity of individual 1's preference for x over y from (i) whether x and y are strictly or not at all separated by z in her ranking (see Example 1), (ii) whether x and y are strictly or weakly separated by z in her ranking (see Example 2), and (iii) whether x and y are weakly or not at all separated by z in her ranking (see Example 3).[4]

[1] I.e., for the case in which, for all i, $\mathcal{R}_i$ consists of only strict preferences.

[2] Maskin (2020b) does define MIIA for the case of weak preferences. However, as I argue in the appendix, the definition in Maskin (2020b) has some shortcomings which MIIA* fixes.

[3] MIIA is obtained from MIIA* by dropping (ii) and (iii) in condition b). IIA is obtained from MIIA* by dropping condition b) altogether. Clearly, when preferences are strict, MIIA and MIIA* coincide. Maskin (2020a) shows that MIIA (and, hence, MIIA*) has "teeth" in that it rules out plurality voting, runoff voting, and vote splitting.

[4] In the context of Example 1, Maskin (2020a) formalises the argument as follows: "Imagine

4 Up-Down Symmetry

From here on, assume that I is finite or equals $[0,1]$. If I is finite, let μ be the counting measure (so that any integrals below become sums); if $I = [0,1]$ let μ be the Lebesgue measure.

The following axiom will play a role in the analysis of the Borda count.

Definition (Up-Down Symmetry). *Suppose* $\succeq \in \bigtimes_{i\in I} \mathcal{R}_i$, $I_1, I_2 \subseteq I$ *(where* $I_1 \cap I_2 = \emptyset$*),* $Y \subset X$ *(where* $Y \neq X$*), and* $x \in X - Y$ *are such that:*

(i) $\mu(I_1) = \mu(I_2)$,

(ii) for all $i \in I$ *and all* $y_1, y_2 \in Y$, $y_1 \sim_i y_2$,

(iii) for all $i \in I$, *all* $w \in Y \cup \{x\}$, *and all* $z \in X - (Y \cup \{x\})$, *we have either* $w \succ_i z$ *or* $z \succ_i w$,

(iv) for all $i, i' \in I$, *all* $w \in X$, *and all* $z \in X - (Y \cup \{x\})$, *we have* $w \succeq_i z \iff w \succeq_{i'} z$,

(v) for all $i \in I_1$ *and all* $y \in Y$, $x \succ_i y$,

(vi) for all $i \in I_2$ *and all* $y \in Y$, $y \succ_i x$,

(vii) for all $i \in I - (I_1 \cup I_2)$ *and all* $y \in Y$, $x \sim_i y$.

Then, for any $y \in Y$, x *is equally good as* y *according to* $F(\succeq)$.

That is, suppose everyone is indifferent between all elements in Y and ranks any element outside of $Y \cup \{x\}$ as either strictly above or strictly below any element in

that, from the perspective of an outside observer (society), each of a voter's utilities $u(x)$, $u(y)$, and $u(z)$ (where u captures preference intensity) is drawn randomly and independently from some distribution. Then, the expected difference $u(x)$–$u(y)$ conditional on z being between x and y in the voter's preference ordering is greater than the difference conditional on z not being between x and y." In Examples 2 and 3, the argument can be formalised in a similar fashion.

$Y \cup \{x\}$. Further, suppose the only difference between individuals is in how they rank x versus any $y \in Y$: all $i \in I_1$ rank x strictly above y, all $i \in I_2$ rank x stricty below y, and all $i \in I - (I_1 \cup I_2)$ are indifferent between x and y. Then, the opposite preferences between x and y of individuals in I_1 and the equal measure of individuals in I_2 cancel out.

5 The Borda Count

Assume that X is finite. Let $b(x, \succeq_i)$ and $e(x, \succeq_i)$ denote the number of elements in X that are strictly below x and equally good as x, respectively, according to $\succeq_i$.

I define the Borda count for the case of possibly weak preferences as:[5,6]

$$B(x, \succeq) = \int_I \left(b(x, \succeq_i) + \frac{1 + e(x, \succeq_i)}{2} \right) d\mu. \tag{1}$$

The following proposition extends one direction of the main result in Maskin (2020a) to the case of possibly weak preferences.

Proposition 1. *The Borda count in (1) satisfies MIIA*, Unrestricted Domain, Anonymity, Neutrality, Positive Responsiveness, the Pareto Property, Nondictatorship, and Up-Down Symmetry.*[7]

Note that the Borda count in (1) would continue to satisfy all the axioms other than Up-Down Symmetry if we replaced $\frac{1+e(x,\succeq_i)}{2}$ in (1) with one of many other functions of $e(x, \succeq_i)$ (such as, say, $\frac{1+e(x,\succeq_i)}{3}$ or $e(x, \succeq_i)$). The importance of Up-Down

[5] I assume the integral in (1) exists.

[6] Note that, if preferences are strict, $e(x, \succeq_i) = 1$, so that we obtain the usual Borda count. Also, note that (1) can equivalently (up to an inconsequential affine transformation) be written as $B(x, \succeq) = \int_I (b(x, \succeq_i) - a(x, \succeq_i))\, d\mu$, where $a(x, \succeq_i)$ denotes the number of elements in X that are strictly above x according to $\succeq_i$.

[7] The axioms other than MIIA* and Up-Down Symmetry are relatively standard and are stated in the appendix. Note that the Pareto Property and Nondictatorship are redundant: Neutrality and Positive Responsiveness imply the Pareto Property, and Anonymity implies Nondictatorship. Also, when preferences are strict, Anonymity and Neutrality imply Up-Down Symmetry.

Symmetry is that it rules out such alternative formulations. The proof of Proposition 1 (which is in the appendix) formalises these points.

6 Related Literature

Several papers analyse the Borda count for the case of strict preferences.[8] Among them, Maskin (2000a) is most closely related because of its emphasis on MIIA. That paper shows that, assuming strict preferences as well as $I = [0, 1]$ and abstracting from what seems like a technical detail, a social welfare function is the Borda count if and only if it satisfies MIIA, Unrestricted Domain, Anonymity, Neutrality, and Positive Responsiveness.

To the best of my knowledge, Young (1974) and Hansson and Sahlquist (1976) are the only papers that define the Borda count for weak preferences. Young (1974) introduces an equivalent version of the Borda count in (1) and characterises it in terms of normatively appealing axioms.[9] These papers differ from the current paper in several respects. First, they allow the set of individuals to vary and they are concerned with social choice functions rather than social welfare functions. Second, the current paper focuses on MIIA*, which isolates the part of IIA that the Borda count satisfies. Young's axioms bear no apparent relation to IIA. Third, the Up-Down Symmetry axiom introduced here is new.

[8] For example, see Nitzan and Rubinstein (1981), Saari (2000), or Dasgupta and Maskin (2000).

[9] Young's characterisation theorem assumes strict preferences, but Young notes that it can be proved in much the same way in the case of weak preferences. Hansson and Sahlquist (1976) provide an alternative proof of Young's theorem (again, assuming strict preferences). These papers assume finitely many individuals.

7 Concluding Remarks

The current paper is concerned with the Borda count in the case when individuals can exhibit indifference. Why is this case important?

First, voters can plausibly exhibit indifference between alternatives, especially when there are many alternatives on the ballot and voters are unfamiliar with some of them. With weak preferences, even if the election rules force voters to state strict rankings, the expectation of the usual Borda count of x computed using the stated strict rankings would equal $B(x, \succeq)$ (where $\succeq$ is the profile of voters' true, weak preferences) as long as each voter votes sincerely and breaks ties randomly.

Second, an economist wishing to aggregate preferences over X may, instead of holding a vote, derive individuals' preferences over X from a deeper model. In that case, dealing with weak preferences may be inevitable.

For example, suppose the economist wishing to aggregate individuals' preferences over labour-income tax schedules derives these preferences based on individuals' (separable) preferences over consumption-leisure bundles and their productivities. If productivities are distributed on $[\underline{w}, \overline{w}]$, all individuals with productivity w (where $\underline{w} \leq w < \overline{w}$) are inevitably indifferent between any two incentive-compatible direct mechanisms that prescribe the same consumption for type $\underline{w}$ and the same amount of labour income for all types in $[\underline{w}, w]$.[10]

By demonstrating that the Borda count in (1) satisfies some normatively appealing axioms, the current paper provides reassurance regarding the use of the Borda count both in actual elections (subject to the caveat regarding sincere voting) and in theoretical work on optimal public policy. An open question is whether the axioms in Proposition 1, possibly augmented by other normatively appealing axioms, imply the Borda count in (1). This question is probably harder to answer and is left for

[10] In an ongoing project, I am trying to derive optimal labour-income tax schedules based on the Borda count.

future research.[11]

References

Arrow, Kenneth J. 1951. “Social Choice and Individual Values.” New York: John Wiley & Sons.

Borda, Jean-Charles. 1781. “Mémoire sur les élections au scrutin.” Histoire de I’Academie Royale des Sciences. 657-665.

Dasgupta, Partha and Eric Maskin. 2000. “Elections and Strategic Voting: Condorcet and Borda.” Working Paper.

Hansson, Bengt, and Henrik Sahlquist. 1976. “A Proof Technique for Social Choice with Variable Electorate.” Journal of Economic Theory 13.2 (1976): 193-200.

Ivanov, Asen. 2021. “Borda-Optimal Taxation of Labour Income,” Working paper.

Maskin, Eric. 2020a. “Arrow’s Theorem, May’s Axioms, and Borda’s Rule.” Working Paper.

Maskin, Eric. 2020b. “A modified version of Arrow’s IIA condition.” Social Choice and Welfare 54.2: 203-209.

Nitzan, Shmuel and Ariel Rubinstein. 1981. “A Further Characterization of the

[11] Maskin (2020a) writes “Allowing for the possibility that an individual is indifferent between two alternatives appears to be significantly more complex.” I suspect most of the complexity lies in proving a kind of converse of Proposition 1.

Scores Method for Multi-Round Tournaments and Social Theory." Public Choice 36, 153-158.

Saari, Donald G. 2000. "Mathematical Structure of Voting Paradoxes: II. Positional Voting." Economic Theory. 15 (1): 55-102.

Young, H. Peyton. 1974. "An axiomatization of Borda's rule." Journal of Economic Theory 9.1: 43-52.

8 Appendix: Shortcomings of MIIA in Maskin (2020b)

Maskin (2020b) provides a definition of MIIA for the case of weak preferences that coincides with MIIA*, except that condition b) in MIIA* is replaced by:

b) for all $z \in X - \{x, y\}$, z lies between x and y in $\succeq_i$ if and only if z lies between x and y in $\succeq_i'$,

First note that, with weak preferences, the statement "z lies between x and y in $\succeq_i$" is imprecise. It could mean (i) "$x \succeq_i z \succeq_i y$ or $y \succeq_i z \succeq_i x$" or (ii) "$x \succ_i z \succ_i y$ or $y \succ_i z \succ_i x$".[12] Under each of these interpretations, MIIA in Maskin (2020b) is too strong.[13] To see this, note that under interpretation (i) (interpretation (ii), respectively) this axiom insists that the social preferences, $F(\succeq)$ and $F(\succeq')$, rank x and y in the same way in Example 2 (Example 3, respectively) in the main text.

Observe also that the Borda count in (1) violates MIIA in Maskin (2020b) under both interpretations. To see this for interpretation (i), consider again Example 2 and suppose there are 35 and 65 individuals like individuals 1 and 2, respectively. We have $B(x, \succeq) = 35 \times 3 + 65 \times 2 = 235 > 230 = 35 \times 1 + 65 \times 3 = B(y, \succeq)$ while $B(x, \succeq') = 35 \times 2.5 + 65 \times 2 = 217.5 < 230 = 35 \times 1 + 65 \times 3 = B(y, \succeq')$.

To see that the Borda count in (1) violates MIIA in Maskin (2020b) under interpretation (ii), consider again Example 3 and suppose there are 45 and 55 individuals like individuals 1 and 2, respectively. We have $B(x, \succeq) = 45 \times 2.5 + 55 \times 2 = 222.5 > 210 = 45 \times 1 + 55 \times 3 = B(y, \succeq)$ while $B(x, \succeq') = 45 \times 3 + 55 \times 2 = 245 < 255 = 45 \times 2 + 55 \times 3 = B(y, \succeq')$.

[12] An analogous remark applies to the statement "z lies between x and y in $\succeq_i'$".

[13] For brevity, I omit the interpretations (iii) "$x \succeq_i z \succ_i y$ or $y \succeq_i z \succ_i x$" and (iv) "$x \succ_i z \succeq_i y$ or $y \succ_i z \succeq_i x$". MIIA in Maskin (2020b) is too strong under these interpretations for similar reasons.

9 Appendix: Definitions of Further Axioms

Let $\succeq_F = F(\succeq)$.

Definition (Unrestricted Domain). *For all $i \in I$, $\mathcal{R}_i$ consists of all transitive and complete preferences over X.*

Definition (Anonymity). *Fix any measure-preserving permutation of society $\pi : I \to I$. For any $\succeq \in \bigtimes_{i\in I} \mathcal{R}_i$, let $\succeq^\pi$ be the profile, such that $\succeq_i^\pi = \succeq_{\pi(i)}$. Then, $F(\succeq^\pi) = F(\succeq)$.*

Definition (Neutrality). *For any permutation $\rho : X \to X$ and any $\succeq \in \bigtimes_{i\in I} \mathcal{R}_i$, let $\succeq^\rho$ be the profile such that, for all $x, y \in X$ and all $i \in I$, $x \succeq_i y \iff \rho(x) \succeq_i^\rho \rho(y)$. Then, for all $x, y \in X$, $x \succeq_F y$ if and only if $\rho(x) \succeq_F^\rho \rho(y)$.*

Definition (Positive Responsiveness). *Suppose $\succeq$ and $\succeq'$ are two preference profiles, such that, for some $x, y \in X$ and all $i \in I$, the following condition $(*)$ holds: $x \succ_i z \implies x \succ_i' z$, $x \sim_i z \implies x \succeq_i' z$, $w \succ_i y \implies w \succ_i' y$, $w \sim_i y \implies w \succeq_i' y$, and $r \succeq_i s \iff r \succeq_i' s$ for all $z, w \in X$ and all $r, s \in X - \{x, y\}$. Then, if $\mu(\{i|(y \succeq_i x \text{ and } x \succ_i' y) \text{ or } (y \succ_i x \text{ and } x \succeq_i' y)\}) > 0$, we have $x \succeq_F y \implies x \succ_F' y$. Furthermore, for the case $I = [0, 1]$, if $y \succ_F x$ and $x \succ_F' y$, then there exists a preference profile $\succeq''$ satisfying condition $(*)$ (with $\succeq_i''$ replacing $\succeq_i'$) such that $x \sim_F'' y$.*

Definition (Pareto Property). *For all $\succeq \in \bigtimes_I \mathcal{R}_i$ and for all $x, y \in X$, if $x \succeq_i y$ for all $i \in I$ and $\mu(\{i|x \succ_i y\}) > 0$, then $x \succ_F y$.*

Definition (Nondictatorship). *There is no $i \in I$ such that, for all $\succeq \in \bigtimes_I \mathcal{R}_i$ and for all $x, y \in X$, $x \succ_i y \implies x \succ_F y$.*

10 Appendix: Proof of Proposition 1

Given $g : \{1, 2, \ldots, |X|\} \to \mathbb{R}$, define a family of "Borda counts" indexed by g by:[14]

$$B_g(x, \succeq) = \int_I (b(x, \succeq_i) + g(e(x, \succeq_i)))\, d\mu. \tag{2}$$

I prove the following two claims which, taken together, imply Proposition 1.

Claim 1. *If g is such that $0 < g(k) \leq k$ for all $1 \leq k < |X|$ and g is strictly increasing on $\{1, 2, \ldots, |X|-1\}$, the Borda count in (2) satisfies MIIA*, Unrestricted Domain, Anonymity, Neutrality, Positive Responsiveness, the Pareto Property, and Nondictatorship.*

Claim 2. *The Borda count in (2) satisfies Up-Down Symmetry if and only if $g(k) = \frac{k+1}{2} + c$ for all $1 \leq k < |X|$, where $-1 < c \leq 0$ is a constant.*[15,16]

10.1 Proof of Claim 1

It is obvious that Unrestricted Domain, Anonymity, Neutrality, and Nondictatorship hold. Note that the condition $0 < g(k) \leq k$ for all $1 \leq k < |X|$ ensures that $b(x, \succeq_i) + g(e(x, \succeq_i)) \geq b(y, \succeq_i) + g(e(y, \succeq_i))$ (i.e., that x receives more "points" than y from individual i in (2)) if and only if $x \succeq_i y$. Given this observation, it is obvious that the Pareto Property also holds. It remains to show that Positive Responsiveness and MIIA* are satisfied.

Proof that Positive Responsiveness holds:

Take $\succeq$, $\succeq'$, x, and y such that condition $(*)$ holds, $\mu(\{i|(y \succeq_i x \text{ and } x \succ'_i y) \text{ or } (y \succ_i x \text{ and } x \succeq'_i y)\}) > 0$, and $B_g(x, \succeq) \geq B_g(y, \succeq)$. Condition $(*)$ and the

[14] I assume the integral in (2) exists.

[15] c can be normalised to 0 without affecting the resulting social welfare function.

[16] $g(|X|)$ can be chosen completely arbitrarily. If some individuals are indifferent between all alternatives, each of them would contribute $g(|X|)$ "points" to the Borda count (2) of each alternative, so that the social preference would not depend on the value of $g(|X|)$.

conditions on g imply that, for all i, $b(x,\succeq_i)+g(e(x,\succeq_i)) \leq b(x,\succeq_i')+g(e(x,\succeq_i'))$ and $b(y,\succeq_i)+g(e(y,\succeq_i)) \geq b(y,\succeq_i')+g(e(y,\succeq_i'))$. Moreover, given $\mu(\{i|(y \succeq_i x \text{ and } x \succ_i' y) \text{ or } (y \succ_i x \text{ and } x \succeq_i' y)\}) > 0$, one of the the two inequalities in the last sentence must be strict for a positive measure of individuals. Thus, $B_g(x,\succeq) \leq B_g(x,\succeq')$ and $B_g(y,\succeq) \geq B_g(y,\succeq')$, with one of these inequalities being strict. Thus, $B_g(x,\succeq) - B_g(y,\succeq) < B_g(x,\succeq') - B_g(y,\succeq')$. Given that $B_g(x,\succeq) \geq B_g(y,\succeq)$, we must have $B_g(x,\succeq') > B_g(y,\succeq')$.

Now let's turn to the existence of the profile $\succeq''$ when $I=[0,1]$. Assume $B_g(x,\succeq) - B_g(y,\succeq) < 0$. Let $\succeq_i^x$ be derived from $\succeq_i$ by placing x strictly at the top of the ranking while leaving $\succeq_i$ otherwise unchanged. For $0 \leq \delta \leq 1$, let $\succeq^\delta$ be derived from $\succeq$ by replacing $\succeq_i$ by $\succeq_i^x$ for all $i \in [0,\delta]$ and leaving $\succeq_i$ unchanged for all $i \in (\delta,1]$. Given that $B_g(x,\succeq^\delta) - B_g(y,\succeq^\delta)$ is continuous in δ,[17] $B_g(x,\succeq^0) - B_g(y,\succeq^0) < 0$, and $B_g(x,\succeq^1) - B_g(y,\succeq^1) > 0$, it must be that $B_g(x,\succeq^{\overline{\delta}}) - B_g(y,\succeq^{\overline{\delta}}) = 0$ for some $\overline{\delta} \in (0,1)$. Note that $\succeq^{\overline{\delta}}$ satisfies condition $(*)$ (with $\succeq_i^{\overline{\delta}}$ replacing $\succeq_i'$). Q.E.D.

Proof that MIIA holds:*

Suppose that $\succeq$, $\succeq'$, x, and y are such that, for all i, conditions a) and b) in MIIA* hold. Fix i and assume without loss of generality that $x \succeq_i y$. Let $Z(x,y,\succeq_i)$ denote the number of alternatives, z, such that $x \succ_i z \succeq_i y$. Note that, by conditions a) and b) in MIIA*, $Z(x,y,\succeq_i) = Z(x,y,\succeq_i')$, $e(x,\succeq_i) = e(x,\succeq_i')$, and $e(y,\succeq_i) = e(y,\succeq_i')$. Thus, we have:

[17] This follows because $B_g(x,\succeq^\delta) = \int_0^\delta (b(x,\succeq_i^x) + g(e(x,\succeq_i^x)))\,d\mu + \int_\delta^1 (b(x,\succeq_i) + g(e(x,\succeq_i)))\,d\mu$ is absolutely continuous in δ (and analogously for $B_g(y,\succeq^\delta)$).

$$b(x,\succeq_i) + g(e(x,\succeq_i)) - (b(y,\succeq_i) + g(e(y,\succeq_i))) =$$

$$Z(x,y,\succeq_i) + g(e(x,\succeq_i)) - g(e(y,\succeq_i)) =$$

$$Z(x,y,\succeq_i') + g(e(x,\succeq_i')) - g(e(y,\succeq_i')) =$$

$$b(x,\succeq_i') + g(e(x,\succeq_i')) - (b(y,\succeq_i') + g(e(y,\succeq_i')))\,.$$

Thus, $B_g(x,\succeq) - B_g(y,\succeq) = B_g(x,\succeq') - B_g(y,\succeq')$. Q.E.D.

10.2 Proof of Claim 2

The proof in the " $\Longleftarrow$ " direction is straightforward. Let us turn to the other direction.

Suppose $\succeq$, I_1, I_2, Y and x be as in the definition of Up-Down Symmetry. The axiom requires that for any $y \in Y$:

$$\int_I (b(x,\succeq_i) + g(e(x,\succeq_i)))\,d\mu = \int_I (b(y,\succeq_i) + g(e(y,\succeq_i)))\,d\mu$$

This can be rewritten as:

$$\int_{I_1} (b(x,\succeq_i)+g(1))d\mu + \int_{I_2} (b(x,\succeq_i)+g(1))d\mu = \int_{I_1} (b(y,\succeq_i)+g(|Y|))d\mu + \int_{I_2} (b(y,\succeq_i)+g(|Y|))d\mu.$$

Rearranging:

$$\int_{I_1} (b(x,\succeq_i) - b(y,\succeq_i) + g(1) - g(|Y|))d\mu = \int_{I_2} (b(y,\succeq_i) - b(x,\succeq_i) + g(|Y|) - g(1))d\mu.$$

For $i \in I_1$, we have $b(x,\succeq_i) - b(y,\succeq_i) = |Y|$. For $i \in I_2$, we have $b(y,\succeq_i) - b(x,\succeq_i$

$) = 1$. Thus, $|Y| + g(1) - g(|Y|)) = 1 + g(|Y|) - g(1)$.[18] This can be written as $g(|Y|) = \frac{1+|Y|}{2} + g(1) - 1$. Given that Y can be chosen so that $|Y|$ equals any integer greater than or equal to 1 and strictly less than $|X|$, the proof is complete. Q.E.D.

[18] I'm assuming $\mu(I_1) = \mu(I_2) > 0$. But I_1 and I_2 always can be chosen so that this holds.

School of Economics and Finance

Queen Mary
University of London

School of Economics and Finance Queen Mary University of London
Mile End Road
London E1 4NS
Tel: +44 (0)20 7882 7356
Fax: +44 (0)20 8983 3580
Web: www.econ.qmul.ac.uk/research/workingpapers/