Uncertainty, Intangible Capital, and Productivity Dynamics

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Abstract

Following an unparalleled rise in uncertainty over the Great Recession, the US economy has been experiencing anaemic productivity growth. This paper offers a quantitative study on the link between uncertainty and low productivity growth. Firstly, using micro level data I show that uncertainty accounts for half of the drop in intangible capital stock during the Great Recession. Secondly, to investigate the effect of uncertainty on productivity growth dynamics, I present a novel general equilibrium endogenous growth model with heterogeneous firms that undertake intangible capital investment subject to non-convex costs and time-varying uncertainty. I show that uncertainty can generate slow recoveries and a persistent slowdown in productivity growth when accounting for the empirical discrepancy between the realised and expected changes to the second-moment of fundamentals.

Keywords: Uncertainty, R&D, Innovation, Productivity, Great Recession, Intangible Capital, Slow Recoveries.
JEL codes: O40, O41, O51.

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1 Introduction

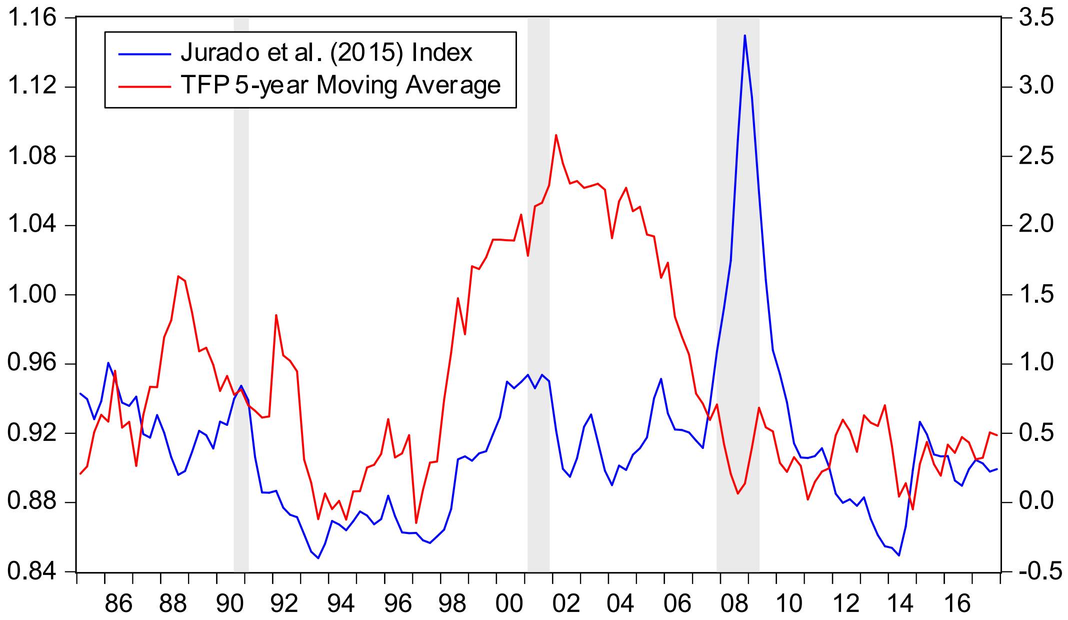
Following an unparalleled rise in uncertainty over the Great Recession, the U.S. economy has been experiencing anaemic productivity growth. This paper offers a quantitative study on the link between uncertainty and low productivity growth. To this end, firstly I investigate the relationship between uncertainty and intangible capital investment using micro-level data. I show that firms reduce intangible investment when uncertainty increases and, specifically, that uncertainty accounts for half of the drop in intangible capital stock during the Great Recession. Secondly, to elicit the effects of uncertainty on productivity dynamics, I develop a novel endogenous growth model with heterogeneous firms, which undertake intangible capital investment under time-varying uncertainty. The model brings about endogenous growth and lumpy intangible capital investment, which is adversely affected by the greater the uncertainty faced by firms. I demonstrate that temporary shocks to uncertainty lead to persistent slumps in productivity growth, when accounting for the empirical discrepancy between realised and expected changes to the second-moment of fundamentals. More precisely, I show that uncertainty decreases productivity growth by 50 basis points per annum through a 30% fall in intangible capital investment. Ultimately, the slow recovery in productivity generates a permanent reduction of 1% in the level of output and productivity. This amounts to a fifth of GDP after the U.S. Great Recession and around a quarter of Total Factor Productivity (TFP) lost during the same period[1].

Three distinguishing features of the Great Recession in the U.S. motivate this research. Firstly, from 2007 to 2010 the U.S. economy experienced an unprecedented rise in uncertainty. Secondly, the economic recovery from the Great Recession has been uncharacteristically slow and weak. Indeed, as illustrated by Figure 1, following the spike in uncertainty index constructed by Jurado et al. (2015), the U.S. economy has been plagued by feeble productivity growth ever since the Great Recession. Thirdly, uncertainty and slow productivity growth have also been accompanied by a collapse of over 60% of intangible capital investment, as calculated by McGrattan (2017) using the U.S. National Income and Products Accounts (NIPA) database. Motivated by these stylised facts, I investigate whether uncertainty has contributed to the slow recovery and weak productivity growth in the aftermath of the Great Recession through intangible capital investment.

The idea that uncertainty drives macroeconomic fluctuations is not new. Work by Bernanke (1983), and subsequently by Bloom (2009) and Bloom et al. (2018), has highlighted the importance of uncertainty shocks in driving economic fluctuations through the negative effects of investment. However, the literature has mainly focused on business

[1]See Ball (2014) and Fernald (2014) for the calculations of the total output and TFP lost by the U.S economy during the Great Recession.

Figure 1. Uncertainty and TFP Growth in the United States



Source: *TFP data is taken from Fernald (2012) accessible via* `https://www.frbsf.org/economic-research/indicators-data/total-factor-productivity-tfp/`. *Uncertainty index is from Jurado et al. (2015) accessible via* `https://www.sydneyludvigson.com/data-and-appendixes/`. *Recession dates are as detailed in the National Bureau of Economic Research (NBER) at* `http://www.nber.org/cycles/cyclesmain.html`.
Notes: *The red line represents a 5-year moving average of the Quarterly, Utilization-Adjusted Series on Total Factor Productivity. The blue line represents the macro uncertainty index at the yearly horizon. The shadows represent the periods of recessions, as identified by the NBER.*

cycle fluctuations and tangible (physical) capital. It has yet to consider the effects of uncertainty on lower frequency fluctuations and intangible capital. Importantly, since the seminal paper by Romer (1986), the endogenous growth literature has placed greater emphasis on intangible capital as a key determinant for growth and productivity, and longer-horizon macroeconomic fluctuations. Empirical evidence has shown that in the last two decades, net investment in intangible capital has overtaken the share of tangible capital net investment. Moreover, such investment accounts for a third of the productivity growth in the U.S. from 1973 to 2003 (see Corrado et al. (2009) and Corrado et al. (2016)). By acknowledging the importance of uncertainty for investment decisions and the role of intangible capital in driving productivity dynamics, this paper is the first to shed light on the contribution of uncertainty in the recent U.S. productivity slowdown through its effects on intangible capital investment.

The empirical analysis in this paper investigates the effects of aggregate uncertainty on firms' intangible capital investment, Research and Development (R&D from hereon in) expenditures and investment intensity. In this effort, I use COMPUSTAT data for publicly traded firms in the U.S. and the macroeconomic uncertainty index developed by Jurado et al. (2015). By exploiting firm-level variation in the dependent variables, I establish a causal link between uncertainty and investment in intangibles, which comprises both of intangible capital and R&D. The identifying assumption relies on the fact that a single firm's investment decisions are unlikely to affect aggregate uncertainty. I document a statistically and economically significant reduction in intangible capital investment, R&D expenditures and R&D intensity, following an increase in the aggregate uncertainty. More precisely, a standard deviation increase in the Jurado et al. (2015) uncertainty index generates a 1.6% decline in the firm's intangible capital, a fall of 0.9% in R&D expenditure and a 1.2% reduction in R&D intensity. It is shown that the increase in uncertainty during the Great Recession can account for roughly half of the drop in intangible capital stock experienced by the U.S. economy.

Once I have empirically established the causal link between uncertainty and lower intangible capital investment, the second contribution of the paper relies on investigating the effects of uncertainty on productivity dynamics through intangible capital investment. In this endeavour, I build a general equilibrium model of endogenous growth with heterogeneous firms. The model is in the same spirit as Comin and Gertler (2006) and Kung and Schimd (2015), where a real business cycle model is augmented to allow for endogenous productivity by introducing industrial innovation in the style of Romer (1990). Productivity growth is generated by the creation of new patented technologies through investment in R&D. In such a framework, patents parsimoniously embody the endogenous stock of intangible capital in the economy. Moreover, to give uncertainty a chance to matter, following Bloom (2009) and Bloom et al. (2018), I model the final good sector of the economy as comprising of heterogeneous firms characterised by idiosyncratic productivity[2] facing non-convex costs of adjusting intangible capital. As shown in Bloom (2009), such costs generate an option value of waiting, which leads firms to halt their investments until uncertainty is resolved.

In this model, uncertainty shocks lead to fluctuations beyond the ordinary business cycle frequencies. A second-moment shock to fundamentals contracts the firms' intangible capital investment through the *real-option* channel, thus confirming the effect of uncertainty of intangible investment found in the empirical investigation. The novelty of the model, as in Comin and Gertler (2006) and Kung and Schimd (2015), focuses on how the initial downturn in intangible capital investment is propagated because R&D investment is the

[2]Note that whilst I refer to idiosyncratic productivity shock, the interpretation should is not so strict. The idiosyncratic productivity shocks are not literally thought as productivity shocks, but rather as shocks to the firms' fundamentals, be it for a demand or supply channel.

main driving force for endogenous fluctuations in aggregate productivity growth. Indeed, due to the nature of intangible capital, the reduction of investment in intangibles leads to a fall in the value of patents, which discourages R&D investment in the economy. As R&D investment is depressed, the growth rate of new patents slows down, resulting in lower productivity growth.

The longer-run dynamics of the model, however, are dominated by the *distributional* effects of uncertainty[3]. Whilst the *real option* effects of the model are generated by the increase in forecast dispersion during a second-moment shock to the fundamentals, the *distributional* effects arise as a direct result of the materialisation of the shocks drawn from a distribution with a higher variance. Due to the convexity of the optimal investment function with respect to fundamentals, the *distributional* effects increase intangible capital investment after the *real option* channel subsides. Ultimately, such *distributional* effects come to dominate and generate a sustained boom following the initial recession. The result is higher productivity growth in the medium-run, which results in a permanent increase in the level of output.

As the result relies on the assumption that forecast dispersion of future shocks is exactly equal to the realised dispersion of such shocks, I empirically test whether during the U.S. Great Recession forecast dispersion of the fundamentals has increased one-to-one with the dispersion of the fundamentals. To this end, I use the Institutional Brokers Estimate System (I—B—E—S) dataset which provides the Earning-Per-Share (EPS) data for each publicly traded company. The novelty of the dataset is that it also contains the analysts' forecast of the EPS for each of these firms. This data is crucial in testing whether the forecast dispersion increased at the same rate as the dispersion of shocks because it provides one of the only sources of data where it is possible to obtain a measure of firm-level forecast dispersion. Empirical evidence suggests that during the U.S. Great Recession within-firm standard deviation of EPS forecasts increased threefold, whilst the within-firm standard deviation of EPS increased by just over a third.

In light of this novel evidence, I modify the modelling of uncertainty shocks to take into account the fact that during the Great Recession forecast dispersion increased ten times more than the fundamental's dispersion. Firms' expectations about uncertainty are decoupled from the shocks' realisations. It is assumed that when an uncertainty shock hits the economy, firms expect the variance of fundamentals to increase by three-fold, however and crucially, the realised dispersion of the fundamental's shocks only increases by a third. Under this new specification of uncertainty shocks, model simulations yield that in the short-term intangible capital investment falls by 30%. More importantly, as the increase in

[3]Bloom (2009) has also called the distributional effect which arise from a second-moment shock to fundamentals, the "volatility effect".

forecast dispersion far outweighs the increase in the dispersion of the shocks' realisation, the *distributional* effects are muted and the *real option* channel generates a slow recovery of intangible capital. Without the investment overshoot, productivity growth rates falls by 50 basis points per annum and result in a permanent fall in the level of output and productivity of 1.0%. This accounts for a fifth of the GDP lost and a quarter of the TFP lost as a result of the U.S. Great Recession.

Literature Review — First and foremost, the paper contributes to the literature on uncertainty, which finds its roots in the early work of Bernanke (1983) and subsequently by Bloom (2009) and Bloom et al. (2018). These studies focus on the response of investment in physical capital following an increase in aggregate and idiosyncratic uncertainty. The interaction between a second-moment shock and non-convex physical capital costs generate the canonical *real-option* channel, whereby firms halt investment and cause a sharp drop in investment and output. However, unlike this paper, this strand of literature focuses only on the effect of uncertainty on physical capital. As a result, these models are unable to generate the productivity dynamics beyond the business-cycle frequencies and explain the productivity slowdown following the Great Recession.

Furthermore, by focusing on medium-run frequencies, this paper highlights the potency of the distributional effects of an uncertainty shock. Indeed, such effects are capable of generating prolonged expansions after the initial recession. Therefore, unlike previous literature, I provide empirical evidence about the relative contribution of distributional effects and the real option effects and study a case whereby forecast dispersion increases more than the actual dispersion of realised shocks. In this manner, the paper is able to calibrated the nature of the distributional effects and finds that they are dominated by the real option channel.

Secondly, this study also relates to the literature on medium-run business cycles pioneered by Comin and Gertler (2006), which aims to understand medium-frequency fluctuations in economic aggregates by studying business cycle shocks in an endogenous growth model. Building on the seminal paper by Comin and Gertler (2006), Kung and Schimd (2015) explore the effects of business cycle fluctuation and the deriving long-run growth fluctuations on asset prices. In such investigations, the roles of uncertainty as a business-cycle shock and firm heterogeneity are ignored, as only first-moment shocks are considered. Seen as this literature analyses business cycle shocks in a representative agent setting, it is unable to study the response of the economy to an uncertainty shock.

Thirdly, this paper speaks to the literature on firms' investment heterogeneity and investment costs. Research by Cooper and Haltiwanger (2006), and then by Khan and Thomas (2008), have shown the importance of non-convex capital adjustment costs in the explaining the lumpy distribution of investment rates in the data. Specifically, they show

how non-convex costs in firm heterogeneous models can generate aggregate dynamics akin to representative agent models with convex capital costs. Again, such papers only focus on physical capital investment, and as such the role for medium-run productivity dynamics is absent. This paper presents a novel calibration of the non-convex costs faced by firms using COMPUSTAT data on intangible capital investment shares.

Fourthly, this investigation contributes to the literature which aims to explain the slow recovery of the U.S. economy following the Great Recession in a heterogeneous firm setting. Such literature (see Garcia-Macia (2017) and Queralto (2019)) investigates how financial crises can provoke slow recoveries though the interaction of higher financial costs, worsening collateral constraints and investment in R&D. Although the role for financial intermediaries is absent, this paper offers a complementary contribution by analysing the effects of uncertainty, embodied by second-order moment shocks, which can generate slow recoveries. In such literature, only open-economies are considered, meaning the effects of general equilibrium are limited as prices are exogenous and as such, they are unable to represent the full dynamics of economies like the United States.

Finally, in this paper I present a general equilibrium with endogenous growth and firm heterogeneity where prices are endogenous and therefore I also follow the strand of literature using the Krusell and Smith (1998) algorithms which help solve such models. The innovation in this paper is that such forward-looking algorithms are applied to models of endogenous growth. Endogenous growth models feature a system of first-order difference equations which are usually solved using Rational Expectations. However, given that the solution to heterogeneous firms' problem rely on Bounded Rationality algorithm similar to Krusell and Smith (1998), I propose a solution algorithm which applies Bounded Rationality also to the system of first-order difference equations. Indeed, the computational challenge tackled in this paper may help subsequent research analyse the effects of firm heterogeneity and uncertainty shock in more complicated settings.

The paper proceeds in the following manner. In Section 2 I provide empirical evidence of the effect of uncertainty on R&D investment. Section 3 describes the model. The model's solution is laid out in Section 4, whilst the model's calibration is explained in Section 5. In Section 6, I elucidate on the effects of uncertainty on productivity dynamics through intangible capital investment using model simulations. Lastly, in Section 7 I conclude the paper summarising its aims and findings, whilst also proposing scope for future research on the matter.

2 Empirical Evidence

In this section, I bring empirical evidence using firm-level data of how higher aggregate uncertainty reduces firms' investment in intangible capital and R&D investment. More precisely, I use data on publicly traded firms in the U.S. provided by COMPUSTAT from 1990 until 2017 for such investigation. In order to proxy uncertainty, I use an aggregate measure of uncertainty constructed by Jurado et al. (2015) for the U.S. economy. I exploit the firm-level variation in intangible capital investment to establish causality.

2.1 Data

For firm-level data on intangible investment and expenditure on R&D, I rely on quarterly COMPUSTAT data of U.S. publicly listed firms from 1990:Q1 until 2017:Q3, accessed via the Wharton Research Data Services. The sample starts in 1990 as before this year only a limited number of companies reported values for R&D expenditure and firms only start reporting intangible capital assets after the year 2000. This panel dataset contains the following variables: global company key (GVKEY), observation date, calendar quarter date, company name, total sales, net income, total assets, total liabilities, common shares outstanding, stock price at quarter close, ISO Country code of incorporation character (FIC), and most importantly the firm's total net value of intangible assets[4] as well as their R&D expenses. Whereas for the price deflator data, in absence of a measure for intangible capital prices, I use the GDP implicit deflator (GDPDEF) for the U.S. obtained via the National Income and Product Accounts of the United States (NIPA) of the U.S. Bureau of Economic Analysis.

In the COMPUSTAT database, intangible capital represents the assets owned by the firm regarding the development of new products or services. Similarly, R&D expenses represent all costs that relate to the development of new products or services. Specifically, R&D expenditure reflects the company's contribution to R&D, whereas intangibles are assets that have no physical existence in themselves but represent rights to enjoy some privilege. Nonetheless, due to U.S. accounting rules, some discrepancies emerge. Specifically, when the firm invests in creating its intangible assets, such cost is expensed on the income statement but it is seldom capitalised on the balance sheet, meaning that such expenditure only shows up as R&D investment. However, if the firm purchases some intangible assets from a third party, like a patent, then the purchase is capitalised on the

[4]The COMPUSTAT database separates the stock of intangible assets into intangible capital (*intanq*) and other intangible capital (*intanoq*). The separate classification of such stock is due to the diverse nature of intangible capital. For the avoidance of doubt, I add these variables together to create the firm's total net value of intangible assets. All regressions are robust even if the dependent variable only contains *intanq*.

balance sheet as intangible assets. To overcome such discrepancies, I analyse the effect of uncertainty on both of these measures for robustness.

To measure uncertainty, I use the macroeconomic uncertainty index constructed by Jurado et al. (2015). In the literature, it is commonplace to use market indexes such as the CBOE Volatility Indexes (VIX or VXO) which track market expectation of near-term volatility conveyed by stock index option prices, a proxy for uncertainty. However, such indexes have been criticised as a significant amount of variations in these market uncertainty indices are forecastable by agents and, as such, cannot be considered as true uncertainty. To avoid this problem, Jurado et al. (2015) construct a measure of macroeconomic uncertainty based on the dispersion of the unforecastable components of a series of macro-variables [5]. Section 2.2 provides a fuller explanation of the reasons for choosing the Jurado et al. (2015) uncertainty index and its construction. In this exercise, the uncertainty index has three different measures which vary according to h, the months of the forecast horizon: 1 for a monthly forecast, 3 for a quarterly forecast and 12 for a yearly forecast.

In the estimation, I will also control for the aggregate economy's activity at quarterly frequency using the U.S. Real Gross Domestic Product, taken from National Income and Product Accounts of the United States (NIPA)[6]. Lastly, I use the excess bond premium introduced by Gilchrist and Zakrajšek (2012) as a control for financial shocks. The measure consists of the average credit spread on senior unsecured bonds issued by non-financial firms. Therefore, any increase in the excess bond premium reflects the deterioration in financial conditions of the economy.

The resulting dataset is composed of an unbalanced panel of $6,551$ firms spanning between 1990 to 2017, where a firm has 26 observations on average. The data cleaning and construction process are detailed in Appendix A. It is possible to summarise the variables of interest in Table 1. Note that for the firm's net value of intangible capital there are fewer observations than other panel variables, since in the COMPUSTAT dataset firms only started to report this variable after the year 2000. However for the uncertainty index, real GDP and the excess bond premium, I only report the time series values.

2.2 Measuring Uncertainty

Since Bloom (2009), a growing body of literature has tried to measure economic uncertainty in the past decade. The most obvious problem with measuring uncertainty is the fact that it is inherently unobservable, and therefore one has to rely on proxies for the latent stochastic

[5] For more details on the construction of the uncertainty measure, please refer to Jurado et al. (2015)

[6] Access can be found at `http://www.bea.gov/national/pdf/nipaguid.pdf`

Table 1. Summary Statistics

Variable	*Observations*	*Mean*	*Standard Deviation*	*Minimum*	*Maximum*
R&D Expenses	170,756	22.15	106.5	0.001	2040
R&D Intensity	170,756	0.050	0.366	4e-06	81
Intangible Capital Net Value	88,276	590.4	3622	0	195791
Total Assets	170,756	1387	6987	0.001	283149
Total Liabilities	170,756	766.6	4316	0.001	162266
Market Capitalisation	170,756	2461	14838	1e-05	750387
Net Income (Loss)	170,756	18.16	247.2	-41847	18024
Sales Growth	170,756	0.022	0.540	-10.0	10.35
Uncertainty Index (h=1)	112	0.647	0.083	0.546	1.052
Uncertainty Index (h=3)	112	0.784	0.087	0.677	1.209
Uncertainty Index (h=12)	112	0.912	0.050	0.847	1.149
Real GDP	112	12991	2232	8865	16851
Excess Bond Premium	112	2.170	0.995	1.140	7.417

Source: *COMPUSTAT database accessed via Wharton Research Data Services. Jurado et al. (2015) data on uncertainty measures. National Income and Product Accounts of the United States (NIPA) of the U.S. Bureau of Economic Analysis for the macroeconomic variables. Gilchrist and Zakrajšek (2012) for the data on excess bond premium.*
Notes: *The table describes all of the variables used in the regression in Section 2. Note the sample starts in 1990:Q1 and ends in 2017:Q3. All monetary variables are deflated using the GDP implicit deflator. Note that there are fewer observations for firm's net value intangible capital since firms only started reporting after the year 2000. For the uncertainty index, the real GDP and the excess bond premium I only report the number of unique values since they do not have a panel dimension. The h in the Jurado et al. (2015) uncertainty index represents the forecast horizon in terms of months used for the calculation: 1 for a monthly forecast, 3 for a quarterly forecast and 12 for a yearly forecast.*

process. As pointed out by Jurado et al. (2015), one approach in measuring uncertainty has been to use common volatility-based proxies such as the implied or realised variance of stock market returns, cross-sectional dispersion of firm profits, sales and productivity, the cross-sectional of subjective forecast, or even amount of times uncertainty related terms appeared in the news.

Nonetheless, the measures, albeit observable, may not only capture uncertainty. Indeed, most of these measures imply a tight link between volatility and uncertainty, when in reality the two concepts are quite distinct. Whilst the notion of volatility is quite straight-forward - the statistical measure of dispersion - the concept of uncertainty is more complex. Uncertainty is typically defined in a juxtaposed manner as the lack of certainty, that is, the inability to *forecast* the future. However, volatility-based measures may increase in relatively certain times because they capture the dispersion of fundamentals, rather than estimate the latent stochastic process of uncertainty.

This key difference is at the basis of the construction of the macroeconomic index of uncertainty by Jurado et al. (2015). Recalling from Jurado et al. (2015), the uncertainty index is constructed as the expected squared error in forecasting tomorrow's fundamentals. Let h denote the forecast horizon in the variable $y_{it} \in Y_t$, and $\mathbb{U}^y_{jt}(h)$ as the conditional volatility in the *unforecastable* component of the future value of the series. The uncertainty over each series can be expressed as

$$\mathbb{U}^y_{jt}(h) = \sqrt{\mathbb{E}\left[\left(y_{jt+h} - \mathbb{E}\left[y_{jt+h}|I_t\right]\right)^2 |I_t\right]}, \tag{1}$$

where expectations are taken with respect to the information set I_t available at time t. A measure of aggregate or macroeconomic uncertainty is constructed as a weighted average of the uncertainty over individual series

$$\mathbb{U}^y_t(h) = \mathbb{E}_w\left[\mathbb{U}^y_{jt}(h)\right], \tag{2}$$

where w are the weights attached to the respective series.

The index has two key characteristics which enable it to capture uncertainty better than other common proxies. Firstly, the index only measures the conditional variance of the *unforecastable* competent of the observed series, be it GDP or stock market returns, etc. By removing the forecastable component, the measure aims to capture uncertainty rather than volatility. This is an essential difference because uncertainty is defined as the *unforecastability* of the economy. Secondly, the index measures the common variation in uncertainty across numerous series of data. This has the advantage of capturing the common aggregate element of uncertainty to which the whole economy is subjected. Only using a single series to represent macroeconomic uncertainty is problematic as the series may capture idiosyncratic variation in uncertainty that does not affect the whole economy.

The differences between proxies based on the notion of volatility, be it related to stock market returns or cross-sectional dispersion of firms' observables, and the aggregate uncertainty index constructed by Jurado et al. (2015) seems to be reflected in the data. Volatility-based proxies appear to vary independently from the aggregate uncertainty measure, as reflected by the correlation between the VXO, the most commonly used volatility based proxy, and the aggregate uncertainty index which is only around 0.45. If one uses a volatility-based measure computed using the stock returns of the firms in the COMPUSTAT database, then the correlation is nearing zero. This is due to two reasons: *i)* volatility-based proxies overestimate the number of uncertainty episodes; *ii)* the aggregate uncertainty index has far more persistence than its counterpart.

The fundamental problem of volatility-based measures is that they pick up changes in the actual dispersion of the distribution of fundamentals, rather than the uncertainty surrounding such dispersion. More specifically, volatility measures also capture the *forecastable* change in the dispersion of the distribution. This points to the conclusion that volatility-based measures of uncertainty may not be well-suited to measure uncertainty because much of the variation seems to be driven by factors other than uncertainty. It is for these reasons that in this empirical investigation I rely on the macroeconomic uncertainty index of Jurado et al. (2015) rather than the common proxies, like the VIX or VXO typically used in the literature.

2.3 Empirical Strategy

The objective of this exercise is to quantify the effect of uncertainty on firms' intangible investment. It could be argued that the analysis suffers from an endogeneity issue, rendering a causal interpretation of the phenomena difficult. However, the firm-level dimension of the dataset allows for a straight-forward identification of uncertainty shocks. Exploiting the firm-level variation in R&D and intangible capital investment, it is possible to impose an identifying restriction without much controversy that the firm's R&D investment, observed at firm-level, does not influence the aggregate measure of uncertainty. In doing so, I am plausibly assuming that the investment decision of a single firm does not cause any variation in aggregate uncertainty.

It is possible to characterise the regression which aims to quantify the effect of uncertainty on firms' intangible capital and R&D investment in the following manner:

$$y_{i,t} = \rho y_{i,t-1} + \beta\sigma_t + \delta\mathbf{m}_t + \gamma\mathbf{x}_{i,t} + \mathbf{f}_{i,sic,q,y,age} + \epsilon_{i,t}. \tag{3}$$

The dependent variable $y_{i,t}$ is one of three different measures of the firm's investment in intangibles: the log of the firm's intangible capital, the log of the firm's R&D expenses, or the firm's R&D intensity calculated as R&D expenditures over total assets. The dependent variable is then regressed on its lagged value to capture the dynamics of intangible capital investment. Investment in R&D and intangibles are subject to high adjustment costs and are highly forward-looking since such investments will not pay-off immediately, hence it is vital to capture such dynamics using the lagged value of the dependent variable. This also may help the estimation if errors are auto-correlated, which in this empirical investigation may well be the case.

Further, the dependent variable is regressed on σ_t, the aggregate Jurado et al. (2015) uncertainty index $\mathbb{U}_t^y(h)$, which is the object of interest in this regression. In the baseline regression, the monthly uncertainty index with a horizon forecast ($h = 1$) will be employed, but I will also show how using different forecast horizons for the regression will not qualitatively alter the results.

The regression also includes two sets of controls: a set of aggregate controls, $\mathbf{m}_t$, which contains the log real GDP and the excess bond premium index developed by Gilchrist and

Zakrajšek (2012); as well as, a set of firm-level controls, $\mathbf{x}_{i,t}$, that include the log of the firm's total assets, the log of the firm's total liabilities, the log of market capitalisation, the firm's net income, and the firm's sales growth.

In this estimation, it is vital to include aggregate controls so to avoid running into problems of omitted variable bias. Since the independent variable of interest, uncertainty σ_{t-1}, does not vary at the firms-level, time fixed effects cannot be applied. Instead, by controlling for the aggregate economic activity, it is possible to capture macroeconomic shocks which may confound the effect of uncertainty on intangible capital and R&D. For this reason, GDP is included as a control in the regression. Similarly, it is a well-established fact in the macroeconomic literature that is extremely difficult to separately identify uncertainty shock from financial shocks, especially in VAR models, due to the similarity they exhibit in the contemporaneous responses of macroeconomic variables (see for example see Caldara et al. (2016)). However, using a regression approach it is possible to disentangle the effects of uncertainty from the financial shocks simply by controlling for the excess bond premium measure build by Gilchrist and Zakrajšek (2012), which in this case is used as a proxy for macroeconomic financial conditions[7].

The specification includes firm (i), industry (sic), quarter (q), year (y), and age (age) fixed effects, all represented by $\mathbf{f}_{i,sic,q,y,age}$. For the industry fixed effects, the 4-digit SIC code is used. Quarter fixed effects are also needed due to the nature of intangible capital investments since companies can exploit tax deductions and tax credits on such investments, which means that data may feature some seasonality. I include the firm's age fixed effects to control for age dynamics that may be present in the investment in intangibles. The regression does not feature a constant due to the inclusion of the fixed effects.

As the levels of fixed effects included in the regression are notable, for the estimation I rely on the method developed by Correia (2016) that proposes a feasible and computationally efficient estimator of linear models with multiple levels of fixed effects. Ultimately, to avoid heteroskedasticity, the standard errors will be computed by clustering at the firm-level.

Finally, a common problem with dynamic panel regressions with firm fixed effects is the Nickell (1981) bias. That is, since one has to control for the lagged dependent variable and firm-level fixed effects, the errors in the regression may potentially be correlated with the lagged dependent variable. This induces a downward bias in the estimation, which according to Nickell (1981) is approximately equal to $\frac{-(1+\beta)}{(T-1)}$. This would be a severe limitation to the empirical strategy if the time dimension of the panel is short, however, the

[7]In this specification, I also avoid the problem of having to take a stance on the contemporaneous effect of uncertainty shocks on financial shocks and *vice versa*, unlike empirical studies using Vector Auto-Regressive models.

bias equation states that as $T \to \infty$ the bias goes to zero, meaning that for a sufficiently long time dimension, the bias is negligible. Indeed, the data employed in this estimation covers the periods from 2000 to 2017 ($T = 72$ quarters) for the specification using intangible capital investment and from 2000 to 2017 ($T = 112$ quarters) for the specification using R&D.

2.4 Empirical Results

Table 2 displays the results from the regression of the firms' investment in intangible capital on uncertainty, as specified in Equation 3. Column (1) reports a preliminary regression using an OLS estimator with no fixed effects, and no correction for heteroskedasticity. In such an exercise I note the negative relationship between the uncertainty measure at the monthly horizon (h=1) and firms' intangible capital, however, no causal interpretation can be deduced. Column (2) shows the same regression with the fixed effects and the negative relationship still holds. In column (3), I illustrate the results of the regression using the aforementioned empirical strategy. This highlights the negative and highly statistically significant effect of macroeconomic uncertainty at the monthly horizon (h=1) on the firm's intangible capital.

A unitary increase in the Jurado et al. (2015) index results in a 19% fall in firms' intangible capital. More precisely, a standard deviation increase (0.083) of the uncertainty index would result in a decrease in the firm's intangible capital by 1.6%. To put this result into perspective, during the last financial crisis the uncertainty index at the monthly horizon increased by 40 basis points, which with a rough calculation translates into a 7.6% fall in the firms' intangible capital. This result would explain roughly half of the drop in aggregate intangible capital during the Great Recession. Moreover, the explanatory variables included in the regression account for 98% of the variation in the firms' investment in intangible capital, showing how well the regression captures the firms' investment behaviour.

These results hold even when I explore the effects of aggregate uncertainty on R&D investment shown in Table 3. Specifically, a unitary increase in the uncertainty index leads to an 11% contraction in R&D expenditure and a 14% decline in R&D intensity. Or equivalently, an increase on one standard deviation (0.083) in the uncertainty index would result in a decrease in the firm's R&D expenditure by 0.9% and a 1.2% fall in R&D intensity. Such a result seems to confirm previous findings by Bloom (2007), who uses volatility-based measures.

Interestingly, if the forecast horizon of the uncertainty measure is extended to a yearly frequency, it is possible to notice that the effect of uncertainty on the firms' investment in intangible strengthens. As in Tables 2 and 3, a unitary increase in the uncertainty measure

Table 2. The Effect of Uncertainty on Firms' Intangible Capital

	(1) Int.Cap.	(2) Int.Cap.	(3) Int.Cap.	(4) Int.Cap.	(5) Int.Cap.
Uncertainty (h=1)	0.103***	-0.204***	-0.190***		
	(0.028)	(0.059)	(0.056)		
Uncertainty (h=3)				-0.189***	
				(0.057)	
Uncertainty (h=12)					-0.326**
					(0.107)
Int. Cap. (t-1)	0.995***	0.910***	0.842***	0.842***	0.842***
	(0.001)	(0.004)	(0.007)	(0.007)	(0.007)
GZ Spread	-0.0310***	-0.0150***	-0.0105*	-0.00950*	-0.00943*
	(0.003)	(0.004)	(0.004)	(0.004)	(0.005)
Real GDP	0.103***	0.475	0.233	0.275	0.425
	(0.025)	(0.393)	(0.377)	(0.376)	(0.375)
Market Cap			0.00527	0.00526	0.00519
			(0.006)	(0.006)	(0.006)
Assets			0.217***	0.217***	0.218***
			(0.016)	(0.016)	(0.016)
Liabilities			0.0399***	0.0399***	0.0399***
			(0.007)	(0.007)	(0.007)
Sales Growth			0.0359***	0.0359***	0.0359***
			(0.005)	(0.005)	(0.005)
Net Income (Loss)			3e-05	3e-05	3e-05
			(0.000)	(0.000)	(0.000)
Constant	-0.953***				
	(0.237)				
Fixed Effects		✓	✓	✓	✓
Observations	62767	62629	62629	62629	62629
Adjusted R^2	0.976	0.978	0.979	0.979	0.979

* $p < 0.05$, ** $p < 0.01$, *** $p < 0.001$.

Notes: *The table shows the result of the regressions for uncertainty on firms' intangible capital. Standard errors are presented in parentheses. The reported standard errors in columns (2),(3),(4), and (5) are clustered within-firm. Int. Cap. is the abbreviation for intangible capital. The h in the Jurado et al. (2015) uncertainty index represents the forecast horizon in terms of months used for the calculation: 1 for a monthly forecast, 3 for a quarterly forecast and 12 for a yearly forecast. The number of observations in the last four columns is lower since the estimator developed Correia (2016) drops singleton observations to compute the fixed effects. Fixed effects include firm, industry, quarter, year and the firm's age.*

Table 3. The Effect of Uncertainty on Firms' R&D Investment

	(1) R&D	(2) R&D	(3) R&D	(4) R&D Int.	(5) R&D Int.	(6) R&D Int.
Uncertainty (h=1)	-0.113**			-0.134***		
	(0.037)			(0.039)		
Uncertainty (h=3)		-0.109**			-0.136***	
		(0.038)			(0.040)	
Uncertainty (h=12)			-0.168*			-0.230**
			(0.074)			(0.078)
R&D (t-1)	0.658***	0.658***	0.658***			
	(0.007)	(0.007)	(0.007)			
R&D Int. (t-1)				0.596***	0.596***	0.596***
				(0.008)	(0.008)	(0.008)
GZ Spread	0.0118***	0.0122***	0.0117***	0.0261***	0.0269***	0.0269***
	(0.003)	(0.003)	(0.003)	(0.003)	(0.003)	(0.003)
Real GDP	-0.382	-0.359	-0.280	-0.129	-0.107	-0.00714
	(0.211)	(0.210)	(0.206)	(0.221)	(0.219)	(0.215)
Market Cap	0.0665***	0.0665***	0.0665***	0.0567***	0.0567***	0.0567***
	(0.003)	(0.003)	(0.003)	(0.003)	(0.003)	(0.003)
Assets	0.145***	0.145***	0.145***	-0.263***	-0.263***	-0.263***
	(0.005)	(0.005)	(0.005)	(0.007)	(0.007)	(0.007)
Liabilities	0.0372***	0.0372***	0.0372***	0.0616***	0.0616***	0.0616***
	(0.003)	(0.003)	(0.003)	(0.004)	(0.004)	(0.004)
Sales Growth	0.0265***	0.0265***	0.0266***	-0.0102**	-0.0102**	-0.0102**
	(0.003)	(0.003)	(0.003)	(0.004)	(0.004)	(0.004)
Net Income (Loss)	-3e-05***	-3e-05***	-3e-05***	-4e-05***	-4e-05***	-4e-05***
	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)
Fixed Effects	✓	✓	✓	✓	✓	✓
Observations	161098	161098	161098	161098	161098	161098
Adjusted R^2	0.966	0.966	0.966	0.884	0.884	0.884

* $p < 0.05$, ** $p < 0.01$, *** $p < 0.001$.
Notes: *The table shows the results for the regression for uncertainty on firms' R&D investment. Standard errors are presented in parentheses. The reported standard errors in columns (2),(3),(4), and (5) are clustered within-firm. R&D Int. is an abbreviation for R&D intensity. The h in the Jurado et al. (2015) uncertainty index represents the forecast horizon in terms of months used for the calculation: 1 for a monthly forecast, 3 for a quarterly forecast and 12 for a yearly forecast. Fixed effects include firm, industry, quarter, year, and the firm's age.*

at the yearly frequency decreases intangible capital by 32%, R&D expenditures by 17% and R&D intensity by 23%. Such a result indicates that longer horizon uncertainty tends to exacerbate the reduction of the firms' intangible investment. This seems to be in-line with the theory on investment because intangible investment is subject to higher sunk costs than physical investment and has a much longer pay-off horizon. Therefore, if uncertainty is expected to be prolonged for some time, the effect on the firms' plans for intangible capital investment intensifies.

Having established the empirical causal effect of uncertainty on intangible capital investment, I turn to the investigation of the effects of uncertainty on productivity growth dynamics through intangible capital investment. To investigate the nexus, I first present a general equilibrium model of endogenous growth augmented with firm heterogeneity and non-convex intangible capital costs of adjustment to capture the effects of uncertainty.

3 Model

In the endeavour to analyse the effects of uncertainty on productivity growth dynamics, I develop a general equilibrium model of endogenous growth with heterogeneous final good firms facing non-convex adjustment costs to intangible capital investment. The model contains two key features which enable it to capture the effects of uncertainty on productivity growth dynamics at medium frequencies through intangible capital investment.

Firstly, the model has endogenous productivity generated by investment in intangible capital and R&D in the spirit of Romer (1990), Comin and Gertler (2006), and Kung and Schimd (2015). Unlike standard business cycle models which assume exogenous technological progress, in this model, growth is sustained through the accumulation of intangible capital. This is defined as a composite of intangible capital goods, each produced with a single patent that facilitates the production of the final output good. Innovation increases the overall stock of intangible capital by creating new patents which require investment in R&D, undertaken by the innovation sector. Intangible capital not only increases the productivity of the single firms that undertake such investment but crucially, it also increases the productivity of every other firm through its spillover effects of innovation.

Secondly, for uncertainty to matter, investment in intangible capital needs to be costly to reverse. Therefore, in the style of Khan and Thomas (2008), Bloom (2009) and Bloom et al. (2018), I model the final good firms as heterogeneous firms characterised by idiosyncratic productivity. These firms face non-convex costs of adjusting intangible capital, comprising of both fixed costs and partial investment irreversibility. Seen as the model is cast in real framework, the idiosyncratic productivity distribution should not be interpreted literally as technology shocks, but more broadly as shocks to the firms' fundamentals regardless of whether they are demand or supply-driven. Indeed, in the paper I refer to idiosyncratic productivity and fundamentals interchangeably. Unlike Bloom (2009) and Bloom et al. (2018), the model does not contain labour adjustment costs for the simple reason that it renders the solution of the model more complex and computationally difficult since the model also incorporates an endogenous growth mechanism. Nevertheless, as noted previously by Bloom (2009), the presence of labour adjustment costs does not fundamentally change the model's behaviour. If anything, the results found for the effect of uncertainty on intangible capital investment and productivity dynamics will prove to be lower-bound estimates, as labour adjustment costs aggravate the response of macroeconomic variables.

3.1 Environment

In the model, time is discrete and the planning horizon is infinite. The economy is populated by a unit measure of households with no population growth. Each period, the households take a standard inter-temporal consumption-saving decision and a static labour supply decision, selling labour services to the final good firms in return for a wage w, which is determined in a perfectly competitive labour market. Furthermore, the economy's production process involves three distinct sectors: a final good sector, an intangible capital good sector and an innovation sector.

In the model, intangible capital is a CES composite of intangible capital goods, defined as

$$m = \left[\int_0^N x_j^\nu dj \right]^{\frac{1}{\nu}}, \tag{4}$$

where $j \in [0, N]$, where x_j is the amount of intangible capital good associated with patent $j \in [0, N]$. Also, N is the measure of patents, and the elasticity of substitution between patents in production is represented by $\frac{1}{1-\nu}$ where $\nu \in (0, 1)$.

Intangible capital production is undertaken by a monopolistic competitive intermediate good sector. Each intangible capital good x_j is produced by the intangible good firm, where patent producers have monopoly power and thus sell the right to use the patent to final good firms at price p_j^x.

Innovation, that is the creation of new patents, is left to the innovation sector, which consists of a representative perfectly competitive firm that increases the total stock of patents defined by N by undertaking investment in R&D, denoted by S. The model, as is standard in the growth-cycle literature, differentiates between R&D and intangible capital. However in the data and in reality, such a distinction is more nuanced. Yet, to model growth-cycles, the separation is necessary[8].

The final good sector is populated by a unit measure of heterogeneous firms, which produce an identical final output good using labour hired from the household and intangible capital goods. These firms purchase intangible capital from intangible capital good firms in an endeavour to increase the productivity of its productive capacity. Final good firms are characterised by a triple of state variables: (i) idiosyncratic productivity, (ii) intangible capital stock, (iii) and finally, a fixed cost associated with intangible capital investment.

The final good firms, at the beginning of the period, will also learn about the next period's level of uncertainty σ', which indicates the dispersion of the idiosyncratic

[8]The separation between producers of patents and the users of the patents goods that arise from such patents does not reflect reality as some firms may develop their patents using R&D investments, whilst others may purchase patented goods from other firms.

productivity $z \in \mathbb{Z}$. Each firm is characterised by an idiosyncratic level of productivity also observed at the beginning of each period which evolves according to an auto-regressive Markov process of order one: $\log(z) = \rho^z log(z_{-1}) + \sigma\epsilon^z \quad \text{where} \quad \epsilon^z \sim \mathcal{N}(0,1)$; where ρ^z is the auto-regressive coefficient of the productivity process and ϵ^z are the disturbances also drawn every period from a Normal distribution with a standard deviation of σ. Notice that the timing of uncertainty shocks implies that firms learn in advance about the next period's distribution from which they will be drawing their idiosyncratic shocks, therefore introducing the concept of uncertainty about next period's fundamentals.

Each final good firms at the beginning of the period also hold a bundle of intangible capital goods $\mathbf{x} = \{x_j\}_{j=0}^{N}$. It will be shown in the model that this bundle of intangible capital goods will boil down to a single representative intangible capital good, that is $\mathbf{x} = x$, using the equilibrium solution in the intangible capital good sector. Such goods facilitate the production of the final output good by increasing the productivity of the other factors of production. Investment in intangible capital goods is undertaken one period in advance, so a firm will purchase the goods this period which will be put into production the next period.

Investment in intangible capital is also subject to non-convex costs, including a fixed cost of adjustment. Such cost is expressed in units of output $y\xi$, where ξ is stochastically drawn each period from the distribution $G(\xi) \sim \mathcal{U}[0,\bar{\xi}]$. The upper bound of the uniform distribution represents the overall level of friction in the economy, and the higher the upper bound the more fictions in intangible capital investment[9].

In this set-up, it is sufficient to describe the population of final good firms with the probability measure $\mu(z, \mathbf{x})$ defined over the Borel algebra $\mathbb{S}$ for the product space $\mathbb{S} = \mathbb{Z} \times \mathbb{R}_+$.

In addition to these idiosyncratic states, I introduce two aggregate state variables. The first aggregate state variable relates to the level of uncertainty faced by the agents in the economy. The uncertainty parameter σ follows a two-state Markov chain which enables the model to generate periods of low and high uncertainty, where shocks to uncertainty are modelled as increases in the variances of the respective stochastic process z.

The second aggregate state, as is common with growth-cycle models, is the state variable is N, the measure of patents in the economy at the end of the period.

[9]This set-up is tantamount to a model whereby firms all face the same deterministic fixed cost, however, in such a model the firms' intangible capital investment policy function is discontinuous. Such discontinuity renders the computational strategy of the model cumbersome. So to avoid such difficulties, the stochastic set-up of the fixed cost used in this model helps smooth-out the resulting discontinuities arising in the firm's policy function.

3.2 Final Good Firm

Each firm produces a final output good using a Cobb-Douglass technology function $y(z,l,m) = z(l^{(1-\zeta)}m^{\zeta})^{\gamma}$ where γ is defined as the span of control parameter and ζ as the intangible capital share of output. Indeed, each firm with productivity z will produce a final good y by hiring labour l and using intangible capital stock m. Since the intangible capital stock m is a CES composite of patents x_j, as in Eq.4, the firm's production technology can be redefined as:

$$y(z,l,\mathbf{x};N) = z\left(l^{(1-\zeta)}\left[\int_0^N x_j^{\nu}dj\right]^{\frac{\zeta}{\nu}}\right)^{\gamma} \quad \text{where} \quad \mathbf{x} = \{x_j\}_{j=0}^{N}. \tag{5}$$

Notoriously, for uncertainty to matter, the objective function of the agent, in the case the firm, needs to be concave. As a result, one condition that the parameters of the production must satisfy is that of decreasing returns to scale, which in this context means I have to impose that the span of control parameter is less than unity, $\gamma < 1$.

Labour is obtained from the household in exchange for the wage w determined in a perfectly competitive market. Intangible capital goods are purchased by the final good firm to be used in production by paying p_j^x for each patent x_j to the intangible good firms every period. The patents have a homogeneous rate of depreciation defined by δ. The investment in intangible capital goods is equal to: $i_j^x = x_j' - (1-\delta)x_j$. The rate of depreciation represents the value of the intangible capital good lost period to period.

Following the literature on non-convex costs of investment and uncertainty, particularly Khan and Thomas (2008), Bloom (2009) and Bloom et al. (2018), investment in intangible capital is not friction-less. Intangible capital investment is subject to non-convex costs of adjusting comprising of a fixed cost and a partial irreversibility cost. To adjust to the optimal intangible capital $\mathbf{x}'$, the firm must first pay a fixed stochastic non-convex adjustment cost. Specifically, the adjustment is defined in output units $\xi y(z,l,\mathbf{x};N)$. Moreover, firms also face a partial irreversibility cost meaning that if they would like to disinvest, denoted by the indicator function $\mathbb{I}(i_j^x < 0)$, then they face a per-unit loss of ω. One can think of the partial irreversibility friction as a cost a firm faces in no longer using a patented good in production.

At the beginning of the period, the firm observes its productivity z and the previous stock of intangible capital $\mathbf{x}$, as well as the aggregate states of the economy (N,σ,μ). At this point it is possible to define v, the value of the final good firm at the beginning of the period, as the expected value of the firm, given the possible values of the stochastic non-convex cost

ξ:

$$
\begin{aligned}
v(z, \mathbf{x}; N, \sigma, \mu) &= \mathbb{E}_{\xi} \tilde{v}(z, \mathbf{x}, \xi; N, \sigma\mu), \\
&= \int_0^{\overline{\xi}} \tilde{v}(z, \mathbf{x}, \xi; N, \sigma, \mu) \; G(d\xi).
\end{aligned} \tag{6}
$$

Here $\tilde{v}$ represents the value of the firm once it has observed the draw of the non-convex adjustment cost. Indeed, once the firm observes the draw of ξ, it decides whether to pay the cost and adjust its intangible capital to its optimal level. Alternatively, it can avoid the fixed cost and operate with this period's intangible capital, net of depreciation. The firm's choice of adjusting the intangible capital can be represented as follows:

$$
\tilde{v}(z, \mathbf{x}, \xi; N, \sigma, \mu) = \max \; \left\{ -\xi y(z, l, \mathbf{x}; N) + v^{A}(z, \mathbf{x}; N, \sigma, \mu) \; , \; v^{NA}(z, \mathbf{x}; N, \sigma, \mu) \right\}. \tag{7}
$$

Here, v^A represents the value of the firm upon adjustment of intangible capital, which equals the discounted flow of profits from production given that the firm optimises over all the factors of production:

$$
\begin{aligned}
v^{A}(z, \mathbf{x}; N, \sigma, \mu) = \max_{\mathbf{x}', l} \; & \left\{ y(z, l, \mathbf{x}; N) - \int_0^{N'} p_j^x [x_j' - (1-\delta) x_j] dj \right. \\
& \left. - \int_0^{N'} \omega \mathbb{I}(i_j^x < 0) dj - w(N, \sigma, \mu) l \right\} \\
& + \mathbb{E} \Omega(N, \sigma, \mu) v(z', \mathbf{x}'; N', \sigma', \mu' | z, N, \sigma, \mu).
\end{aligned} \tag{8}
$$

Whereas, v^{NA} is the value of the firm upon not adjusting. This equals the discounted flow of profits from production, given that the firm optimises only over labour and retains its predetermined level of intangible capital:

$$
\begin{aligned}
v^{NA}(z, \mathbf{x}; N, \sigma, \mu) = \max_{l} \; & \{ y(z, l, \mathbf{x}; N) - w(N, \sigma, \mu) l \} \\
& + \mathbb{E} \Omega(N, \sigma, \mu) v(z', (1-\delta)\mathbf{x}; N', \sigma', \mu' | z, N, \sigma, \mu).
\end{aligned} \tag{9}
$$

In this set-up, $\Omega(N, \sigma, \mu)$ represents the state-contingent discount factor used by firms to discount future flows, since the firms are ultimately owned by the households. Note that expectations for the next period are taken with respect to the exogenous processes of productivity Γ_z, as well as to the endogenous aggregate distributional state μ. Note that due to the timing assumption of the uncertainty state, the firms do not take expectations with respect to σ since they learn about the future dispersion of productivity in advance.

Seen as each firm has to choose whether to adjust the level of intangible capital based on the draw of the fixed adjustment cost, after having observed all other aggregate and idiosyncratic states, it is possible to define for every firm a threshold adjustment cost $\underline{\xi}(z, \mathbf{x}; N, \sigma, \mu)$ such that the value of adjusting its intangible capital stock is equal to the value of not adjusting:

$$-\underline{\xi}(z, \mathbf{x}; N, \sigma, \mu) y(z, l, \mathbf{x}; N) + v^A(z, \mathbf{x}; N, \sigma, \mu) = v^{NA}(z, \mathbf{x}; N, \sigma, \mu). \tag{10}$$

Equivalently, the threshold for adjusting capital can be expressed as an explicit function:

$$\underline{\xi}(z, \mathbf{x}; N, \sigma, \mu) = \frac{v^A(z, \mathbf{x}; N, \sigma, \mu) - v^{NA}(z, \mathbf{x}; N, \sigma, \mu)}{y(z, l, \mathbf{x}; N)}. \tag{11}$$

In this set-up, the firm's optimal intangible capital investment policy decisions can be represented by the following piecewise function:

$$X(z, \mathbf{x}, \xi; N, \sigma, \mu) = \begin{cases} \mathbf{x}'(z, \mathbf{x}; N, \sigma, \mu) & \text{if} \quad \xi \leq \underline{\xi}(z, \mathbf{x}; N, \sigma, \mu), \\ \mathbf{x}(1-\delta) & \text{otherwise.} \end{cases} \tag{12}$$

Whereby $X(z, \mathbf{x}, \xi; N, \sigma, \mu)$ is equal to the optimal level of capital if the adjustment cost is below the threshold $\underline{\xi}(z, \mathbf{x}; N, \sigma, \mu)$. Otherwise, it is equal to the previously accumulated level of intangible capital minus its depreciation.

Denote $L(z, \mathbf{x}; N, \sigma, \mu)$ and $Y(z, \mathbf{x}; N, \sigma, \mu)$ as the employment and output policy functions respectively. It is also possible to denote the total amount spent on non-convex costs by firms as

$$\Xi(N, \sigma, \mu) = \int_0^{\bar{\xi}} \left[\int_0^{N'} y(z, l, \mathbf{x}; N) \mathbb{I}(i_j^x \neq 0) + \omega \mathbb{I}(i_j^x < 0) dj \right] G(d\xi) \mu(d[z \times \mathbf{x}]), \tag{13}$$

where $\mathbb{I}(i_j^x \neq 0)$ is an indicator function. It takes the value 0 if the firm does not invest, that is, $i_j^x = X(z, x_j, \xi; N, \sigma, \mu) - (1-\delta)x_j = 0$, and $\mathbb{I}(i_j^x \neq 0) = 1$ if the firm chooses to invest, hence $X(z, x_j, \xi; N, \sigma, \mu) \neq (1-\delta)x_j$.

Notice that as a result of the introduction of the fixed cost of adjustment, the intangible investment policy function will exhibit an inaction region. In the inaction region, firms will find it more advantageous not to invest and instead wait and see until uncertainty subsides or capital depreciates beyond a certain point, rather than continue investing. Such inaction region is the basis for the *real option* channel and will be paramount for uncertainty to have an effect on macroeconomics dynamics. Indeed, an exogenous second-moment shock to fundamentals will generate the dynamics of aggregate productivity through the canonical *real options* channel, whereby uncertainty reduces the firms' demand of intangible capital, slowing down its accumulation.

3.3 Intangible Good Firms

Intangible good firms are monopolists which turn one patent j into an intangible capital good x_j using one unit of final output as a factor of production. Monopoly power is a crucial assumption in this framework as it is needed to create positive profits in equilibrium. Giving a positive value to patents provides incentives for the innovation sector to produce new patents, which will be described in Section 3.4. Since intangible good firms are monopolist, they maximise profits π_j by optimally setting prices p_j^x:

$$\pi_j(N,\sigma,\mu) = \max_{p_j^x} \left\{ p_j^x x_j'(p_j^x; N,\sigma,\mu) - x_j'(p_j^x; N,\sigma,\mu) \right\}. \tag{14}$$

Note that $x_j'(p_j^x; N,\sigma,\mu)$ represents the demanded quantity for patent $j \in [0,N]$, which is a function of its price and the aggregate states (N,σ,μ).

The value of owning exclusive rights to produce the intangible capital good x_j' is equal to the present discounted value of profits obtained from its sale

$$f_j(N,\sigma,\mu) = \pi_j(N,\sigma,\mu) + (1-\phi)\beta\mathbb{E}\Omega(N,\sigma,\mu) f_j(N',\sigma',\mu'|N,\sigma,\mu), \tag{15}$$

where ϕ is the patents' obsolescence rate. When a patent becomes obsolete, it provides no further value as it cannot be used for final good production[10].

Assuming a symmetric equilibrium a lá Dixit and Stiglitz (1977), the monopolist competitive characterisation of the intangible good sector implies that all monopolists in the sector will choose the same price and quantity:

$$x_j'(N,\sigma,\mu) = x' = X(N,\sigma,\mu) \quad \text{and} \quad p_j^x = p^x = \frac{1}{\nu}. \tag{16}$$

Thus, the profits for each monopolist can be written as:

$$\pi_i(N,\sigma,\mu) = \pi(N,\sigma,\mu) = \left[\frac{1}{\nu} - 1\right] X(N,\sigma,\mu). \tag{17}$$

Note that I have substituted in the total amount of intangible capital goods demanded $X(N,\sigma,\mu)$. Moreover, it is possible to re-formulate the present discounted value of profits by dropping the j subscript:

$$f(N,\sigma,\mu) = \pi(N,\sigma,\mu) + (1-\phi)\beta\mathbb{E}\Omega(N,\sigma,\mu) f(N',\sigma',\mu'|N,\sigma,\mu). \tag{18}$$

[10] Note that ϕ, the patents' obsolescence rate, is different from δ, the depreciation rate of the intangible capital good. Whilst the former refers to the process by which a patent becomes obsolete, the latter refers to the value lost in the intangible capital good produced with a certain patent. For example, imagine a computer. The patent that allowed the construction of the computer may become obsolete rendering the technology no longer useful for production, or the computer itself through wear-and-tear may depreciate and thus need replacing.

As a result of the symmetric equilibrium, one can define the quantity of intangible capital goods demanded by the final good firms equivalently as a single patented good $\mathbf{x}' = x'$. The simplification of the symmetric equilibrium allows the model not to track all the different varieties of patents and assume that there is only one equally diversified patent. Furthermore, the overall intangible capital stock can be taken to be: $m(N,\sigma,\mu) = N^{\frac{1}{\nu}} X_{-1}(N,\sigma,\mu)$.

3.4 Innovation Sector

The innovation sector is perfectly competitive and so can be formulated by a single representative innovation firm. In contrast to the intangible good firms who transform final outputs goods into existing patents, the innovation firm generates new patents by investing amount S in R&D. As a result, R&D expenditures by the innovation firms increase the total number of patents available in the economy N, whose law of motion can be written as:

$$N'(N,\sigma,\mu) = \theta(N,\sigma,\mu)S(N,\sigma,\mu) + (1-\phi)N. \tag{19}$$

The term $\theta(N,\sigma,\mu)$ represents the innovation firm's productivity of R&D expenditures

$$\theta(N,\sigma,\mu) = \frac{\chi N}{S(N,\sigma,\mu)^{1-\eta} N^{\eta}}, \tag{20}$$

where $\eta \in [0,1]$ is the elasticity of new patents with respect to R&D and χ is a scaling parameter[11]. Such specification postulates a positive externality whereby a higher stock of intangible capital makes innovation more productive, hence $\left(\frac{\delta\theta}{\delta N} > 0\right)$ as in Romer (1990). In fact, as the stock of intangible capital increases, the creation of new patents is facilitated. Additionally, this specification also exhibits a congestion externality whereby R&D investment has decreasing marginal returns, that is $\left(\frac{\delta\theta}{\delta S} < 0\right)$ as in Comin and Gertler (2006) and Kung and Schimd (2015). This means that additional R&D spending leads to a less than proportional increase in the total stock of intangible capital.

Assuming that the sector is characterised by free entry, one can deduce that the value of a new patent is equal to the value of the patent to the intangible good firms[12]. Therefore, the innovation firm's problem is to optimally choose the amount of R&D expenditures S to maximise profits, which consists of the present discounted value of the revenues generated

[11]The scaling parameters will be useful to calibrate the Balanced Growth Path growth rate of the model.

[12]It is for this reason that it is essential to have the monopolistic competition assumption in the intangibility good sector, as it ensures that there is always a non-negative value to producing new patents.

by the new patents minus the costs of R&D expenditures:

$$\max_{S} -S(N,\sigma,\mu) + \theta(N,\sigma,\mu)S(N,\sigma,\mu)\mathbb{E}\Omega(N,\sigma,\mu)f(N',\sigma',\mu'|N,\sigma,\mu). \tag{21}$$

Given the free entry condition, the optimal level of R&D expenditures can be obtained via the zero profit condition which yields:

$$S(N,\sigma,\mu) = \theta(N,\sigma,\mu)S(N,\sigma,\mu)\mathbb{E}\Omega(N,\sigma,\mu)f(N',\sigma',\mu'|N,\sigma,\mu). \tag{22}$$

3.5 Households

A unit mass of identical households populates the economy and there is no population growth. I assume that households also have access to a complete set of state-contingent claims, which are not modelled here since there is no heterogeneity in households, and as such these assets are in zero net supply in equilibrium. As a result, it is possible to cast the household's problem in terms of a representative household for simplicity.

The household has the following preferences with respect to consumption C and labour L^h

$$U(C,L^h) = log(C) - \varphi(1-L^h), \tag{23}$$

where φ is the parameter for the relative dis-utility of labour. Define β as the discount factor. The use of this utility function is dictated by computational reasons. The model's solution is complicated, so to avoid having to also find the wage that clears the labour market, I use this utility function because its linearity with respect to labour allows us to obtain a closed-form solution for the wage rate.

The household can store wealth as one-period shares in firms denoted with the measure λ. The household is the ultimate owner of the intangible good firms and innovation firms, and so every period it receives the profits from their activities. Note that the per-period profits obtained by the household will be equal to $\Pi(N,\sigma,\mu) = N\pi(N,\sigma,\mu) - S(N,\sigma,\mu)$. Given the prices for their current shares $\varrho(z,\mathbf{x}|N,\sigma,\mu)$ and the real wage they receive for their labour $w(N,\sigma,\mu)$, the household optimally chooses their current consumption $C(N,\sigma,\mu)$, the labour supply $L^h(N,\sigma,\mu)$, as well as the number of shares λ' to purchase at price $\varrho'(z',\mathbf{x}'|N',\sigma',\mu')$. It does so by maximising its lifetime utility subject to their budget constraint:

$$H(\lambda;N,\sigma,\mu) = \max_{C,\lambda',L^h} U(C(N,\sigma,\mu),L^h(N,\sigma,\mu)) + \beta\mathbb{E}H(\lambda';N',\sigma',\mu'|N,\sigma,\mu) \tag{24}$$

s.t.

$$C(N,\sigma,\mu)+\int_{\mathbb{S}}\varrho'(z',\mathbf{x}';N,\sigma,\mu)\lambda'(d[z'\times\mathbf{x}'])\leq$$
$$w(N,\sigma,\mu)L^h(N,\sigma,\mu)+\int_{\mathbb{S}}\varrho(z,\mathbf{x};N,\sigma,\mu)\lambda(d[z\times\mathbf{x}])+\Pi(N,\sigma,\mu).$$

Denoting $p(N,\sigma,\mu)$ as the marginal utility of consumption and $\Omega(N,\sigma,\mu)$ as the stochastic discount factor, one can derive the household's first-order conditions as follows:

$$p(N,\sigma,\mu)=\frac{1}{C(N,\sigma,\mu)},\tag{25}$$

$$w(N,\sigma,\mu)=\frac{\varphi}{p(N,\sigma,\mu)},\tag{26}$$

$$\Omega(N,\sigma,\mu)=\frac{\beta\mathbb{E}p'(N',\sigma',\mu')}{p(N,\sigma,\mu)}.\tag{27}$$

Let $C(\lambda;N,\sigma,\mu)$ describe the household's current consumption policy function, where $L^h(\lambda;N,\sigma,\mu)$ is the labour supply policy function, and I denote $\lambda(z',\mathbf{x}',\lambda;N,\sigma,\mu)$ as the policy function for the quantity of shares purchased in firms with productivity z' and intangible capital stock $\mathbf{x}'$.

3.6 Aggregate Constraint

The aggregate constraint, derived from the household's budget constraint, specifies that the total resources of the economy must be equal to the total expenditures. In this model, the aggregate constraint of the economy specifies that the total final output produced by the final good firms, $Y(N,\sigma,\mu)=\int_{\mathbb{S}}y(z,l,\mathbf{x};N)d([z\times\mathbf{x}])$, must be equal to the total consumption by the household, $C(\lambda;N,\sigma,\mu)$, the total investment in intangible capital, $N'X(N,\sigma,\mu)$, and the expenditure on R&D by the innovation sector, $S(N,\sigma,\mu)$, and finally the amount spent by firms in non-convex adjustment costs $\Xi(N,\sigma,\mu)$. The aggregate constraint can be expressed as

$$Y(N,\sigma,\mu)=C(\lambda;N,\sigma,\mu)+N'X(N,\sigma,\mu)+S(N,\sigma,\mu)+\Xi(N,\sigma,\mu),\tag{28}$$

where the strict equality in the aggregate constraint is guaranteed by the properties of the utility function.

3.7 Recursive Competitive Equilibrium

The model comprises of three exogenous idiosyncratic states: the idiosyncratic productivity (z), the firm's intangible capital stock $(\mathbf{x})$ and the non-convex cost (ξ). Furthermore, there is one exogenous aggregate state, the aggregate uncertainty level (σ), and two endogenous aggregate states: the joint distribution of firms over the intangible capital holdings and productivity $\mu(z, \mathbf{x})$, and the measure of intangible capital (N).

Given any initial conditions $\{N_0\}$ and the law of motion for the productivity process z and the uncertainty process of σ, a recursive competitive equilibrium is defined as a set of value functions $\{v, \tilde{v}, v^A, v^{NA}, f, H\}$, prices $\{w, (\Omega)_{j=1}^{\mathbb{S}}, \varrho, \varrho', p^x\}$, and quantities $\{Y, X, S, \Omega, L, C, N\}$ that solve the final good firms' problem, the intangible firm problem, the innovation sectors' problem, and the household problem, as well as clearing the market for assets, intangible capital goods, labour, and output:

(i) $v, \tilde{v}, v^A, v^{NA}$ satisfies the final good firms' problem in Eq.(6)-Eq.(9) and $(Y, L, X, \underline{\xi})$ are the associated policy functions.

(ii) f satisfies the intangible good firm's problem in Eq.(18) and X is the associated policy function.

(iii) S is the innovation sector's policy function which solves Eq.(21).

(iv) H satisfies the household's problem in Eq.(24) and (Ω, C, L^h) are the associated policy functions.

(v) The asset market clears:

$$\Lambda(z_m, \mathbf{x}; N, \sigma, \mu) = \mu'(z_m, \mathbf{x}; N, \sigma, \mu) \quad \forall (z_m, \mathbf{x}) \in \mathbb{S} \quad \text{and} \int_{\mathbb{S}} \mu'(z_m, \mathbf{x}; N, \sigma, \mu) = 1. \tag{29}$$

(vi) The labour market clears:

$$L^h(N, \sigma, \mu) = \int_{\mathbb{S}} L(z, \mathbf{x}; N, \sigma, \mu)\mu(d[z \times \mathbf{x}]). \tag{30}$$

(vii) The intangible capital market clears[13]:

$$M'(N, \sigma, \mu) = N'X(N, \sigma, \mu) = X(z, \mathbf{x}, \xi; N, \sigma, \mu). \tag{31}$$

[13]The simplification of the intangible capital markets uses the symmetric equilibrium solutions of the intangible good sector.

(viii) The output market clears:

$$Y(N,\sigma,\mu) = C(\lambda;N,\sigma,\mu) + N'X(N,\sigma,\mu) + S(N,\sigma,\mu) + \Xi(N,\sigma,\mu). \tag{32}$$

(ix) The distributional state $\mu(z,\mathbf{x})$ evolves according to the law of motion Γ_μ

$$\mu'(z',\mathbf{x}') = \Gamma_\mu(\mu(z,\mathbf{x}),N,\sigma,\mu), \tag{33}$$

where the decision rule $X(z,\mathbf{x},\xi;N,\sigma,\mu)$ together with the exogenous stochastic process for σ and the endogenous process for N, are consistent with the transitional rule Γ_μ.

3.8 Decision Rules

Using C and L to describe the market-clearing value of the household consumption and labour which satisfy the equilibrium conditions, it can be shown that market-clearing requires the household first-order conditions to be equal to:

$$w(N,\sigma,\mu) = \frac{U_L(C,L;N,\sigma,\mu)}{U_C(C,L;N,\sigma,\mu)}, \tag{34}$$

$$\Omega(N,\sigma,\mu) = \frac{\beta \mathbb{E} U_C(C',L';N',\sigma',\mu')}{U_C(C,L;N,\sigma,\mu)}. \tag{35}$$

It is now possible to simplify the model and solve it by directly substituting the equilibrium conditions for the household maximisation problem into the final good firms' problem. By representing $(V,\tilde{V},V^{NA},V^A)$ as the final good firm value functions expressed in terms of marginal utility of consumption units, the final good firms' problem can be stated as:

$$y(z,l,x;N) = z\left(l^{(1-\zeta)}N^{\frac{\zeta}{\nu}}x^{\zeta}\right)^{\gamma} \quad \text{where} \quad x = \mathbf{x}, \tag{36}$$

$$\begin{aligned} V(z,x;N,\sigma,\mu) &= \mathbb{E}_\xi \tilde{V}(z,x,\xi;N,\sigma,\mu), \\ &= \int_0^{\bar{\xi}} \tilde{V}(z,x',\xi;N,\sigma,\mu)\ G(d\xi), \end{aligned} \tag{37}$$

$$\tilde{V}(z,x,\xi;N,\sigma,\mu) = \max\Big\{-\xi y(z,l,x;N)p(N,\sigma,\mu) + V^A(z,x;N,\sigma,\mu)\ , \\ V^{NA}(z,x;N,\sigma,\mu)\Big\}\ , \tag{38}$$

$$\begin{aligned} V^A(z,x;N,\sigma,\mu) = \max_{x',l}\ & p(N,\sigma,\mu)\left\{y(z,l,x;N) - p^x[N'x' + (1-\delta)Nx]\right. \\ & \left. -\omega N'\mathbb{I}(i^x<0) - w(N,\sigma,\mu)l\right\} \\ & + \beta\mathbb{E}V(z',x';N',\sigma',\mu'|z,N,\sigma,\mu), \end{aligned} \tag{39}$$

$$\begin{aligned} V^{NA}(z,x;N,\sigma,\mu) = \max_{l}\ & p(N,\sigma,\mu)\left\{y(z,l,x;N) - w(N,\sigma,\mu)l\right\} \\ & + \beta\mathbb{E}V(z',(1-\delta)x;N',\sigma',\mu'|z,N,\sigma,\mu). \end{aligned} \tag{40}$$

Note that the expressions have been simplified by substituting for $\mathbf{x} = x$ given the result of the symmetric equilibrium in the intangible good sector.

Moreover, it is possible to simplify the intangible good firm's problem and the innovator sector's problem similarly, by substituting for the equilibrium conditions for the household maximisation problem. Let's denote F as the value of a patented good in terms of marginal utility consumption units, then:

$$F(N,\sigma,\mu) = p(N,\sigma,\mu)\Pi(N,\sigma,\mu) + (1-\phi)\beta\mathbb{E}F(N',\sigma',\mu'|N,\sigma,\mu), \tag{41}$$

$$S(N,\sigma,\mu) = \frac{\theta(N,\sigma,\mu)\beta\mathbb{E}F(N',\sigma',\mu'|N,\sigma,\mu)}{p(N,\sigma,\mu)}, \tag{42}$$

$$\theta = \frac{p(N,\sigma,\mu)}{\beta\mathbb{E}F(N',\sigma',\mu'|N,\sigma,\mu)}. \tag{43}$$

This concludes the description of the model. I turn to the theoretical analysis of the endogenous growth mechanism and description of the Balanced Growth Path (BGP from hereon-in).

3.9 Endogenous Growth and Balanced Growth Path (BGP)

Unlike real business cycle frameworks, the model developed in this paper exhibits an endogenous growth process driven by the accumulation of patents. Unfortunately, due to the lack of an analytic solution to the final good firm's problem, it is not possible to write an equation for the endogenous productivity process of the economy. However, it is possible to show analytically how the growth rate of the total stock of intangible capital is ultimately a function of the value of the exclusive rights to patents.

Let's derive an expression for the growth rate of the total stock of intangible capital N. By taking the law of motion of intangible capital stock in Equation 19 and substituting for the productivity of R&D in Equation 20, the following equation is obtained:

$$N'(N,\sigma,\mu) = \chi N^{1-\eta} S(N,\sigma,\mu)^{\eta} + (1-\phi)N. \tag{44}$$

Using the above equation, I substitute out the variable for the R&D investment using the first-order condition of the innovation firm in Equation 42, to get the growth rate equation. Denoting $\Delta N'(N,\sigma,\mu) = 1 + g'(N,\sigma,\mu)$ as the gross growth rate of the total stock of intangible capital and dividing the equation above by N yields:

$$g'(N,\sigma,\mu) = -\phi + \left[\chi^{\frac{1}{\eta}} \frac{\beta \mathbb{E} F(N',\sigma',\mu'|N,\sigma,\mu)}{p(N,\sigma,\mu)}\right]^{\frac{\eta}{1-\eta}}. \tag{45}$$

Ultimately, the growth rate of the intangible capital, and therefore of the economy, arises at any given time from two possible channels: (i) the stochastic discount factor and (ii) the demand for intangible capital. On the one hand, any increases in the stochastic factor will increase the growth rate as a lower consumption growth will drive up savings and investment. On the other hand, an increase in the demand for intangible capital will boost the value of patents, which will increase the economy's growth rate. Specifically, more valuable patents imply higher monopoly profits which induce the innovation sector to increase R&D expenditure, thus increasing the stock of intangible capital at a faster rate. Interestingly, since the monopoly profits depend on the demand for intangible capital, an uncertainty shock may reduce the economy's growth rate endogenously through the higher real option of deferring intangible capital investment.

The Balanced Growth Path (BGP from hereon-in) is characterised by the constant growth rate in the variables of the model. In the BGP, the model has a constant level of patented goods X, but the stock of patents available N grows at the economy's growth rate g. In the model, Equation 45 disciplines the growth rate of the economy and illustrates that for a BGP to exist, I necessitate two conditions: a constant stochastic discount factor given by a constant interest rate, and a constant value of patents. Firstly, since in the BGP all variables grow at a constant rate so too will consumption, thus resulting in a constant ratio of marginal utilities which by the household's first-order conditions will mean a constant stochastic discount factor. Secondly, homogeneity of degree one in the accumulating factor of the production function (N) is sufficient for a constant value of patents, which the following condition must hold $\frac{\zeta\gamma}{\nu} = 1$.

4 Model Solution

The solution of the model poses some major obstacles to overcome and in doing so, I present a novel approach to solve models featuring heterogeneous agents, aggregate uncertainty, and a system of non-linear first-order difference equations which describes the model's endogenous growth mechanism. Interestingly, the same approach could be used to solve any model with both heterogeneous agents and a system of non-linear first-order difference equations. Such a solution method would be adopted to investigate questions so far inaccessible to researchers as they are too complex to solve computationally.

The first obstacle arises from the introduction of heterogeneous final good firms under aggregate uncertainty in a general equilibrium setting. This feature of the model means that the solution of the model relies on a variant of the Krusell and Smith (1998) algorithm. Specifically, I make use of a similar algorithm to Young (2010), which increases the accuracy and efficiency of the solution by using a histogram approach to the distribution of firms over productivity and capital, instead of using firm-level simulations which induce sampling errors. Indeed, firm heterogeneity means that prices in the model are not only a function of the aggregate states but also a function of the endogenous high-dimensional distributional state $\mu(z, x)$. Since agents in the model take expectations with respect to tomorrow's prices, the solution needs to estimate the transitional rule Γ_μ which governs the evolution of the joint distribution of firms over productivity and intangible capital holdings. The details on how the model is solved are deferred to Appendix C.5. Table 4 displays the solution to the forecasting rules needed to estimate the law of motion of Γ_μ.

The second, and most important, obstacle in this model when computing the simulations, is that not only does one have to solve the firms' value function optimisation problem as in standard models featuring heterogeneous firms, but also the system of non-linear first-order difference equations which encompasses the endogenous growth mechanism in the model. Indeed, after the firms' optimal policy functions are computed, to generate the growth cycle dynamics it is necessary to solve a system of non-linear first-order difference equations which comprise the set of equations relating to the intermediate good sector and the innovation sector. In representative agents models, these systems of equations are solved under Rational Expectations, however, since the final good firms' problem cannot be solved under Rational Expectations as explained above, I will use the same Bounded Rationality approach to solve the system of non-linear equations. This means that the forecast law of motion for aggregate intangible capital X is used to compute the value for the next period's value of patents (F'). Further details of the model's simulation are outlined in Appendix C.5.

Finally, since the model's endogenous mechanism for growth renders the model non-stationary, for the model to be solved, I need to stationarise the model equations by

Table 4. Forecasting Rules for Model Solution under Aggregate Uncertainty

Intangible Capital Forecasting Law of Motion				
	α	β	R^2	S.E
$X(\sigma^L, \sigma^L)$	-0.135	0.949	0.999	0.006
$X(\sigma^L, \sigma^H)$	-0.252	0.906	0.999	0.002
$X(\sigma^H, \sigma^L)$	-0.255	0.904	0.999	0.001
$X(\sigma^H, \sigma^H)$	-0.183	0.931	0.999	0.005
Price Forecasting Law of Motion				
	α	β	R^2	S.E
$p(\sigma^L, \sigma^L)$	0.641	-0.681	0.999	0.004
$p(\sigma^L, \sigma^H)$	0.833	-0.609	0.999	0.001
$p(\sigma^H, \sigma^L)$	0.936	-0.569	0.999	0.001
$p(\sigma^H, \sigma^H)$	0.665	-0.670	0.999	0.003

Notes: *This table presents the forecasting rules obtained when solving the baseline model. Each forecasting rule takes the following form:* $log(\hat{m}) = \alpha^m_{i,j} + \beta^m_{i,j} log(\hat{X}_t)$ *where* $i, j = \{\sigma_L, \sigma_H\}$ *and* $\hat{m} = \{X_{t+1}, p_t\}$. *The* α *refers to the constant of each regression, whilst the* β *refers to the coefficient with respect to the independent variable. The* R^2 *is the coefficient of determination and refers to the accuracy of the regression and S.E. refers to the standard error of the* β *coefficient to gauge the precision of the estimation. The Den Haan (2010) check results are: X max error = 0.010; X average error = 0.002; p max error = 0.008; p average error = 0.001.*

removing trend growth. In Appendix C.2, I stationarise the model around a BGP and in Appendix C.3, I derive the non-stochastic BGP equations.

5 Calibration

The model is calibrated to the U.S. economy for the pre-2008 Great Recession period at a quarterly frequency. The parameters are calibrated by matching moments generated by the model in a state of non-stochastic BGP assuming that the aggregate state of uncertainty is in the low state (see Appendix C.3 for further details). There are three sets of parameters: *i)* the standard parameters of an endogenous growth model; *ii)* the parameters relating to the heterogeneous firms model with non-convex adjustment costs; *iii)* and finally, the parameters governing the uncertainty state stochastic process of uncertainty.

5.1 Standard Endogenous Growth Model Parameters

The selection for the parameters relating to the endogenous growth model is illustrated in Table 5. Starting with the household sector, the discount factor β is set to match an annual interest rate of 5%, as obtained from the Penn World Table. Whilst the parameter pertaining to the relative disutility of labour φ is chosen to achieve a 64% labour share of output, calculated from the Penn World Table[14] As for the final good firms, the parameter for elasticity of intangible capital to output ζ is set to achieve an overall intangible capital good elasticity of output to half in Kung and Schimd (2015)[15]. The parameter relating to the intermediate good firms, that is, the inverse mark-up of patents parameter ν is picked to ensure the BGP condition holds[16]. Finally, the innovation sector I calibrate the scale parameter χ to match a 1.4% Balanced Growth Path growth, as calculated from the World Bank national accounts data. The elasticity of new patent creation with respect to R&D η is set in accordance to estimates by Kung and Schimd (2015), and the patents obsolescence rate ϕ matches the 15% annual rate used by the Bureau of Labor Statistics.

5.2 Final Good Firm-Specific Parameters and Non-Convex Adjustment Costs

There are six parameters directly pertaining to the final good firm-specific parameter and non-convex intangible capital adjustment parameters: *i)* the persistence of the productivity process (ρ); *ii)* the standard deviation of the productivity process in the low uncertainty state (σ^L); *iii)* the span of control parameter (γ); *iv)* the intangible good depreciation rate (δ); *v)* the upper limit of the distribution of the stochastic fixed costs $(\bar{\xi})$; *vi)* the per-unit loss associated with intangible capital disinvestment (ω).

[14] The Penn World Table data I use is the share of labour compensation in GDP at current national prices for the United States.

[15] Note that the intangible capital goods elasticity of output is equal to $\zeta\gamma$ due to the decreasing return to scale of the production function.

[16] See Section 3.9 for more details.

Table 5. Growth Model Parameter Calibration

Parameter	*Value*	*Target*	*Source*
Households			
β	0.987	Annual interest rate of 5%	Penn World Table
φ	6.461	Labour share of output to 64%	Penn World Table
Final Good Firms			
ζ	0.587	Elasticity of x to y	Kung and Schimd (2015)
Intermediate Good Firms			
ν	0.500	Inverse mark-up	BGP condition
Innovation Sector			
χ	0.966	BGP Growth of 1.4%	World Bank
η	0.835	Elasticity of N' to S	Kung and Schimd (2015)
ϕ	0.036	Patent obsolescence rate	BLS

Notes: *The table shows the calibration of the standard parameters of the model with endogenous growth, where the parameters, its value, the description, and their respective targets and sources are presented in order.*

To calibrate these parameters, I use quarterly intangible capital investment data for publicly traded firms from COMPUSTAT, which I have already utilised in the empirical section 2[17]. Firstly, I select only the pre-Great Recession period, from 1990 to 2007, to build the data moments needed for calibration. Secondly, I clean and deflate the data as in Section 2. Thirdly, I construct the intangible capital investment rate for each firm i at period t accordingly:

$$I_{i,t}^{X} = \frac{intanq_{i,t} - (1-\delta^{BLS})intanq_{i,t-1}}{intanq_{i,t-1}}, \tag{46}$$

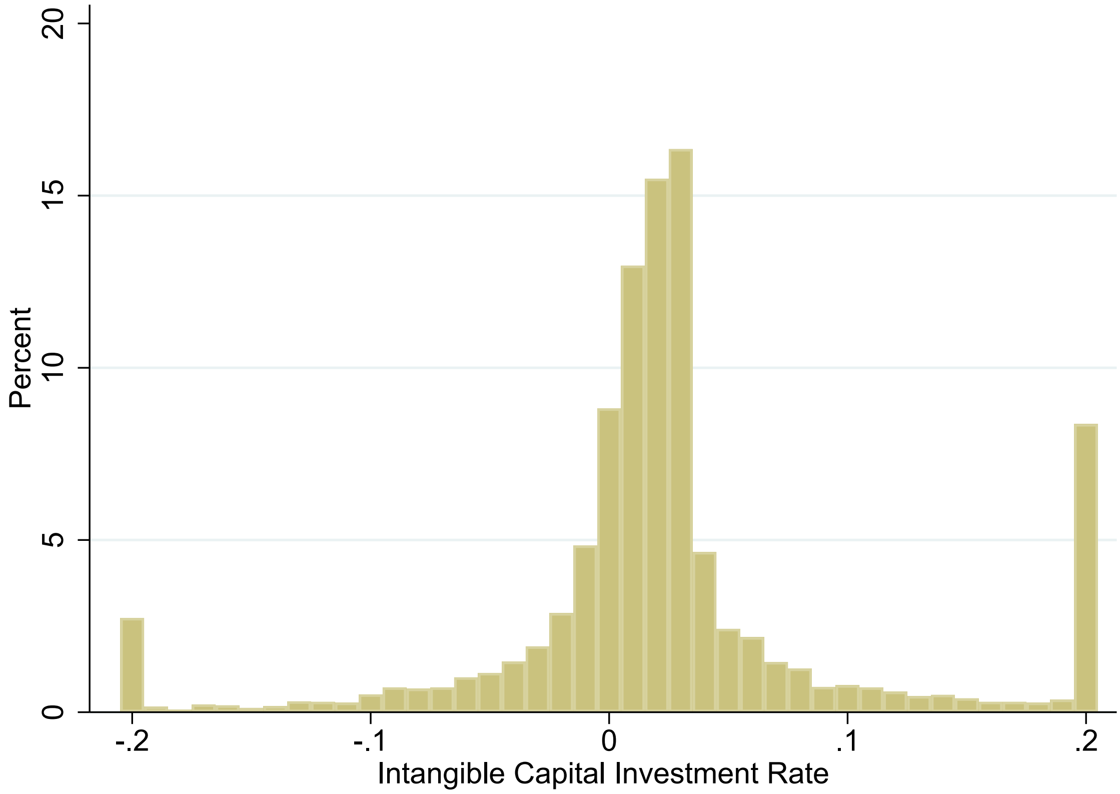
where the variable $intanq$ is the intangible capital held by the firm and δ^{BLS} is the obsolescence rate of intangible capital which is chosen following the BLS estimates provided[18].

The histogram of intangible capital investment rates resulting from the COMPUSTAT

[17] I use quarterly data for two reasons. Firstly, it matches the frequency of the model. Secondly, quarterly data better identifies the fraction of firms that are inactive with respect to intangible capital investment, as yearly data may be too coarse. Using quarterly data to construct the distribution for intangible capital investment rates also means recognising that the data is affected by the seasonality of such investment. The problem lies in the fact that intangible investment is subject to tax deductions, thus firms have incentives to move the accounting of their intangible capital investment towards the fourth fiscal quarter. However, to make sure that the seasonality does not affect key moments of the distribution, I also construct the distribution using yearly data and check that the moments are not altered.

[18] In the data there is no distinguishing between the depreciation of the intangible capital good (δ) and the obsolescence rate of the patent of that good (ϕ), as a result, I use the most commonly used rate for the calculation of the investment rate. Moreover, this also allows the model to replicate the average investment rate moments using δ.

Figure 2. Intangible Capital Investment Distribution



Source: *COMPUSTAT accessed via Wharton Research Data Services and author's calculations.*
Notes: *The figure shows the distribution of the firms' quarterly intangible capital investment rates from 1990 to 2007. The distribution is cut off at the specified intervals to better represent the density of the positive and negative spikes, defined as investment rates* $\pm 20\%$*. The density within investment rates between* -1% *and* 1% *represent the percentage of firms that are classified as inactive.*

dataset is reported in Figure 2. It is immediately noticeable that the distribution of investment rates for intangible capital is highly asymmetric, with a positive slowness and significant excess kurtosis. Indeed, the majority of firms exhibit positive investment rates of around 0% to 5%. Moreover, there is a considerable mass of firms around 0% and at the extremities, with a larger share of firms amassing above the 20% investment rate. The distribution of intangible capital investment is remarkably similar to the distribution of physical investment constructed by Cooper and Haltiwanger (2006) for the U.S. economy.

It is also possible to analyse the moments selected for calibration in Table 6 where the data moment is displayed in the second column, along with the standard errors. The moments confirm the intuition given by the distribution in Figure 2. The average intangible capital investment rate is 2.6%. The fraction of firms deemed to be inactive with

Table 6. Intangible Capital Investment Moments

Moment	*Data*	*Model*
Average investment rate	2.6% (0.011)	2.6 %
Inaction investment rate	17.4% (0.002)	15.6%
Positive investment rate	64.9% (0.003)	59.5%
Negative investment rate	17.8% (0.003)	20.1%
Spike: positive investment	8.3 % (0.001)	6.7%
Spike: negative investment	2.7 % (0.001)	2.1%

Source: *COMPUSTAT accessed via Wharton Research Data Services and author's calculations.*
Notes: *The moments are built using quarterly firm-level data on intangible capital investment rates. Firms with intangible capital investment rates between* -1% *and* 1% *are deemed as inactive. The positive (negative) spike moment is defined as firms with investment rates* $\pm 20\%$*. Such definitions are in line with work by Cooper and Haltiwanger (2006). The bootstrapped standard errors are presented in parentheses and are expressed in the same units as the moments.*

respect to the investment in intangible capital is 17%, where a firm is deemed inactive if it has an investment rate between -1% and 1%[19]. Furthermore, the data show that firms engage in sudden bursts of investments, with over 8% of firms featuring an investment rate of over 20% and nearly 3% having an investment rate below 20%. Given to the dimension of the panel dataset, the investment rate moments are estimated with a high degree of precision, as shown by the minuscule standard errors. However, this does not mean there is not plenty of heterogeneity in the firm-level dimension. Indeed, as already indicated by the distribution of investment rates in Figure 2, there is a wide dispersion of investment rates in the distribution given that the standard deviation of the average investment rate of intangible capital is 42%.

The selection of these moments for the calibration of the parameters that pertain to the intangible capital investment (ρ, σ_L, γ, δ, $\bar{\bar{\xi}}$, and ω) follows the tradition of lumpy investment literature[20]. The distinction is that whilst previous literature focuses on physical capital investment, this paper aims to calibrate the parameters to match key moments of the distribution of intangible capital investment rates. Specifically, the average investment rate informs the calibration of the patent's depreciation rate δ, since it governs the ratio of investment to intangible capital stock. The inaction investment rate is extremely useful in choosing the correct value of the fixed cost parameter $\bar{\bar{\xi}}$ which governs the firm's capital adjustment choice. The share of positive and negative investment rates disciplines the standard deviation in the low uncertainty state σ_L. Notably, as already highlighted by

[19]These discretionary bounds are set according to the lumpy investment literature (see Cooper and Haltiwanger (2006) and Khan and Thomas (2008)). The needs for such bounds arises from the fact that in such datasets it is impossible to encounter observation with exactly 0% investment rates, thus a proxy is necessary.

[20]See Cooper and Haltiwanger (2006) and Khan and Thomas (2008).

Table 7. Final Good Firm-Specific Parameters and Non-Convex Adjustment Costs Calibration

Parameter	*Value*	*Description*
ρ	0.950	Persistence of productivity process
σ_L	0.041	Standard deviation in low uncertainty state
δ	0.026	Intangible capital goods' depreciation rate
$\bar{\xi}$	0.005	Upper limit of the fixed cost distribution
ω	0.560	Partial irreversibility per unit loss
γ	0.762	Span of control parameter

Notes: *The table displays the values of the parameters calibrated using the Simulated Method of Moments (SMM) with the data moments displayed in Table 6. The only exception is the persistence of the productivity process, rho, which is chosen in accordance to previous literature on firm heterogeneous models (i.e. Khan and Thomas (2008) and Bloom (2009)) in absence of relevant data.*

Cooper and Haltiwanger (2006) and Khan and Thomas (2008), one cannot rely solely on the fixed cost parameter to match the entire moments of the investment rate distribution. The spikes in the investment rate distribution shape the size of the parameter for the unit loss for disinvestment ω and the span of control parameter γ. Unfortunately, the data available is not able to inform the persistence parameter of the productivity process, hence I follow the literature on heterogeneous firms to set ρ to 0.95, in accordance with Khan and Thomas (2008) and Bloom (2009).

I undertake the calibration using the Simulated Method of Moments (SMM) and the resulting values for the parameters are displayed in Table 7. The details of the method are laid out in Appendix B. Remarkably, the results of the calibration produce model moments very close to the data, as illustrated in Table 6. The model, although not explicitly built to replicate the distribution of investment rates in intangible capital, manages to capture the key features of such distribution. Notably, if one compares the results of the calibration to the literature on uncertainty and physical capital investment, I notice some similarities and some differences driven by the nature of intangible capital. Specifically, the variance of productivity shocks in the low state of uncertainty and the upper limit of the fixed cost distribution are quite similar to the estimates of Cooper and Haltiwanger (2006), which reflect the similarities in the dispersion and inaction of firms between intangible and tangible capital investment rates. The differences arise when calibrating the partial irreversibility parameter which is double the estimates of Bloom (2009), thus acknowledging the difficulty of reversing intangible capital investment, notoriously much harder than physical capital investment. Note that, unlike previous lumpy investment literature (see Khan and Thomas (2008)), the model does not require a region of investment

Table 8. Uncertainty Stochastic Process Parameter Calibration

Parameter	*Value*	*Description*	Source
σ_H/σ_L	1.370	Increase in σ in high uncertainty state	(I—B—E—S)
$\pi_{L,H}$	0.026	Prob. of high uncertainty shock	Bloom et al. (2018)
$\pi_{H,H}$	0.943	Persistence of high uncertainty state	Bloom et al. (2018)

Notes: The table displays the values of the parameters relating to the aggregate stochastic process of uncertainty. The parameters are set in accordance to Bloom et al. (2018), apart from the increase in σ in high uncertainty state, which is set according to the increase in the average firm-level standard deviation of EPS taken from the Institutional Brokers Estimate System (I—B—E—S) dataset.

for which the non-convex costs of adjustment are not enforced in order to capture both the inaction region and the spikes in investment rates. This is due to the inclusion of partial irreversibilities.

5.3 Uncertainty State Stochastic Process Parameters

In this section, I present the calibration of the uncertainty state stochastic process which consists of two sets of parameter: *i)* the probabilities that govern the stochastic process of uncertainty; and, *ii)* the increase of the second-moment of the fundamental's distribution in the high uncertainty state.

5.3.1 Probabilities of the Stochastic Process of Uncertainty

The uncertainty state of the economy is represented by a two-state Markov process whereby there is a low uncertainty state σ_L and a high uncertainty state σ_H. The underlying stochastic process can be described by the following transition matrix:

$$\begin{aligned} &\Gamma_\sigma(\sigma' = \sigma_i | \sigma = \sigma_q) = \\ &\qquad \begin{array}{lcc} \downarrow \sigma', \sigma' \rightarrow & \sigma_L & \sigma_H \\ \sigma_L & \multirow{2}{*}{$\left(\begin{array}{cc} 1-\pi_{L,H} & \pi_{L,H} \\ 1-\pi_{H,H} & \pi_{H,H} \end{array}\right)$.} \\ \sigma_H & \end{array} \end{aligned} \tag{47}$$

As a result of this setup, there are two probabilities to calibrate: *i)* the probability of a transitioning to a high uncertainty state ($\pi_{L,H}$); and, *ii)* the probability of remaining in the high uncertainty state ($\pi_{H,H}$). The calibrated parameters are express in Table 8.

I rely on estimates by Bloom et al. (2018) to inform $(\pi_{L,H})$ and $(\pi_{H,H})$. They have estimated the stochastic process of aggregate uncertainty for the U.S. economy from 1972 to 2010. Bloom et al. (2018) estimated the variance of both idiosyncratic and aggregate shock to productivity, as well as the transitional probabilities for such a process. They have assumed that the underlying stochastic process which governs the transition between low and high uncertainty states is the same for both aggregate and idiosyncratic uncertainty. As such, I can use their estimates to inform the probabilities in this model even if I do not model the differences between aggregate and idiosyncratic uncertainty. The Markov process that is estimated tells us that if the economy is in the low uncertainty state, there is a 2.6% probability of a high uncertainty shock and that the persistence of such a shock is 94.3%.

5.3.2 Increase in Uncertainty

To inform the increase of the second-moment of the distribution of fundamentals, I make use of Institutional Brokers Estimate System (I—B—E—S) data[21]. The I—B—E—S contains data about the earnings-per-share (EPS) of U.S. publicly traded companies, as well as the analysts' forecast of each firm's EPS. The data is cleaned to only include U.S. based companies and I only keep firms that report EPS data throughout the period 2005-2009. Like Bloom et al. (2018), I have chosen to calibrate the increase in uncertainty using data 2 years either side of the 2007 Great Recession. Usually, the literature relies on sales data or TFP data to calibrate the second-moment of the distribution of fundamentals. In contrast, I have chosen to use EPS data for the simple reason that in the Sections 6.3 and 6 I will use the I—B—E—S forecast data on firms' EPS to disentangle the increase in agents' forecast from the realised dispersion of the firms' EPS.

To calibrate the second-moment of the distribution of fundamentals, I use the I—B—E—S data to construct a measure of realised dispersion of firms' EPS. Notably, building the firm-level dispersion of the realised EPS poses a major difficulty in that one can only observe a single realisation of the EPS per firm at any given point in time, which means that calculating any dispersion measure is impossible. Since the I—B—E—S data reports at a quarterly frequency, to overcome this issue, I take the yearly dispersion of the firm's EPS over four quarters. As such, one can calculate the measure for realised dispersion at the firm level as

$$\sigma^r_{i,t} = \frac{D_{i,t}}{|E\bar{P}S_{i,t}|}, \tag{48}$$

where $D_{i,t}$ is the yearly standard deviation of the EPS for each firm $i \in [0, J]$, and the

[21] Accessible via Wharton Research Data Services.

measure is normalised by the median of the firm's EPS at the yearly frequency ($E\bar{P}S_{i,t}$). I then take the average firm-level dispersion of the EPS, $\sigma^r_{i,t}$, to build the aggregate measure $\sigma^r_t = \frac{1}{I}\sum_{i=1}^{I}\sigma^r_{i,t}$. The realised standard deviation of EPS displays an increase of 37% from 2007 to 2009[22]. I have used the EPS data from the I—B—E—S dataset for consistency. Since the I—B—E—S data also provides forecasts about the firms' EPS, later this will allow me to empirically quantify the difference in the increase in forecast EPS dispersion and the realised EPS dispersion.

Once I have calibrated the model to the U.S. economy pre-Great Recession, I move to the quantitative analysis of the model. Specifically, I investigate the growth effects of an uncertainty shock, to then delve deeper into the inspection of the mechanisms involved.

[22]The increase in the measure is robust in changing the start date to 2006 or 2005.

6 Quantitative Analysis

The quantitative analysis of the model is laid out in this section. Firstly, I will analyse the model's response to an uncertainty shock and elicit the effects on productivity dynamics. Next, I will make a clear distinction between the different channels at work when an uncertainty shock hits the economy. Specifically, I will differentiate between the *expectation* and *distributional* effects of an uncertainty shock. I will bring empirical evidence that questions the assumption that the dispersion of firms' expectations increases one-to-one with the realised dispersion of shocks during an uncertainty shock. Finally, I will model uncertainty by separately calibrating the dispersion of the firms' expectations and the dispersion of realised shocks, such that a shock to uncertainty can replicate the slow recovery and weak productivity growth experienced by the U.S. economy after the Great Recession.

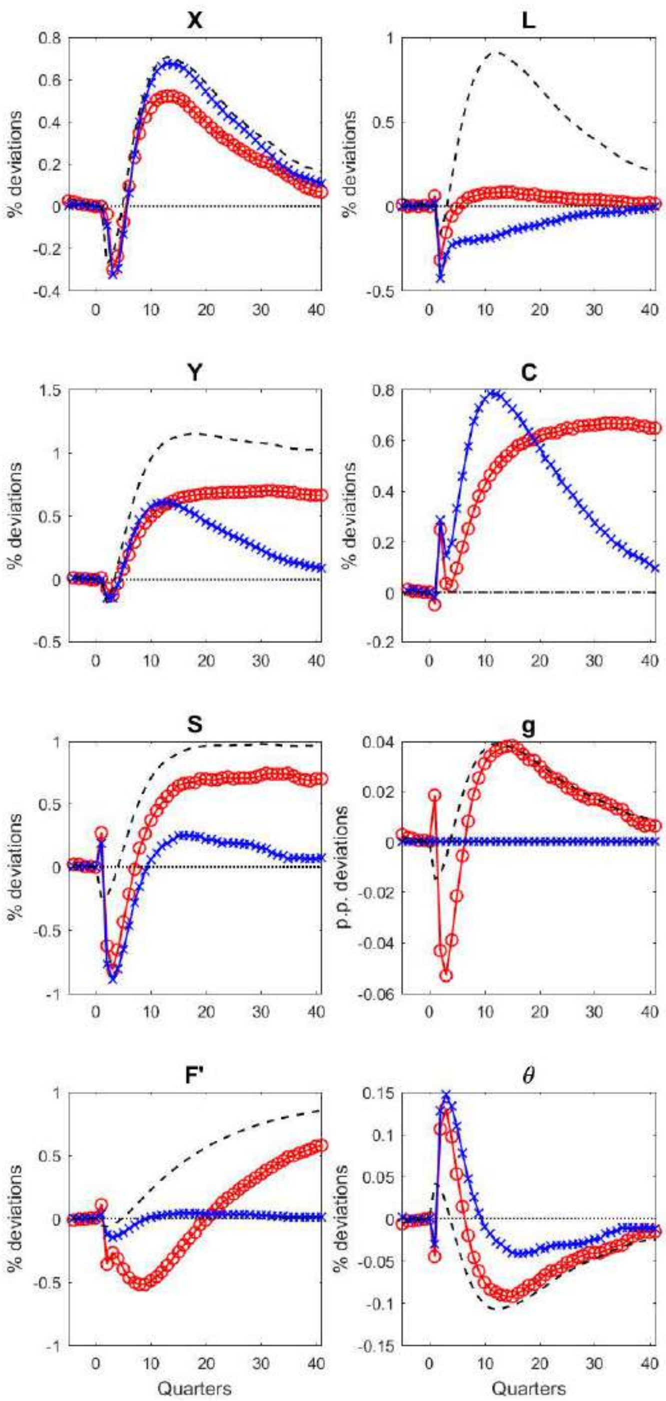
6.1 The Growth Effects of an Uncertainty Shock

I turn to the analysis of the model's response following a second-moment shock to the fundamentals. After having solved the model under aggregate uncertainty using a variant of the Krusell and Smith (1998) algorithm, where I leave the elucidation of the algorithm employed for the Appendix C.5, the model is subjected to a second-moment shock to the productivity process of the final good firms. As per the timing established in the model, the firm learns next period's productivity distribution one period in advance. The response of the economy is calculated as follows. I independently simulate 1000 economies for 200 quarters each, where for the first 100 quarters the exogenous processes of productivity and uncertainty evolve as indicated by their respective stochastic processes. Then, for all economies at the 101^{st} quarter, I shock the second-moment of the productivity distribution and impose it be in the higher level $\sigma = \sigma_H$, regardless of the histories. Following the homogeneous shock to the economies, each economy's stochastic process will evolve normally until the 200^{th} quarter. I compute the model's response to the shock as the average impulse response functions for each variable across the different economies[23]. More details of the conditional simulation can be found in Appendix C.5.4.

Figure 3 displays the model's response to an increase in the second-moment of the firms' fundamentals. Note that I have normalised the shock period to the first quarter. The red line with *o* symbols represents the model's response to the second-moment shock, the blue line with $\times$ symbols represents the response to the model without the endogenous growth

[23] Robustness of the results is carried out by increasing the number of economies and the total length of each economy, as well as varying the shock period. Safe to say that the results are robust to all these checks.

Figure 3. Growth Effects of an Uncertainty Shock



Notes: *The figure shows the response of the endogenous growth model to a second-moment shock to the fundamentals. The red line with o symbols displays the model's response with the endogenous growth mechanism, whilst the blue line with × symbols represents the model's response without endogenous growth, and the black dashed line shows the model's response in partial equilibrium (PE) setting. The horizon is in quarters. Note that all plots are in percentage deviations from the Balanced Growth Path, except the growth rate plot* (g) *and the inaction plot, which are in percentage points deviations.*

mechanism in place[24], and the black dashed line represents the model's response in a partial equilibrium setting. As previously mentioned, note that the Total Factor Productivity does not have a closed-form expression. However, following previous literature, I define it as $TFP = Y/\left(L^{(1-\zeta)}N^{\frac{\zeta}{\nu}}X^{\zeta}\right)^{\gamma}$, where Y, L, and X are the aggregates of output, labour, and intangible capital goods.

In Figure 3 it is immediately noticeable the similarity of the response of this model to exercises undertaken by Bloom (2009) concerning physical capital, most starkly, in the response of aggregate capital and output. The initial shock provokes a severe downturn in investment in intangible capital goods (I_x) of around 10% and a decrease in the intangible capital goods (X) by 0.35%, followed by a sharp upturn which *overshoots* above its initial BGP level before converging to the original level after 40 quarters. The reaction of final good firms to the heightened levels of uncertainty causes a sharp decline in intangible capital investment.

The endogenous growth mechanism in the model operates through two key channels: *i)* the stochastic discount factor channel; *ii)* and the intangible capital demand channel. I discuss the mechanism of each of these in turn.

***Stochastic Discount Factor Channel* —** The endogenous growth mechanism, as shown by the different responses of the two models in the first quarter, produces an increase in growth on impact due to the increase in consumption that generates a one-time increase in the stochastic discount factor. The increase in the stochastic discount factor is a by-product of the uncertainty shock in a general equilibrium setting. As already noted by Bloom et al. (2018), uncertainty generates misallocation of the factor of inputs as firms do not adjust their intangible capital optimally. This misallocation acts similarly to a negative first-moment shock to aggregate productivity. As the household observes this *pseudo*-first-moment shock, it lowers the expected return on savings, therefore discouraging saving and rendering consumption more attractive, at least in the first period. The consumption is also allowed to increase due to the timing of the uncertainty shock. At the period of the shock, fundamentals remain unaltered, so while intangible capital investment will be lower today, the direct effects will only show up next period. Hence, as output remains more or less unchanged, lower investment frees up resources for consumption.

By contrasting the general equilibrium response (the red line with *o* symbols) and the partial equilibrium response (the black dashed line) in Figure 3, then it is obvious that without the general equilibrium effects where, by definition, the stochastic discount factor channel is switched off as the household is not modelled, this one-time increase in growth disappears. This possibly unattractive feature of the model is given by the simplicity of

[24]Such a model is computed by setting the scaling parameter χ to zero, which effectively shuts off the mechanism for endogenous accumulation of intangible capital.

how the household is modelled. Without repetition, Bloom et al. (2018) provide some useful suggestions on possible remedies, all of which would require significantly increasing the computational complexity of the model.

Intangible Capital Demand Channel — Any temporary perturbations to intangible capital dynamics produce medium-term demand in the growth of productivity (g) and lasting effects on output (Y). Indeed, the initial fall in intangible capital investment causes a collapse in the present discounted value of patents after the initial impact, which discourages temporarily patent production and innovation for the first year. Figure 3 highlights a downturn in the number of patents produced (N) and a sharp decline in R&D (S), which are absent in the model without endogenous growth. This temporary contraction results in a fall in the growth rate of productivity and the economy of 0.05 percentage points. Nevertheless, the contraction in investment in the first few quarters is followed by an upsurge in the periods following . As demand increases when uncertainty subsides, the higher intangible capital goods cause a subsequent boom in the economy and the endogenous growth mechanism produces a long-lasting expansion which lasts for almost 10 years. This *overshoot* phenomenon of investment, thanks to the positive spillover effects of intangible capital, generates an above BGP growth of productivity for nearly 30 quarters, driven by higher intangible capital demand. This increases the discounted value of patents and results in higher $R\&D$ expenditures. Overall, the *overshoot* culminates with a permanent level increase in output and productivity of around 0.6%.

Overall, the model augmented with endogenous growth features a propagation mechanism which generates dynamics that would be absent in a model without growth. This provides evidence of the importance of investigating the drivers of intangible capital investment, which can generate dynamics in productivity growth and permanent change in output. Specifically, the uncertainty shock is propagated thanks to the endogenous growth mechanism which generates aggregate productivity dynamics that permanently affect the level of output and productivity by altering the returns to producing patents.

6.2 Disentangling the Effect of Uncertainty on Intangible Capital Investment

The initial collapse of intangible capital investment to a second-moment shock, which confirms the empirical findings in Section 2, seems to suggest that uncertainty causes a brief recession in the economy, but it is immediately followed by a lasting period of higher growth and increase productivity due to the *overshooting* of investment in intangible capital. This indicates an inconsistency between the model's response and the story of the U.S. economy following the Great Recession, which has experienced a prolonged period of feeble below trend productivity growth. To understand the discrepancy between the model and the data, one must look at disentangling the different effects of a second-moment shock to the fundamentals.

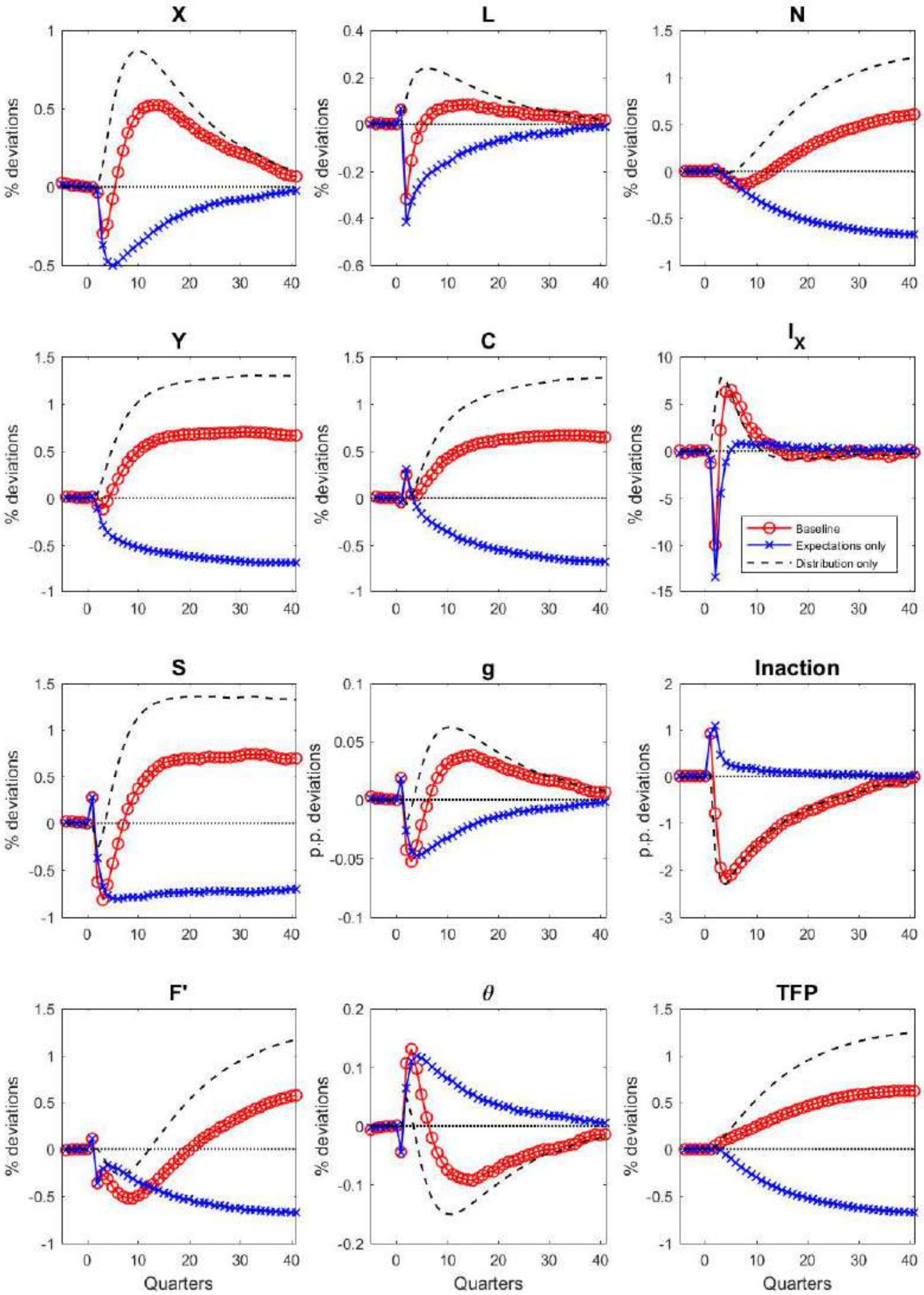
Whenever modelling uncertainty shocks, there are two vital but juxtaposed effects which generate the notorious J-shaped response of investment. An announced increase in the second-moment of the fundamental distribution has *expectations effects*, which means that firms' forecast a more dispersed distribution of future fundamentals shocks; and *distributional effects*, which affects the firm's investment decisions through the realisation of more dispersed shocks. This paper is not the first one to make the distinction between these effects, indeed Bloom et al. (2018) have called the former "pure uncertainty shocks" and the latter the "volatility effects". I will discuss each effect in the context of endogenous growth, unlike the previous literature, so as to further understand the growth effects of a second-moment shock.

To disentangle the different effects, that is, the *expectation* effects and the *distribution* effects, Figure 4 presents the contribution of each of the different channels to a second-moment shock to productivity. The red line with *o* symbols depicts the model's response to a second-moment shock as before, the blue line with $\times$ symbols plots the model's response only with the expectation effects, and the black dashed line illustrates the model's response only with the distributional effects. To disentangle the two effects from each other. I proceed as follows: to account only for the expectation effects of uncertainty, firms in the model, upon news of the future increase in their productivity's variance, will take expectations under the new high uncertainty state, however, the shock that realises will be taken from the low uncertainty state. Conversely, by assuming that firms' expectations remain unaltered, whilst the economy is hit by the shocks drawn from the high productivity state, it is possible to capture the distributional effects of uncertainty.

Expectation Effects — When firms forecast a higher dispersion of future productivity shocks, two types of expectation effects underline the response of the firms' investment decision. Firstly, and most notably, the *real option* channel generates a sharp fall in investment when higher forecast dispersion of shocks interacts with non-convex costs of capital adjustment. When firms expect a wider distribution of fundamentals, they will find it more beneficial to wait and see instead of investing or dis-investing. The option value of waiting is created by the interaction of the expected higher variance of shocks and the presence of non-convex costs of adjusting investment. Since it is costly to reverse any investment decision thanks to these costs, some firms will ride out the uncertainty by not adjusting their investments. Such inaction, however, causes investment in intangibles to fall.

A secondary but less powerful effect that arises when firms expect higher dispersion of

Figure 4. Disentangling Uncertainty Effects from the Distributional Effects



Notes: *The figure shows the decomposition of the response of the endogenous growth model to a second-moment shock to the fundamentals. The red line with o symbols displays the model's response with the endogenous growth mechanism, whilst the blue line with* $\times$ *symbols represents the baseline model's response only with expectation effects, and the black dashed line shows the model's response only with the distributional effects. The horizon is in quarters. Note that all plots are in percentage deviations from the Balanced Growth Path, except the growth rate plot* (g) *and the inaction plot, which are in percentage points deviations.*

shocks is the *Oi-Hartman-Abel* effect[25]. Seen as the optimal choice of intangible capital is a convex function with respect to productivity, an increase in the variance of the underlying stochastic process of the fundamentals may induce firms to expand. This effect only operates in the medium-term, because in the short-term the option value of waiting for the resolution of uncertainty dominates thanks to the non-convex costs of adjustment. Since such non-convex costs become less important in the medium-term, the increase in the variance of the expected productivity shocks produces an increase in the expected average of productivity, which induces an increase in investment.

In Figure 4, the blue line with $\times$ symbols shows that the *Oi-Hartman-Abel* effects are dominated by the *real option* channel, which means that overall the expectation effects produce a downturn in investment and output, leading to a prolonged recession and a drop in the level of output and productivity through the endogenous growth mechanism. Indeed, expectation effects alone through the *real option* channel produce and increase in the share of inactive firms of 1 percentage point. The rise of inaction causes an immediate fall in the intangible capital investment of 0.5%, leading to a fall in the growth rate of the economy of 0.05 percentage points. As investment slowly reverts to its trend, growth is depressed beyond the business cycle frequencies, and, eventually this leads to a 0.75% permanent loss of output and productivity.

***Distributional Effects* —** After the announcement one period ahead of a future increase in the variance of the stochastic process of productivity, the firms not only are affected by the expectation of this increase but are also directly affected by the distributional effects of the realised variance. Firstly, an increase in the realised variance of productivity means that after the announcement period, the firms will draw their productivity shocks from a distribution with a larger variance. The distributional effect of a larger variance causes an increase in investment and production, due to the convexity of the optimal investment function with respect to productivity. This means that the new wider distribution causes some firms to draw high productivity shocks and thus increase production and investment disproportionately more than the firms that draw low productivity shocks. Although similar to the aforementioned *Oi-Hartman-Abel* effects, the realised increased in variance directly induces higher levels of production and investment through the realisation of the shock, rather than the mere expectation.

A second reason, which has been less explored by the literature on uncertainty literature, is the *indirect realised first-moment shock* created by an increase in the second-moment of the productivity distribution. The *indirect realised first-moment shock* is a by-product of how uncertainty has been modelled thus far, that is an anticipated increase in the second-moment of the distribution of productivity shocks. More specifically, since

[25] See Oi (1961), Hartman (1972), and Abel (1983) for further clarification.

productivity in the model is distributed as a log-normal, any increase in the variance will be accompanied by an increase in the mean. That is, given that the productivity is distributed as follows $z \sim log\mathcal{N}(0, \sigma)$, then the mean of the productivity process is $\mu = exp\{0 + \frac{\sigma^2}{2}\}$, meaning that an increase in σ generates an increase in the realised mean of productivity shocks. Hence, if the uncertainty shock increases the realised variance of the distribution, not only may the firm directly increase its investment and output due to the aforementioned realised variance, but the firm may also increase its investment because it will be hit by a higher productivity shock on average.

In Figure 4, the black dashed line shows that these distributional effects have a positive impact on intangible investment from the second period onwards when the shocks begin to materialise. As firms are hit by shocks with higher variance, and to a smaller extent with a higher mean, firms that receive high idiosyncratic productivity shocks begin to increase their investment and their production. In contrast to the expectation effects, the distributional effects generate a fall in the share of inactive firms of more than 2 percentage points. In fact, without the change in expectations, the *real option* channel disappears. The distributional effects cause investment to increase by 7.5% and output by 1.25%. As investment begins to decline, the endogenous growth mechanism pushes the growth rate up by 0.06 percentage points. Overall, these effects lead to a permanent increase in output and productivity of 1.25%.

6.3 Firm-level Evidence of the Components of an Uncertainty Shock

As demonstrated by the exercise in Section 6.2, it is paramount to understand whether this uncertainty shock is driven by expectation or distributional effects to establish the permanent effects of uncertainty. That is, whether the effect of uncertainty derives from an increase in the expected variance of future shocks ($\mathbb{E}\sigma'$), or an increase in the realised variance of the shocks (σ'). Seen as these sources produce opposed responses to investment and therefore productivity dynamics, it is necessary to understand the relevant contribution of these effects in the data to understand the effects of uncertainty.

So far, when modelling uncertainty, I have assumed, along with the literature on uncertainty, that an announced increase in the variance of productivity is fully materialised, that is, $\mathbb{E}\sigma' = \sigma'$. This supposes that the forecast variance of the firms increases one-to-one with the variance of the realised shocks. To discern whether this happens in reality, I turn to the data.

The novelty of the Institutional Brokers Estimate System (I—B—E—S) dataset is that apart from containing data on the Earning-Per-Share of the publicly listed U.S. companies, it also contains point forecast of the EPS made by individual analysts. Forecast data on EPS provides a simple, yet intuitive, manner to measure the dispersion of the firm's expected distribution of fundamentals, and as such, I can identify it separately from the dispersion of realised shocks. Indeed, EPS forecasts yield *ex-ante* data on the agents' beliefs of the firm's fundamentals, whilst the realised EPS provides information on the actual fundamentals.

Consequently, using the same data as the one utilised for the calibration of the uncertainty shock, I build an aggregate measure of within-firm forecast dispersion in the following manner. I first calculate the quarterly dispersion measure

$$\sigma^f_{i,q} = \frac{D^f_{i,q}}{|\bar{EPS}^f_{i,q}|}, \tag{49}$$

where $D^f_{i,q}$ is the quarterly standard deviation of the EPS forecast made by the analysts for the firm $i \in [0, J]$ at the forecast period t, and it is normalised by the quarterly median forecast of Earning-Per-Share, $\bar{EPS}^f_{i,q}$. To be consistent with the earlier measure, I construct the measure at the yearly frequency by taking the yearly average across quarters and firms[26], and it can be expressed as

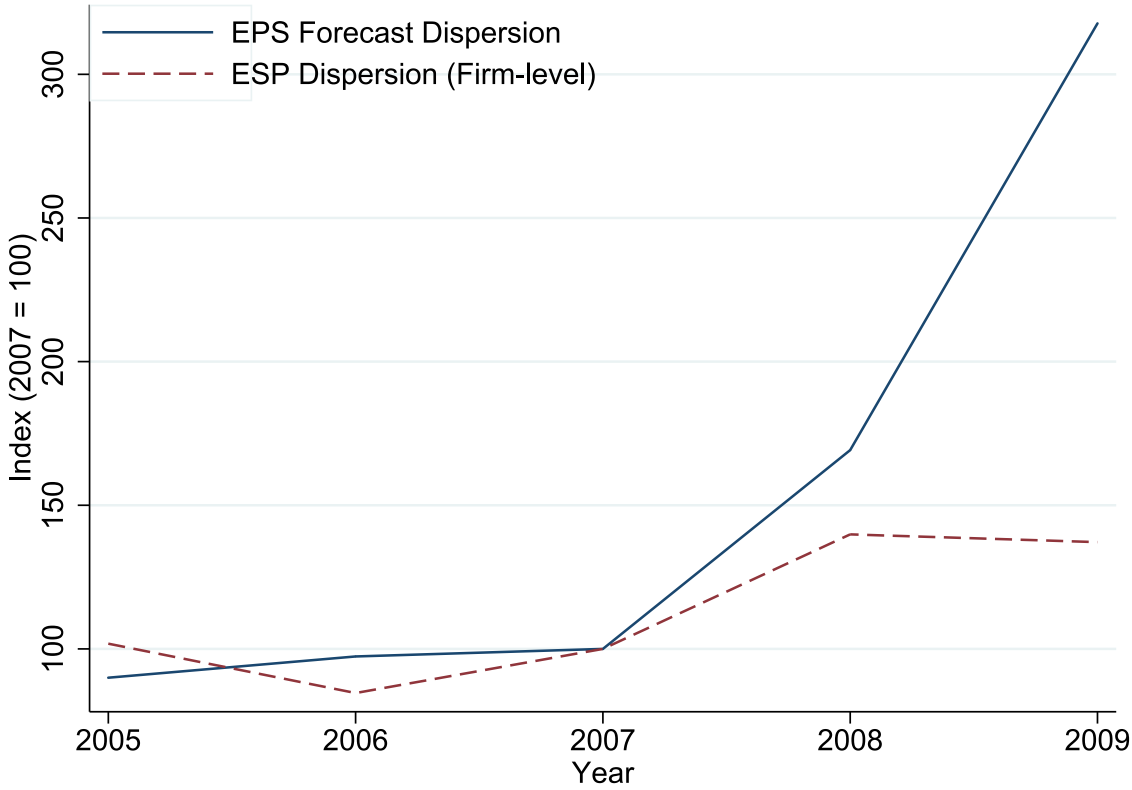
$$\sigma^f_t = \frac{1}{Q}\sum_{q=1}^{Q}\frac{1}{I}\sum_{i=1}^{I}\sigma^f_{i,q}, \tag{50}$$

where t is the yearly frequency.

Figure 5 illustrates the dispersion of forecasted σ^f_t and realised earnings-per-share (EPS) ratios σ^r_t. The solid blue line displays the EPS forecast dispersion measure (σ^f_t), whilst the red dashed line illustrates the realised EPS dispersion (σ^r_t). The measures are indexed at the pre-Great Recession levels with the base year 2007. As evidenced by Figure 5, during the Great Recession forecast dispersion has increased at a greater rate than the realised dispersion. If one compares the measures, the realised dispersion increased by just over a third, whereas forecast dispersion increase more than three-fold, precisely by 3.17%. The repercussions of these finding are vital to this paper and this line of research as a whole. It is shown that during the Great Recession, firms expected a higher dispersion of future shocks which only partially materialised. This evidence provides support to the thesis that during the Great Recession the *expectation effects* of uncertainty have been far greater than the *distributional effects*, and one should not assume that the forecast variance of the firms increases one-to-one with the variance of the realised shocks.

[26] Such adjustment also corrects for any seasonality within the year.

Figure 5. Evidence of Expectation and Distributional Effects



Source: *Institutional Brokers Estimate System (I—B—E—S) accessed via Wharton Research Data Services at* `https://wrds-www.wharton.upenn.edu/`.

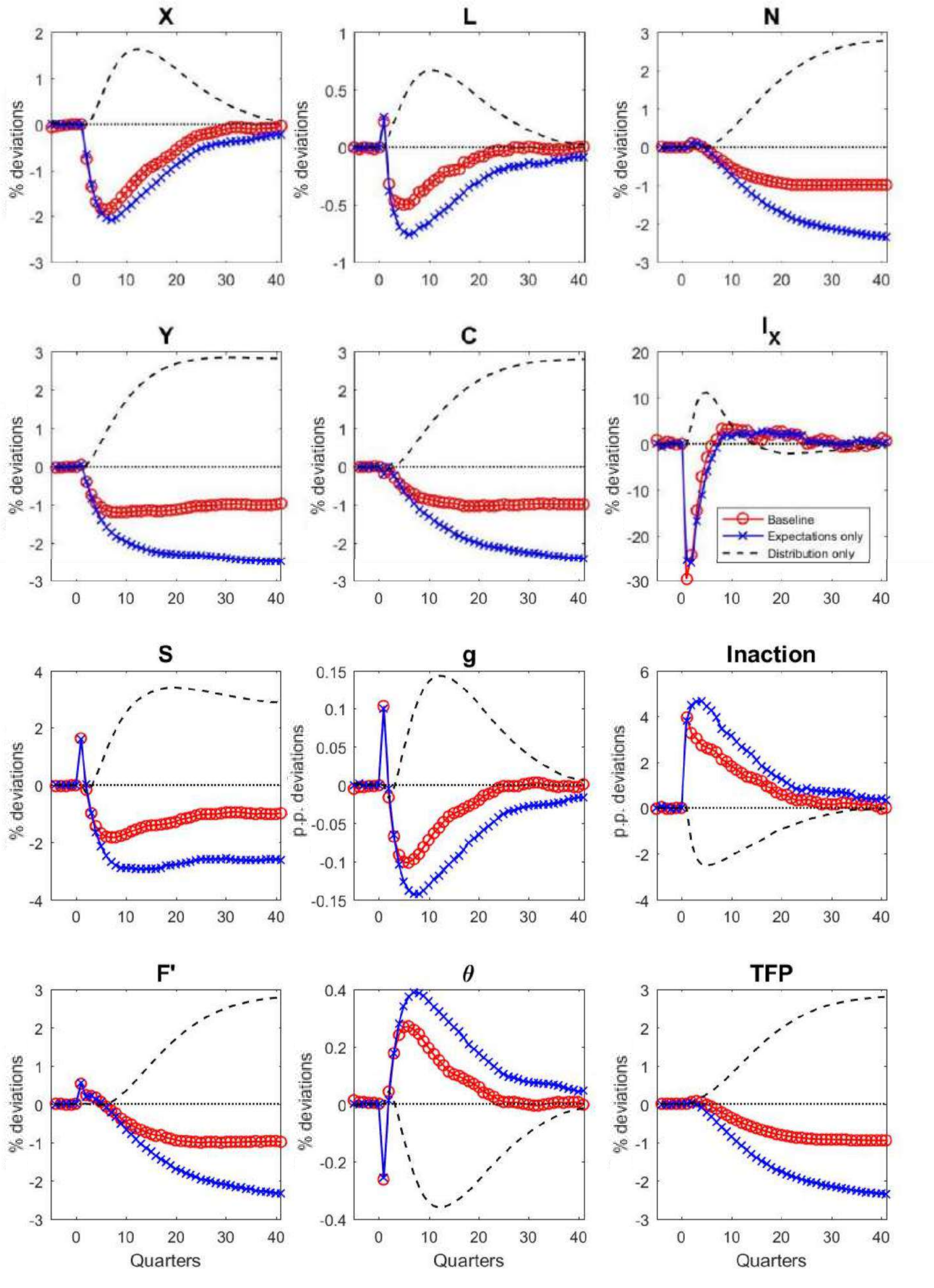
Notes: *The figure displays the within-firm dispersion of Earning-Per-Share* σ_t^r *and the dispersion of its forecast* σ_t^f. *The solid blue line displays the EPS forecast dispersion measure, whilst the red and dashed line illustrates the realised EPS dispersion. The measures have been indexed at pre-Great recession level using the year* 2007 *as the base year.*

6.4 The Scarring Effects of Uncertainty during the Great Recession

In light of this evidence, I update the assumptions in modelling uncertainty to calculate the scarring effects of uncertainty brought about by the Great Recession. As mentioned above, so far when modelling an uncertainty shock it has been assumed that the dispersion expected by firms is exactly equal to the realised dispersion. As highlighted by Section 6.3, this assumption is not borne out in the data. As a result, in this section, I present an alternative modelling of uncertainty, where I decouple the stochastic process of realised shocks (σ) and the firms' expectations of such shocks (σ^f).

To model such shock, I assume that in the low uncertainty state, the stochastic process of firms' fundamentals and the firms' expectations have the same variance, that is $\sigma_L^f =$

Figure 6. Uncertainty Shock during the Great Recession



Notes: *The figure shows the response of the model to an uncertainty shock during the Great Recession when calibrating separately the firms' expectation variance and the shock realisation variance. The red line with o symbols displays the baseline model's response with the endogenous growth mechanism. The blue line with × symbols represents the model's response only with the expectation effects, and the black dashed line shows the model's response only with the distributional effects. The horizon is in quarters. Note that all plots are in percentage deviations from the Balanced Growth Path, except the growth rate plot* (g) *and the inaction plot, which are in percentage points deviations.*

σ_L. However, in the high uncertainty this equality breaks down. Specifically, I assume that whilst the variance of the shock's stochastic process increases by a just over a third ($\sigma_H = 1.37 * \sigma_L$), the variance of firms' forecast of such shocks increases more than three-fold ($\sigma_H^f = 3.17 * \sigma_L$).

Figure 6 plots the model's response to an uncertainty shock when expected dispersion and the realised dispersion of fundamentals are decoupled. Immediately, one notices that the model can replicate the slow recovery in productivity growth experienced by the U.S. economy post-Great Recession. Indeed, as the uncertainty shock hits the economy, in the short-run the model behaves as before, that is, the *real option* effects generate an option value of waiting to invest for firms. The percentage of inactive firms, firms that freeze their investment decisions, spikes by 4 percentage points with respect to the pre-shock period, and only slowly converges back to its usual level after around 40 quarters. The higher investment inactivity by firms means that intangible capital investment falls by 30% causing lower growth of productivity, which is around half of the drop in intangible capital investment in the Great Recession. The lower intangible investment generates a quarterly loss of 11 percentage point in productivity growth, which amounts to 50 basis points per annum. Even if investment slight overshoots as the distributional effects are still present but are not as strong as before, there is a slow decline in output which culminates to a permanent fall of 1.0% produced by the endogenous growth mechanism. Similarly, whilst the growth rate of productivity reverts to its BGP levels, there is a permanent effect on the level of productivity of around 1.0%. The model simulations indicate that uncertainty has accounted for a fifth of the GDP lost and a quarter of the loss in TFP in the U.S. as a consequence of the Great Recession.

As before, general equilibrium effects, through the stochastic discount factor channel, cause a one-period increase in growth rates, which reverses as soon as the stock of intangible capital begins to decline. As fundamentals do not change when the uncertainty shock hits the economy, the stochastic discount factor increases because the marginal utility of consumption (p) needed to clear the final good market increases. Thereby increasing the present discounted value of patents for one-period, which fuels innovation and R&D spending. The slight increase in labour and output is again due to the lack of labour frictions, seen as the uncertainty shock lowers wages, so firms will increase labour demand and production in the first period.

Figure 6 also depicts the model's responses with just the *expectation* effects (the blue line with $\times$ symbols) and just the *distributional* effects (the black dashed line). What is evident is that, under this new specification of an uncertainty shock, where the forecast variance is decoupled from the variance of the realisation of the shock, the expectation effects through the *real option* channel dominates the model's overall response. Most strikingly, the difference is made by the share of firms that exhibit an inactive behaviour with respect to intangible capital investment.

Interestingly, by separately calibrating the increase in the firms' forecast variance about future shock and the realised dispersion of such shocks, the model can resolve the overshooting problem that uncertainty shocks can generate. The empirical uncertainty literature, most notably Jurado et al. (2015), has found empirical evidence of the output and other macroeconomic variable overshooting following an uncertainty shock to be lacking. This paper demonstrates that the divergence between the empirical data and the theory rests on the fact that, whilst in model forecast and realised variance increase one-to-one, in the data, forecast variance increases more than realised variance. Therefore by decoupling the two processes, I can replicate the empirical evidence of the literature.

7 Conclusion

During the last two decades, the U.S. economy has undergone a great transformation: the amount of net investment in intangible capital has overtaken the net investment in tangible (physical) capital. However, unlike tangible capital, investment in intangible capital is a key driver of productivity growth. As a result of this transformation and the nature of intangible capital, understanding the drivers of investment in intangibles can rationalise recent trends in productivity growth, such as the post-Great Recession slowdown in productivity growth. A vital determinant of investment is uncertainty, however, recent literature has focused on the effects of uncertainty on physical capital. The objective of this paper was to investigate the effects of uncertainty on productivity growth dynamics through intangible capital investment.

The paper generated two key insights. Firstly, I provided empirical evidence of uncertainty reducing investment in intangible capital. Using data on publicly traded firms, and exploiting the firm-level variations on investment, I establish a causal link between uncertainty and firms' investment decisions in intangible capital and R&D. I find that a standard deviation increasing the Jurado et al. (2015) uncertainty index causes a fall in intangible capital of 1.6% and a reduction in R&D expenditures of 0.9%.

Secondly, I developed a general equilibrium growth model with heterogeneous firms to understand the effects of uncertainty on productivity dynamics. Calibrating the model to the U.S. economy pre-Great Recession, and simulating an uncertainty shock, modelled as an increase in the dispersion of the final good firms' fundamentals, the model produces an immediate recession followed by a prolonged expansion. Investment falls due to the *real option* channel on impact, causing a recession and a downturn in productivity growth. However, after the initial fall, *distributional effects* produce a sustained recovery in intangible capital and therefore productivity growth. Although the model confirms the empirical findings which see uncertainty reducing investment in intangible capital at least in the short-term, it is not able to produce the feeble productivity growth experienced by the U.S. economy post-Great Recession.

Nevertheless, using forecast dispersion earnings-per-share data on publicly traded firms, the paper has questioned the validity of a key assumption when modelling uncertainty shocks. Specifically, when modelling uncertainty it is normally assumed that firms' expectations of the variance of the shocks increase on-to-one with the dispersion of realised shocks. However, I provide empirical evidence that during periods of high uncertainty, firms expect a higher dispersion of possible shocks in contrast to the shocks that are realised. When allowing firms' beliefs to differ from the realised shocks in the second-moment of the distribution of fundamentals, more precisely, when imposing that firms expect a greater variance of the distribution than the realised shocks, the model is also able to account for the weak productivity growth that has plagued the U.S. economy after the financial crisis of 2008. The model estimates that uncertainty during the Great Recession has slowed down productivity growth by 50 basis points per annum, causing a permanent loss of output and productivity equal to 1.0%. The result is driven by the fact that *expectation* effects, which drive the *real option* channel, are stronger when modelling firms' beliefs separately from the realised shocks to the fundamentals. The permanent loss of output generated by a temporary shock to uncertainty accounts for a fifth of the GDP permanently lost by the U.S. economy since the Great Recession; whilst the loss in productivity amounts to a quarter of the permanent TFP lost during the same period.

Finally, a vital contribution of this paper, aside from its stated objective, has been the ability to solve a model with both a non-trivial distribution of heterogeneous agents and a first-order difference system of equations. Such computational achievement can help researchers tackle economic questions that require models thus-far deemed too complex to solve computationally.

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Appendices

A Data and Empirics

The COMPUSTAT data requires some attention before the estimation. In this section, I describe the steps I have taken to clean and construct the data. First, I remove all firms which are not based in the U.S. by only keeping those firms that have "USA" as their ISO Country code of incorporation character. Subsequently, all duplicates are eliminated by dropping the observations that contain "PRE-FASB" in their company name. I also remove all the observations which have missing data for any of the variables I employ in the estimation, except for the firm's net value intangible capital since it is only reported from the year 2000 onward. Further, I drop the firms which do not report consecutive quarterly data. Consequently, I deflate the R&D expenses using the GDP implicit deflator. Since the model does not take into account the firms' dynamics - that is the entry and exit of firms in the market - to make the dataset and model as compatible as possible I drop the first and last observation from every firm in the dataset. Finally, another problem of data is that there are observations that show enormous increases and decreases in intangible investment of an order of ten times the stock of the firm's intangible capital stock. This problem arises due to firms merging or purchasing other firms and integrating the capital stock. Since the model is not capable of capturing these dynamics, I drop these data points which are observed in the top and bottom 1% of the dataset.

I proceed in constructing three extra variables necessary for the empirical strategy: market capitalisation is built by multiplying the firm's total outstanding common shares with the price of the firm's stock at quarterly close; sales growth is the difference in logs between the firm's total sales at period t and period $t-1$; research intensity is expressed as the firm's research expenses over total assets.

B Method of Simulated Moments

The Method of Simulated Moments (MSM) as introduced by McFadden (1989) obtains the parameters which minimise the sum of squared residuals between the data moments and the model's moments, which can be represented as:

$$\Theta = \arg\min_{\Theta} \mathbf{d}(\Theta)' W \mathbf{d}(\Theta), \tag{51}$$

where Θ is a $N \times 1$ vector of parameters, $\mathbf{d}(\Theta)$ is a $M \times 1$ vector of residuals, and W is a $M \times M$ weighting matrix. There is requirement that there are as many parameters (N) and moments (M), that is $N \geq M$. In the case that $N = M$ then the model is just-identified, whereas if $N > M$ the model is over-identified. Note that setting W as a diagonal matrix with the bootstrapped standard deviation of each moment in the data ensures that when solving Equation 51 minimises the distance between data moments and the model's moments in the most efficient manner.

In order to solve the MSM I rely on the root-finding method of Nelder and Mead (1965). Since I use a local root-finding method, I conduct robust checks by altering both the initial starting values and the step factor and I find that results do not change.

C Model and Computational Strategy

C.1 State Space Discretization

The model contains a total of three states to discretise: idiosyncratic productivity (z), idiosyncratic capital good (x), uncertainty states (σ). The discretization of the three states is as follows:

- The idiosyncratic productivity (z) is discretised into a grid $\mathbf{z} \in \{\overline{z}_1, ..., \overline{z}_{N_z}\}$ containing of $N_z = 5$ log-linearly spaced points.
- The idiosyncratic intangible capital good (x) is discretized into a grid $\mathbf{x} \in \{\overline{x}_1, ..., \overline{x}_{N_x}\}$ containing of $N_x = 25$ log-linearly points between $1 \times e^{-5}$ and $1 \times e^{1}$.
- The aggregate states (σ) are four ($N_\sigma = 2$): the low uncertainty state (σ_L) and the high uncertainty state (σ_H). These aggregate states can be represented into the following grid $\boldsymbol{\sigma} \in \{\overline{\sigma_L}, \overline{\sigma_H}\}$.
- The stochastic process of the aggregate states can be represented by the transition matrix Γ_σ of size $N_\sigma \times N_\sigma$ where $\sum_{l=1}^{N_\sigma} \pi^\sigma j, l = 1$ for all $j \in \{1, ..., N_\sigma\}$. The transition matrix probabilities are displayed in Equation 47.

Overall, the state space used for the numerical method used for the computational purposes of the model is $N_z \times N_x \times N_\sigma \times N_\sigma$, or more specifically $5 \times 25 \times 2 \times 2$.

C.2 Stationary Model

Given that the model features an endogenous growth rate g, in order for the model to be solved, one needs to stationarise the model by removing the trend. For the economy to exhibit a constant growth rate, the model is assumed to have homogeneity of degree one in the production function with regards to the accumulating factors of production and a constant interest rate. In this case, then all accumulating variables will growth at the same rate g. Here I present the stationarised model equations.

Household

$$\hat{p}(N,\sigma,\mu) = \frac{1}{\hat{C}(N,\sigma,\mu)}, \tag{52}$$

$$\hat{w}(N,\sigma,\mu) = \frac{\varphi}{\hat{p}(N,\sigma,\mu)}, \tag{53}$$

$$\hat{\Omega}(N,\sigma,\mu) = \frac{\beta}{1+g'(N,\sigma,\mu)}. \tag{54}$$

Final Good Firm

$$\hat{y}(z,l,x;N) = z\left(l^{(1-\zeta)}x^{\zeta}\right)^{\gamma}, \tag{55}$$

$$\hat{V}(z,x;N,\sigma,\mu) = \int_0^{\bar{\xi}} \hat{\hat{V}}(z,x,\xi;N,\sigma,\mu)G(d\xi), \tag{56}$$

$$\begin{aligned}\hat{\hat{V}}(z,x;N,\sigma,\mu) = \max\Big\{&-\xi\hat{y}(z,l,x;N,\sigma,\mu) + \hat{V}^{A}(z,x;N,\sigma,\mu),\\ &\hat{V}^{NA}(z,x;N,\sigma,\mu)\Big\},\end{aligned} \tag{57}$$

$$\begin{aligned}\hat{V}^{A}(z,x;N,\sigma,\mu) = \max_{x',l}\ &\hat{p}(N,\sigma,\mu)\left\{\hat{y}(z,l,x;N,\sigma,\mu) - p_x[(1+g')x' + (1-\delta)x]\right.\\ &\left.-\hat{w}(N,\sigma,\mu)l - \omega\mathbb{I}(i^x<0)\right\}\\ &+\beta\mathbb{E}\hat{V}(z',x';N',\sigma',\mu'|z,N,\sigma,\mu),\end{aligned} \tag{58}$$

$$\begin{aligned}\hat{V}^{NA}(z,x;N,\sigma,\mu) = \max_{l}\ &\hat{p}(N,\sigma,\mu)\left\{\hat{y}(z,l,x;N,\sigma,\mu) - \hat{w}(N,\sigma,\mu)l\right\}\\ &+\beta\mathbb{E}\hat{V}(z',x(1-\delta);N',\sigma',\mu'|z,N,\sigma,\mu),\end{aligned} \tag{59}$$

$$\hat{\underline{\xi}}(z,x;N,\sigma,\mu) = \frac{\hat{V}^{A}(z,x;N,\sigma,\mu) - \hat{V}^{NA}(z,x;N,\sigma,\mu)}{\hat{y}(z,l,x;N)}. \tag{60}$$

Intangible Good Sector

$$p_x = \frac{1}{\nu}, \tag{61}$$

$$\Pi(N, \sigma, \mu) = (p_x - 1)X(N, \sigma, \mu), \tag{62}$$

$$\hat{F}(N, \sigma, \mu) = \hat{p}(N, \sigma, \mu)\Pi(N, \sigma, \mu) + (1-\phi)\beta\mathbb{E}\frac{\hat{F}(N', \sigma', \mu' | N, \sigma, \mu)}{1 + g'(N, \sigma, \mu)}. \tag{63}$$

Innovation Sector

$$\hat{\theta}(N, \sigma, \mu) = \chi \hat{S}(N, \sigma, \mu)^{(\eta - 1)}, \tag{64}$$

$$\hat{\theta}(N, \sigma, \mu) = \frac{\hat{p}(N, \sigma, \mu)(1 + g'(N, \sigma, \mu))}{\beta \mathbb{E}\hat{F}(N', \sigma', \mu')}, \tag{65}$$

$$[1 + g'(N, \sigma, \mu)] = \hat{\theta}(N, \sigma, \mu)\hat{S}(N, \sigma, \mu) + (1 - \phi). \tag{66}$$

Market Clearing Conditions

$$\hat{Y}(N, \sigma, \mu) = \hat{C}(N, \sigma, \mu) + (1+g)X(N, \sigma, \mu) + \hat{S}(N, \sigma, \mu) + \hat{\Xi}(N, \sigma, \mu) \tag{67}$$

$$\hat{\Lambda}(z_m, x, \mu; N, \sigma, \mu) = \mu'(z_m, x) \quad \forall (z_m, x) \in \mathbb{S} \quad \text{and} \int_{\mathbb{S}} \mu'(z_m, x) = 1, \tag{68}$$

$$\hat{L^h}(N, \sigma, \mu) = \int_{\mathbb{S}} L(z, x; N, \sigma, \mu)\mu(d[z \times \tilde{x}]), \tag{69}$$

$$X(N, \sigma, \mu) = \int_{\mathbb{S}} \int_0^{\bar{\xi}} X(z, x, \xi; N, \sigma, \mu)G(d\xi)d([z \times x]), \tag{70}$$

$$\hat{Y}(N, \sigma, \mu) = \int_{\mathbb{S}} y(z, l(z, x; N, \sigma, \mu), x; N)\mu(d[z \times x]), \tag{71}$$

$$\hat{\Xi}(N, \sigma, \mu) = \int_0^{\bar{\xi}} \left[(1+g)\hat{y}(z, l, x; N)\mathbb{I}(i^x \neq 0) + \omega\mathbb{I}(i^x < 0)\right] G(d\xi)\mu(d[z \times x]). \tag{72}$$

C.3 Non-Stochastic Balanced Growth Path Equations

To calculate the Non-Stochastic Balanced Growth Path (NSBGP), I assume that economy is not hit by any aggregate shocks, so that the aggregate state does not change ($\sigma' = \sigma$). It is assumed that the growth rate of the economy is constant, thus $g' = g$, such that the economy is on the balanced growth path. As such, I assume that any aggregate variable $\hat{a}' = \hat{a}$ if the variable is a growing variable at rate g, and $a' = a$ if it is constant. Here I present the model's stationarised equations for the Non-Stochastic Balanced Growth Path.

Household

$$\hat{p} = \frac{1}{\hat{C}}, \tag{73}$$

$$\hat{w} = \frac{\varphi}{\hat{p}}, \tag{74}$$

$$\hat{d} = \frac{\beta}{1+g}. \tag{75}$$

Final Good Firm

$$\hat{y}(z,l,x) = z\left(l^{(1-\zeta)}x^{\zeta}\right)^{\gamma}, \tag{76}$$

$$\hat{V}(z,x) = \int_0^{\bar{\bar{\xi}}} \hat{\hat{V}}(z,x,\xi)G(d\xi), \tag{77}$$

$$\hat{\hat{V}}(z,x) = \max\{-\xi\hat{y}(z,l,x) + \hat{V}^A(z,x), \hat{V}^{NA}(z,x)\}, \tag{78}$$

$$\begin{aligned}\hat{V}^A(z,x) = \max_{x',l}\ & \hat{p}\{\hat{y}(z,l,x) - p_x[(1+g)x' + (1-\delta)x] - \hat{w}l - \omega\mathbb{I}(i^x < 0)\} \\ & + \beta\mathbb{E}\hat{V}(z',x'),\end{aligned} \tag{79}$$

$$\hat{V}^{NA}(z,x) = \max_{l}\ \hat{p}\left\{\hat{y}(z,l,x) - \hat{w}l\right\} + \beta\mathbb{E}\hat{V}(z',(1-\delta)x), \tag{80}$$

$$\underline{\hat{\xi}}(z,x) = \frac{\hat{V}^A(z,x) - \hat{V}^{NA}(z,x)}{\hat{y}(z,l,x)}. \tag{81}$$

Intangible Good Sector

$$p_x = \frac{1}{\nu}, \tag{82}$$

$$\Pi = (p_x - 1)X, \tag{83}$$

$$\hat{F} = \frac{\hat{p}\Pi(1+g)}{1-(1-\phi)\beta} \tag{84}$$

Innovation Sector

$$\hat{\theta} = \chi \hat{S}^{(\eta-1)}, \tag{85}$$

$$\hat{\theta} = \frac{\hat{p}(1+g)}{\beta \hat{F}}, \tag{86}$$

$$(1+g) = \hat{\theta}\hat{S} + (1-\phi). \tag{87}$$

Market Clearing Conditions

$$\hat{Y} = \hat{C} + (1+g)X + \hat{S} + \hat{\Xi}, \tag{88}$$

$$\hat{\Lambda}(z_m, x, \mu) = \mu'(z_m, x) \quad \forall (z_m, x) \in \mathbb{S} \quad \text{and} \int_{\mathbb{S}} \mu'(z_m, x) = 1, \tag{89}$$

$$\hat{L^h} = \int_{\mathbb{S}} l(z, x)\mu(d[z \times x]), \tag{90}$$

$$X = \int_{\mathbb{S}} \int_0^{\bar{\bar{\xi}}} X(z, x, \xi)G(d\xi)d([z \times x]), \tag{91}$$

$$\hat{Y} = \int_{\mathbb{S}} y(z, L(z, x), x)\mu(d[z \times x]), \tag{92}$$

$$\hat{\Xi} = \int_0^{\bar{\bar{\xi}}} \left[\hat{y}(z, l, x)\mathbb{I}(i^x \neq 0) + \omega\mathbb{I}(i^x < 0)\right] G(d\xi)\mu(d[z \times x]). \tag{93}$$

C.4 Balanced Growth Path Algorithm

In this section, I explain how the model's balance growth path is solved, where the objective is to find the economy's growth rate g. When solving for the Non-Stochastic Balanced Growth Path I abstract from aggregate uncertainty and I fix the aggregate state of the economy ($\sigma' = \sigma$).

The solution involves bisecting the price of the consumption good p (the marginal utility of consumption) around the excess consumption demand. Even though the model features multiple prices, I only bisect the price of consumption due to some simplifications which render the solution computationally feasible. Firstly, thanks to the functional form of the utility function that has been assumed, I only have to bisect about the price of the consumption good, since the wage w is only functions of parameters and of p. Secondly, due to the symmetric equilibrium assumed in the intangible good sector I know that the price of the intangible capital goods, p^x is only a function of the elasticity of substitution of patents, which means it is constant.

As a result the algorithm for finding the model's growth rate in the Balanced Growth Path is the following.

C.4.1 Bisection Algorithm

1. Start by guessing an upper and lower bound of price p: p^+ and p^-. Let $\gamma^b \in (0, \infty]$ be the bounds updating parameter.
2. Check that the upper bound price p^+ yields an excess supply of consumption by solving the model in BGP (see C.4.2 BGP Model Algorithm), if not update $p^+ = (1 + \gamma^p)p^+$ and repeat step.
3. Check that the lower bound price p^- yields an excess demand of consumption by solving the model in BGP (see C.4.2 BGP Model Algorithm), if not update $p^- = (1 - \gamma^b)p^-$ and repeat step.
4. Once I have the upper and lower bound of price, p^+ and p^-, I can bisect the price accordingly: $p^* = (p^+ + p^-)/2$, and solve the model in BGP (see C.4.2 BGP Model Algorithm):
    (a) If p^* yields excess consumption supply set $p^+ = p^*$ and repeat Step 4.
    (b) If p^* yields excess consumption demand set $p^- = p^*$ and repeat Step 4.
    (c) If p^* clears the consumption good market within a set tolerance, exit the algorithm.

C.4.2 BGP Model Algorithm

The solution for the model given a certain guess on the consumption price p is found as follows.

1. Given the price p, use the household's first order condition with respect to labour (Equation 74) to find the wage rate w.

2. Solve the final good firm's value function using the value function iteration technique to find the optimal policy functions $Y(z,l,x)$, $L(z,x)$, $X(z,x)$, $\xi^T(z,x)$ given the price p:

   (a) Guess an initial next period's value function $V(z',x') = Y(z,l,x)$[27].

   (b) Using the guess for $V(z',x')$, solve the firm's value function conditional on adjusting (Equation 79) and not adjusting capital (Equation 80) and retrieve both the value functions $V^A(z,x), V^{NA}(z,x)$ and the policy functions $Y(z,l,x), L(z,x), X^A(z,x), X^{NA}(z,x)$.

   (c) Using the threshold's policy function (Equation 81) find the policy function for adjusting capital or not $\xi^T(z,x)$.

   (d) Find the value for today's value function $V(z,x)$ using the policy functions $\xi^T(z,x)$ and value functions $(V^A(z,x), V^{NA}(z,x)$ (Equation 77 and Equation 78).

   (e) If the error between the guess $V(z',x')$ and $V(z,x)$ is within a set tolerance, then exit the loop; otherwise, set $V(z',x') = V(z,x)$ and repeat steps (b) to (d).

3. Once the final goods firm's problem is solved retrieve the policy functions and solve for the ergodic stationary distribution $\mu(z,x)$:

   (a) Guess an initial distribution $\mu(z,x)$.

   (b) Using the policy functions $X(z,x)$, $\xi^T(z,x)$ and the stochastic transitional matrix for the productivity process Γ_z find next period's distribution $\mu(z',x')$.

   (c) If the error between the guess $\mu(z,x)$ and next period's distribution $\mu(z',x')$ is within a set tolerance, then exit the loop; otherwise, update the distribution $\mu(z,x) = \mu(z',x')$ and repeat step (b).

4. Solve the intangible good sector's problem with the value function iteration technique and using the policy functions and the stationary distribution obtained:

[27] I use this guess as it has the advantage of skipping the initial iteration when one uses $V(z',x') = 0$ as a guess.

(a) Using the intangible good policy function $X(z, x)$, the capital adjustment policy function $\xi^T(z, x)$, and the stationary distribution $\mu(z, x)$, find the total intangible capital good X.

(b) Calculate the current period profits (Π^X) for intangible capital goods (Equation 82 and Equation 83).

(c) Guess initial next period value function $F' = \Pi^X$.

(d) Solve the innovation sector's system of non-linear equations (Equations 85, 86, and 87) using the Nelder and Mead (1965) root-finding method. Obtain the optimal growth rate g given the guess on F'.

(e) Using g and F' solve the intangible good firm's value function and obtain F.

(f) If the error between the guess F and next period's value function F' is within a set tolerance, then exit the loop; otherwise, update the guess $F' = F$ and repeat steps (d) to (e).

5. Calculate the consumption supplied C^s using the aggregate constraint (Equation 88), the policy functions and the stationary distribution.

6. Using the household' first order condition with respect to consumption (Equation 73) calculate the consumption demanded C^d given the price p.

C.5 Stochastic Model Solution Algorithm

The simulation algorithm for the model under aggregate uncertainty follows the method of Krusell and Smith (1998). The model solution is based on iterating on the beliefs of the laws of motion of agents regarding the aggregate intangible capital level and the price of consumption. However, seen as I am working with a model with an endogenous growth rate, some modification are needed to the original algorithm. Indeed, the original Krusell and Smith (1998) algorithm is augmented with a section which solves for the growth rate of the economy using a system of non-linear first order difference equations. What follows is an elucidation of the algorithm used to solve the model under aggregate uncertainty.

C.5.1 Outer Loop Algorithm

1. Guess the coefficients for the endogenous law of motion of the joint distribution $\mu(z, X)$ using the first moment $\hat{X}$ as its characterisation:

$$log(\hat{X}') = \alpha^X_{i,j} + \beta^X_{i,j} log(\hat{X}) \quad \text{where} \quad i, j = \{\sigma_L, \sigma_H\}. \tag{94}$$

2. Guess the coefficients for the forecast of the price of consumption p:

$$log(p) = \alpha^p_{i,j} + \beta^p_{i,j} log(\hat{X}) \quad \text{where} \quad i, j = \{\sigma_L, \sigma_H\}. \tag{95}$$

3. Solve the model for all states and for all aggregate intangible capital grid points to obtain the final good firms value function $V(z, x; \sigma_i, \sigma'_j, \tilde{X})$, the intangible good firm value function $F(\sigma_i, \sigma'_j, \tilde{X})$, where $\tilde{X}$ represents the aggregate intangible capital grid[28]. See Algorithm C.5.2 for details.

4. Having obtained the value functions for all states and aggregate grid points over the aggregate intangible capital grid, I can now unconditionally simulate the economy for T periods. See Algorithm C.5.3 for details.

5. Figure out whether the time series of X, p are generated from the unconditional simulation are the same as the time series $X^{f'}, p^f$ generated by the beliefs of the laws of motions.

    (a) Regress the time series $X^{f'}$ and p^f on X separately for each state $i, j = \{\sigma_L, \sigma_H\}$:

$$log(\hat{X}') = \hat{\alpha}^X_{i,j} + \hat{\beta}^X_{i,j} log(X), \tag{96}$$

[28] This is how the highly dimensional object μ is discretized, by assuming that only the mean, that is the aggregate intangible level of capital matters.

$$log(p) = \hat{\alpha}^p_{i,j} + \hat{\beta}^p_{i,j} log(X), \quad (97)$$

(b) Calculate errors between the time series generated, Err^X, Err^p.

(c) Execute the Den Haan (2010) check.

6. If the errors Err^X, Err^p are within a set tolerance, then exit the loop.

7. Otherwise, let γ^{KS} be the coefficient's updating parameter and update the coefficients as follows:

$$\alpha^m_{i,j} = \gamma^{KS}\hat{\alpha}^m_{i,j} + (1-\gamma^{KS})\alpha^m_{i,j} \quad \forall \quad m = \{X, p\} \quad \forall \quad i, j = \{\sigma_H, \sigma_L\}, \quad (98)$$

$$\beta^m_{i,j} = \gamma^{KS}\hat{\beta}^m_{i,j} + (1-\gamma^{KS})\beta^m_{i,j} \quad \forall \quad m = \{X, p\} \quad \forall \quad i, j = \{\sigma_H, \sigma_L\}. \quad (99)$$

8. Repeat steps 3 to 5.

C.5.2 Inner Loop Algorithm

This algorithm solve the model given a set of beliefs on the laws of motion for X and p.

1. Using the coefficients guessed $\{\alpha^m_{i,j}, \beta^m_{i,j}\}$ $\forall$ $m = X, p$ and $\forall$ $i, j = \{\sigma_L, \sigma_H\}$, obtain the forecasts: $\tilde{X^f}(\sigma_i, \sigma'_j, \tilde{X})$ and $p^f(\sigma_i, \sigma'_j, \tilde{X})$.

2. Using the household's labour first order condition obtain the wage forecast $w^f(\sigma_i, \sigma'_j, \tilde{X})$.

3. Use the intangible good firm's symmetric equilibrium result for the price of intangible capital to get p_x.

4. Solve the final good firm's problem using value function iteration to obtain $V^f(z, x; \sigma_i, \sigma'_j, \tilde{X})$, as well as the policy function $L(z, x; \sigma_i, \sigma'_j, \tilde{X})$, $X(z, x; \sigma_i, \sigma'_j, \tilde{X})$, $Y(z, x; \sigma_i, \sigma'_j, \tilde{X})$. The algorithm for the value function iteration is similar to C.4.2 BGP Model Algorithm.

5. Solve the intangible good firm's problem using value function iteration and solving a system of non-linear equation to obtain $F^f(\sigma_i, \sigma'_j, \tilde{X})$..

C.5.3 Unconditional Simulation Algorithm

This algorithm solve the unconditional simulation for the stochastic model given a set of beliefs in the law of motion with respect to X and p, and the resulting value functions $V(z,x;\sigma_i,\sigma_j',\tilde{X})$ and $F^f(\sigma_i,\sigma_j',\tilde{X})$ for each state $i,j=\{\sigma_L,\sigma_H\}$. T refers to the time period of the simulation.

For every $t=1,...,T$:

1. Draw an aggregate state σ_{t+1} according to the exogenous process of Γ_σ.
2. Start by guessing an initial distribution $\mu_1(z,x)$ and an initial total number of patents N_1.
3. Using the initial distribution obtain the staring level of intangible capital goods X_1.
4. Using the coefficients for the law of motion for joint distribution $\mu(z,x)$ get the forecast $\hat{X}^f_{t+1}$, given the draw of the aggregate shock σ_{t+1}.
5. Solve the model for the state σ_{t+1} by bisecting about p_t similar to Algorithm C.4.1 Bisection Algorithm, and calculating the next period's value functions by interpolating $V(z,x;\sigma_i,\sigma_j',\tilde{X})$ and $F^f(\sigma_i,\sigma_j',\tilde{X})$ on the forecast of $\hat{X}^f_{t+1}$.
6. Using the policy functions $X_t(z,x)$, $\xi^T_t(z,x)$ from the previous step, and the stochastic transitional matrix for the productivity process Γ_z find next period's distribution $\mu(z',x')$.
7. Obtain the aggregate intangible capital good demanded X_{t+1}, the growth rate g_{t+1}, and work out N_{t+1}.

C.5.4 Conditional Simulation Algorithm

This algorithm solve the conditional simulation for the stochastic model given a set of beliefs in the law of motion with respect to X and p, and the resulting value functions $V(z,x;\sigma_i,\sigma_j',\tilde{X})$ and $F^f(\sigma_i,\sigma_j',\tilde{X})$ for each state $i,j=\{\sigma_L,\sigma_H\}$. T refers to the time period of the simulation and N_e to the number of economies simulated.

For every $i=1,...,N_e$:

1. For every $t=1,...,T$:
    (a) Draw an aggregate state $\sigma_{i,t+1}$ according to the exogenous process of Γ_σ.

(b) Start by guessing an initial distribution $\mu_{i,1}(z, x)$ and an initial total number of patents $N_{i,1}$.

(c) Using the initial distribution obtain the staring level of intangible capital goods $X_{i,1}$.

(d) Using the coefficients for the law of motion for joint distribution $\mu(z, x)$ get the forecast $\hat{X}^{f}_{i,t+1}$, given the draw of the aggregate shock $\sigma_{i,t+1}$.

(e) Solve the model for the state $\sigma_{i,t+1}$ by bisecting about $p_{i,t}$ similar to Algorithm C.4.1 Bisection Algorithm, and calculating the next period's value functions by interpolating $V(z, x; \sigma_i, \sigma'_j, \tilde{X})$ and $F^f(\sigma_i, \sigma'_j, \tilde{X})$ on the forecast of $\hat{X}^{f}_{i,t+1}$.

(f) Using policy functions $X_{i,t}(z, x), \xi^{T}_{i,t}(z, x)$ from the previous step, and the stochastic transitional matrix for the productivity process Γ_z find next period's distribution $\mu(z', x')$.

(g) Obtain the aggregate intangible capital good demanded $X_{i,t+1}$, the growth rate $g_{i,t+1}$, and work out $N_{i,t+1}$

2. Once you have done this for all N_e economies, the impulse response functions will be the the average of the variables of interest:

$$x^{IRF}_t = \frac{\sum_1^{N_e} x_{i,t}}{N_e} \tag{100}$$

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