Strategic Default in Financial Networks

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Abstract. This paper investigates a model of strategic interactions in financial networks, where the decision by one agent on whether or not to default impacts the incentives of other agents to escape default. Agents' payoffs are determined by the clearing mechanism introduced in the seminal contribution of Eisenberg and Noe (2001). We first show the existence of a Nash equilibrium of this default game. Next, we develop an algorithm to find all Nash equilibria that relies on the financial network structure. Finally, we explore some policy implications to achieve efficient coordination.

JEL classification: C72, D53, D85, G21, G28, G33.

Keywords: systemic risk, default, financial networks, coordination games, central clearing counterparty, financial regulation.

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1. Introduction

Financial institutions carry out various transactions with each other, including risk–sharing and insurance. The architecture of the network of transactions between institutions can support financial stability because it enables them to share funding or transfer risk. But these linkages can also facilitate the diffusion of shocks through the system, due to chains of default and the domino effect. This is referred to as systemic risk. Systemic risk is costly for individuals, institutions and economies, as demonstrated by the last financial crisis. The obvious need for a stable financial system has led to a significant interest in policies that could reduce systemic risk and mitigate contagion.

This paper introduces a model of strategic interactions in financial networks. We study a two-period economy where agents have a positive endowment in each period. The endowment represents agents' cash flows from outside the financial system. We assume that agents hold each other's financial liabilities and that this constitutes the network between them. These liabilities mature in the second period, and we assume that agents' second-period endowments are small and deterministic, so that they face a risk of default. More specifically, the liabilities structure results in cyclical payments interdependencies that are simultaneously computed according to the clearing mechanism described in the seminal contribution of Eisenberg and Noe (2001). The clearing vector satisfies three criteria:

- debt absolute priority, which stipulates that liabilities are paid in full in order to have positive equity;
- limited liability, which means that the payment made by each agent cannot exceed its inflows;
- equal seniority of all creditors, which implies pro rata repayments.

Agents can avoid default by storing part of their first-period endowment. Due to complementarities in the payments, the decision taken by one agent to store part of his endowment exerts a positive externality on the other agents to whom he is connected.

We show that the strategic interactions in the financial system modelled here can be investigated as a coordination game, called the default game, where agents' decisions are simply whether to default or not. It is well known in the literature that coordination games will in general yield multiple pure–strategy Nash equilibria and that the set of pure–strategy Nash equilibria has a lattice structure—in particular, there are two extreme pure–strategy Nash equilibria. In our setting, the best equilibrium is the one where the largest number of agents choose the maximal action Non-Default and the worst equilibrium is the one where the largest number of agents choose the minimal action Default. In the paper, we develop a simple algorithm for finding all Nash equilibria of the default game. While there are easy algorithms for finding the maximal and minimal equilibria and relatively easy algorithms to compute all Nash equilibria in coordination games such as the default game (see Echenique (2007)), the advantage of the algorithm developed in this paper is that it relies on the financial network structure to inform the computation of Nash equilibria. Algorithms that exploit the financial network structure such as the algorithm developed in this paper, as well as quickly computing all Nash equilibria, provide useful information on the strategic interactions between agents.

In this paper, we show that the problem of inefficient coordination may arise in financial networks. Similar to other areas in economics, the strategic complementarities of payments due to the cyclical financial interconnections allow for the existence of multiple Nash equilibria. This gives rise to the question of which one of these equilibria will be the outcome of the underlying default game. From a policy perspective, given that inefficient coordination might pose a severe economic problem, there is a need for financial institutions fostering efficient coordination of agents' decisions. Recently, central clearing has become the cornerstone of policy reform in financial markets since it limits the scope of default contagion. Our analysis shows that introducing a central clearing counterparty (henceforth, CCP) also allows agents to coordinate on the efficient equilibrium at no additional cost. As a consequence, our result reinforces the key role CCP plays in stabilising financial markets.

This paper is structured as follows. In Section 2, we go over the related literature. Then we describe the model and show the existence of a Nash equilibrium in Section 3. We develop an algorithm to find all Nash equilibria in Section 4 and Section 5 provides some policy implications. Section 6 concludes the paper and Section 7 is an appendix devoted to the proofs.

2. Related Literature

The impact of the financial network structure on economic stability has been a subject of ongoing interest since the last financial crisis (of 2008). The seminal contributions of Allen and Gale (2000) and Eisenberg and Noe (2001) were first to acknowledge that the financial network structure determines default contagion, and would serve as a basis for many subsequent contributions.

Allen and Gale (2000) investigate how symmetric financial networks lead to contagion, where links represent sharing agreements. Their key finding is that incomplete financial networks are less resilient and more vulnerable to contagion than their complete counterparts. Eisenberg and Noe (2001) develop a static model of default contagion in a financial network where agents hold each other's financial liabilities and the activities and operations of each agent are condensed into one value: the operational cash flow. The repayment of liabilities will be interdependent, since whether an agent defaults or not is a result of his operational cash flow as well as the payments he receives from other agents. Eisenberg and Noe first prove the existence of a clearing payment vector that is unique under mild conditions. They also provide an algorithm to compute the clearing vector, which is important to predict chains of defaults.

Acemoglu, Ozdaglar and Tahbaz-Salehi (2015) extend the Eisenberg–Noe model to accommodate agent exposure to outside shocks. They establish that up to a certain magnitude of shocks, the more connected the financial network is, the more stable it is; beyond this threshold, the connectedness of the network makes it more prone to contagion and thus more fragile. Elliott, Golub and Jackson (2014) introduce two concepts of cross-holdings that have distinctive and non-monotonic impact on default cascades. Integration, which measures the dependence on counterparties, expands the extent of default contagion but reduces the probability of the first failure; while diversification, which measures the heterogeneity of cross-holdings, increases the propagation of failure cascades but decreases the exposure level among pairs of financial institutions. Cabrales, Gottardi and Vega-Redondo (2017) investigate the optimal network structure that maximizes risk-sharing benefits among interconnected firms while decreasing their risk exposure. Other recent contributions include Teteryatnikova (2014) and Csóka and Herings (2016).

Several approaches have been investigated to mitigate the domino effect in the financial network, such as central clearing and identifying the most systemically relevant financial institutions and then targeting them through cash injections. For instance, Demange (2017), following a similar approach to Eisenberg and Noe (2001), develops a new measure, called the *threat index*, which identifies the most systemically relevant agents for optimal targeted cash injection.

3. The Model

Consider a two-period ($t = 1, 2$) economy with $N = \{1, 2, ..., n\}$ agents. Agent i's endowment in the first period is z_i^1 and in the second period is z_i^2. The endowment of agent i in each period denotes the cash flows arriving from outside the financial system. We assume that agents hold each other's liabilities, which mature in the second period. More specifically, given two agents $i, j \in N$, let $L_{ij} \in \mathbb{R}^+$ denote the liability that agent i owes agent j. Then, agent i's total liabilities are $L_i = \sum_{j \in N} L_{ij}$. Meanwhile, $\sum_{j \in N} L_{ji}$ is the total assets of agent i. Let $\boldsymbol{\alpha} = (\alpha_{ij})_{i,j \in N}$ denote the matrix of relative liabilities, with entries $\alpha_{ij} = \frac{L_{ij}}{L_i}$ representing the ratio of the liability agent i owes to agent j over the total amount of agent i's liabilities.

The utility function of agent i is $U_i(e_i^1, e_i^2)$, where e_i^1 is the equity of agent i at $t = 1$, e_i^2 is the equity of agent i at $t = 2$, and U_i is an increasing and continuous function from $\mathbb{R}_+^2$ to $\mathbb{R}_+$.

Each agent i can store an amount $x_i \in [0, z_i^1]$ from his first-period endowment and receives an interest rate $r > 0$ on his storage. Given the storage strategies of agents $\mathbf{x} = (x_i)_{i \in N}$, let $\boldsymbol{\pi}^{\mathbf{x}} = (\pi_i^{\mathbf{x}})_{i \in N}$ denote the clearing payment vector, uniquely[1] defined as in Eisenberg and Noe (2001), such that for each agent i it holds that

$$\pi_i^{\mathbf{x}} = \min \left\{ z_i^2 + (1 + r)\, x_i + \sum_{j=1}^{n} \alpha_{ji} \pi_j^{\mathbf{x}}; L_i \right\}.$$

This means that $z_i^1 - x_i$ denotes the equity of agent i in the first period and $z_i^2 + (1 + r)x_i + \sum_{j=1}^{n} \alpha_{ji} \pi_j^{\mathbf{x}} - \pi_i^{\mathbf{x}}$ denotes the equity of agent i in the second period. Therefore, the

[1] Under mild assumptions.

utility function of agent i, given the storage strategies of agents $\mathbf{x} = (x_i)_{i \in N}$, is

$$U_i(z_i^1 - x_i, z_i^2 + (1+r)\, x_i + \sum_{j=1}^{n} \alpha_{ji} \pi_j^{\mathbf{x}} - \pi_i^{\mathbf{x}}).$$

Observe that agent i will choose to store a positive amount of his first-period endowment if and only if he prefers (is better off) not to default. If he prefers not to default, he will store enough endowment to avoid it; otherwise he will store nothing. Similarly, it is only the decision of an agent to default or not, rather than the amount of storage, that affects the other agents. This is because, if he defaults, he will pay out his total second-period equity and, if he does not default, he will pay his total liability, neither of which is directly affected by his level of storage.

Therefore, the strategic interaction of agents in the economy can be investigated as a binary coordination game with two actions (Default) = 0 and (Non-Default) = 1 among which agents must choose.

Define a threshold $T_i(\mathbf{a}_{-i})$ as the minimum amount agent i must store to avoid default, given other agents' actions $\mathbf{a}_{-i} \in \{0,1\}^{N-1}$. Define also $\hat{z}_i^1$ as i's first-period endowment equivalent, which satisfies

$$U_i(z_i^1, 0) = U_i(0, \hat{z}_i^1).$$

Proposition 1. *The best reply function of agent i can be written as follows:*

$$\Psi_i(\mathbf{a_{-i}}) = \begin{cases} 0 & \text{if } (1+r)\, z_i^1 - T_i(\mathbf{a_{-i}}) < \hat{z}_i^1, \\ 1 & \text{otherwise.} \end{cases}$$

Proof. The proof of Proposition 1, together with all our other proofs, appears in the Appendix.□

Observe that the default game corresponds to a binary game of strategic complements or equivalently a coordination game, since the decision of an agent not to default makes it easier for other agents not to default too. As defined in Bulow, Geanakoplos and Klemperer (1985), strategic complementarities arise if an increase in one agent's strategy increases the optimal strategy of the other agents.

Observe also that agents can choose to default strategically. More precisely, agent i has nonnegative equity when he stores and still chooses not to store since his equity does not exceed his first-period endowment equivalent. That is, agent i chooses not store when

$$0 \leq (1+r)\, z_i^1 - T_i\left(\mathbf{a}_{-\mathbf{i}}\right) < \hat{z}_i^1.$$

Theorem 1. *There exists a pure–strategy Nash equilibrium of the default game.*

Theorem 1 shows the existence of a pure–strategy Nash equilibrium. Understandably, the existence of a pure–strategy Nash equilibrium follows from the strategic complementarities between agents' actions.

4. Nash Equilibria of the Default Game

The default game introduced above corresponds to a binary game of strategic complements—see Topkis (1979), Sobel (1988), Milgrom and Roberts (1990), Vives (1990), Echenique and Sabarwal (2003), Amir (2005), Echenique (2007) and Barraquer (2013) for other economic applications of games of strategic complements. It is well known in the literature that the set of Nash equilibria of a binary game of strategic complements will in general have multiple pure–strategy Nash equilibria with a lattice structure. In particular, this class of games has two extreme equilibria: the best equilibrium is the equilibrium where the largest number of agents choose the maximal action (Non-Default) $= 1$ and the worst equilibrium is the equilibrium where the largest number of agents choose the minimal action (Default) $= 0$.

The following result highlights the connection between the multiplicity of equilibria and the structure of the financial network.

Proposition 2. *If the default game has multiple Nash equilibria then the financial network has cyclical obligations.*

Proposition 2 shows that the presence of a cycle of financial obligations is necessary for the multiplicity of Nash equilibria. Eisenberg and Noe (2001) term this phenomenon cyclical interdependence and illustrate it as follows: "*A default by Firm A on its obligations to Firm B may lead B to default on its obligations to C. A default by C may, in turn have a feedback effect on A.*"

In the following, we will show that the close relationship between the multiplicity of Nash equilibria and the cyclical financial interconnections is useful to solve for pure–strategy

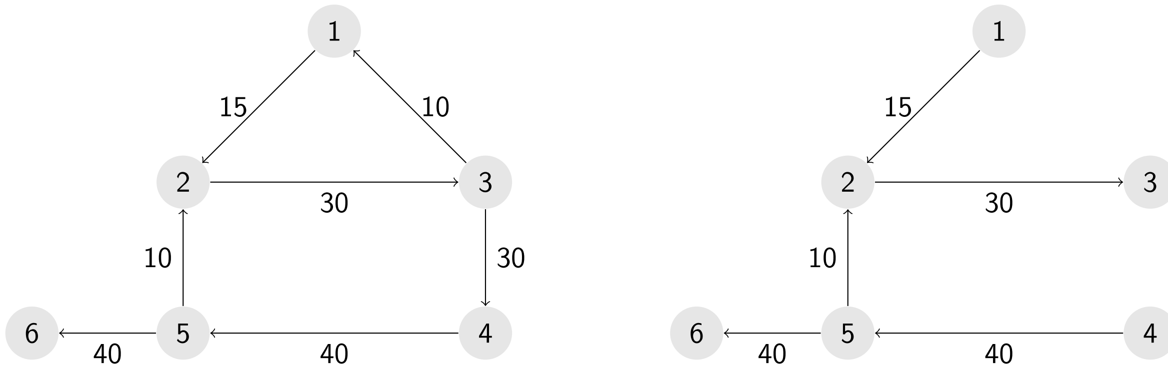

Figure 1. Cyclical obligations — Unidirectional obligations

Nash equilibria of the default game. More specifically, we will provide an algorithm to find all pure–strategy Nash equilibria of the default game. Recall that the financial network is strongly connected if there is a path of obligations between all pairs of agents. A strongly connected component (henceforth, SCC) of the financial network is a maximal[2] strongly connected subnetwork.

4.1. **A financial network with a unique SCC.** First, for simplicity, we consider the case of a financial network with a unique strongly connected component. We will use the following notion of *ear decomposition* of a network, which is useful given its close relationship to network connectivity. An ear decomposition of a network is a partition of the set of agents into an ordered collection of agent-disjoint simple paths, called ears. More precisely, an ear decomposition of a network is a partition of the agents into $E_0, E_1, \ldots, E_p$ such that

- $E_0 = \{v_0\}$ is a single agent;
- for each $h = 1, \ldots, p$, it holds that $E_h = \{v_{1_h}, \cdots, v_{k_h}\}$ is a directed path such that the endpoints of each E_h—that is, v_{1_h} and v_{k_h}—are in $E_1 \cup \ldots \cup E_{h-1}$ but the internal agents of E_h—that is, $v_{2_h}, \cdots, v_{(k-1)_h}$—are not in $E_1 \cup \ldots \cup E_{h-1}$.[3]

A financial network is strongly connected if and only if it has an ear decomposition. In the following, we will refine further the concept of ear decomposition. Given an ear E_h, we say a subset of consecutive internal agents $R_{t_h} = \{v_{t_h}, \cdots, v_{s_h}\}$ is a *rim* of the ear if $v_{(t-1)_h}$ is an ear's first agent and v_{s_h} is either an ear's first agent or E_h's penultimate

[2]In the sense that it is not properly contained in a larger SCC.

[3]Each E_h $(h = 1, \ldots, p)$ is called an ear.

agent and none of the other agents in the rim is an ear's first agent. Hence the internal agents of each ear can be partitioned into a collection of rims. Observe also that the last ear always has a unique rim.

In the following, we will rely on this refinement of the ear decomposition to provide an algorithm to find all pure–strategy Nash equilibria of the default game of a financial network with a unique SCC.

The algorithm, which we call USCCNE, goes as follows:

(1) For each rim in the network, *assume* that each agent in the rim is the last non–defaulting agent or that all agents in the rim are defaulting.

(2) For every case in (1), start from the last ear E_p and repeat the following until reaching the first ear E_0: for each ear delete, the internal agents and update the inflows of the affected (intercepting) agents.

(3) For every case of assumed actions in (1), start from the single agent v_0 in E_0 and move along all agents in every ear in the opposite direction; for each agent compute, the optimal action while taking feedback into consideration.[4]

In interpretation, the USCCNE algorithm assumes for each rim that a particular agent is the last non-defaulting agent or that all agents in the rim default. Then start from the last ear and repeat the following until reaching the first ear: delete all the internal agents of each ear and update the inflows of all affected (intercepting) agents. Finally, the algorithm navigates every ear in the opposite direction computing the optimal actions of all agents. The Nash equilibria correspond to the iterations where all the *assumed* actions are satisfied.

As a consequence, the USCCNE algorithm also provides a bound on the number of Nash equilibria.

Corollary 1. *The maximal number of Nash equilibria for a financial network with a unique SCC is* $\Pi_{R_{t_h}}|R_{t_h}+1|$.

The next example illustrates the default game.

[4]That is, for agent i

$$\text{inflow}_i = a_i\pi_i + b_i,$$

which vary according to the case considered.

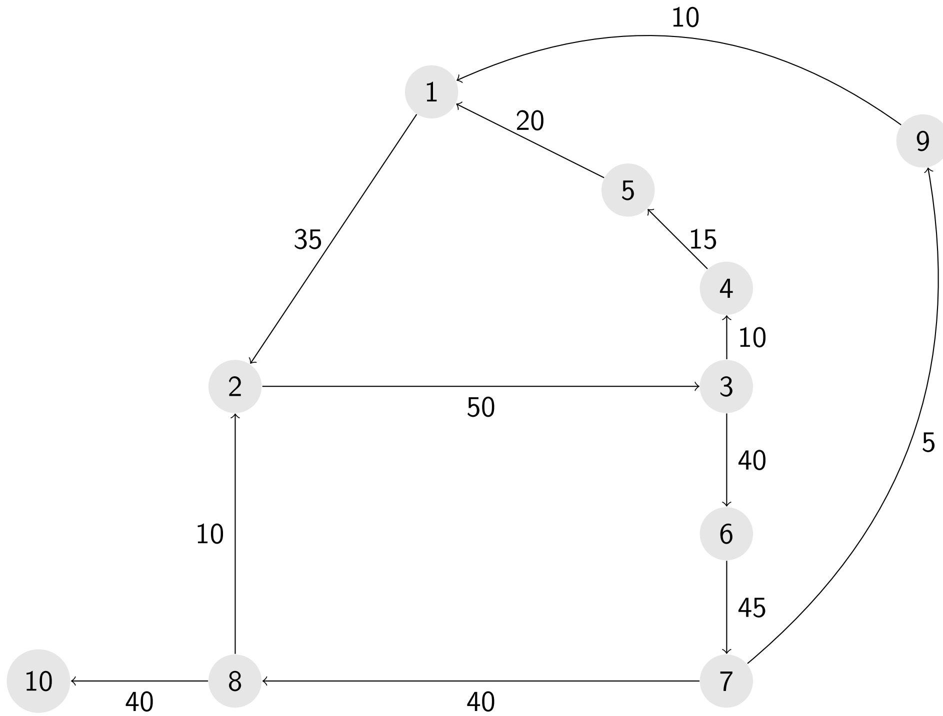

Figure 2. A financial network with ten agents

Example 1. Consider an economy of ten agents connected through their ownership of each other's liabilities, among which only the first nine agents are strategically relevant. Agents' endowments in the first period are $\mathbf{z}^1 = (25, 25, 40, 40, 60, 40, 40, 70, 24)$ and in the second period are $\mathbf{z}^2 = (3, 3, 3, 3, 3, 3, 3, 3, 3)$ and the interest rate is $r = 0.1$. All agents have the same utility function $U_i(e_i^1, e_i^2) = e_i^1 + e_i^2$. The financial liabilities of agents to each other are illustrated in the financial network in Figure 2.

This financial network contains a unique SCC, $\{1, 2, 3, 4, 5, 6, 7, 8, 9\}$, which has four ears, $E_0 = \{1\}$; $E_1 = \{1, 2, 3, 4, 5, 1\}$; $E_2 = \{3, 6, 7, 8, 2\}$; and $E_3 = \{7, 9, 1\}$, and five rims, $R_2 = \{2, 3\}$; $R_4 = \{4, 5\}$; $R_6 = \{6, 7\}$; $R_8 = \{8\}$; and $R_9 = \{9\}$.

To compute the Nash equilibria, we apply the USCCNE algorithm described above. We find three Nash equilibria–the best equilibrium $1, 1, 1, 1, 1, 1, 1, 1, 1$, the intermediate equilibrium $0, 0, 0, 1, 1, 0, 0, 0, 0$, and the worst equilibrium $0, 0, 0, 0, 0, 0, 0, 0, 0$–which we illustrate in Figures 3-5.

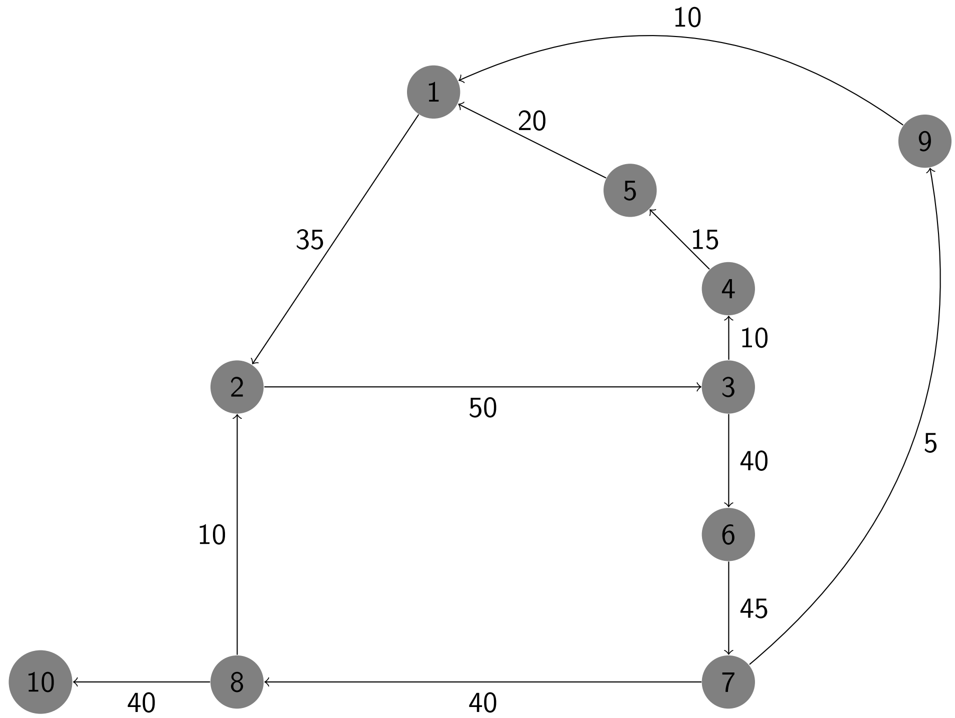

Figure 3. The best equilibrium

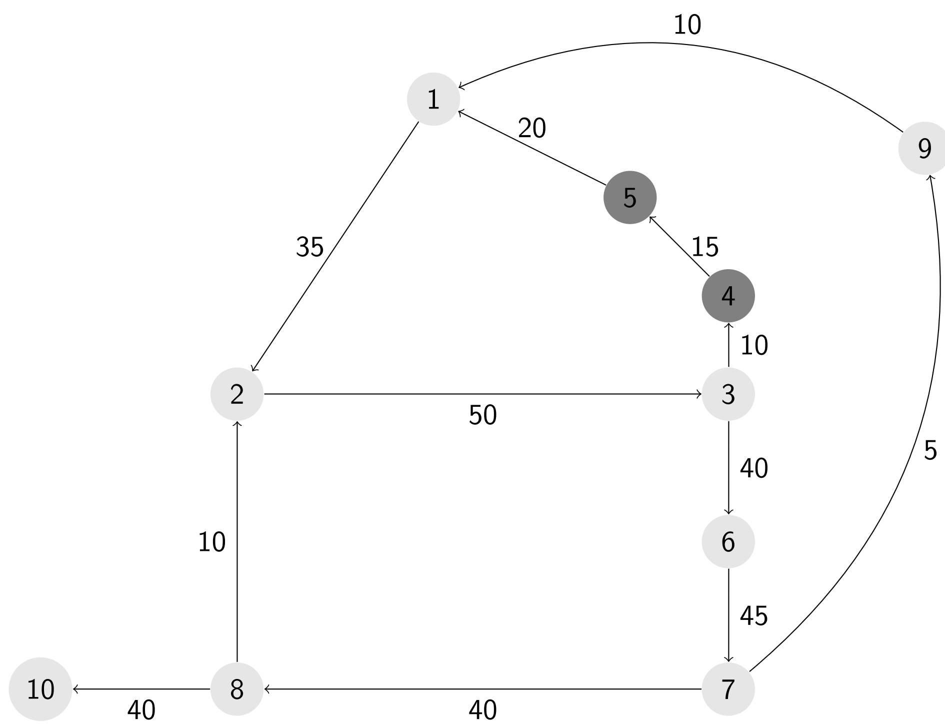

Figure 4. The intermediate equilibrium

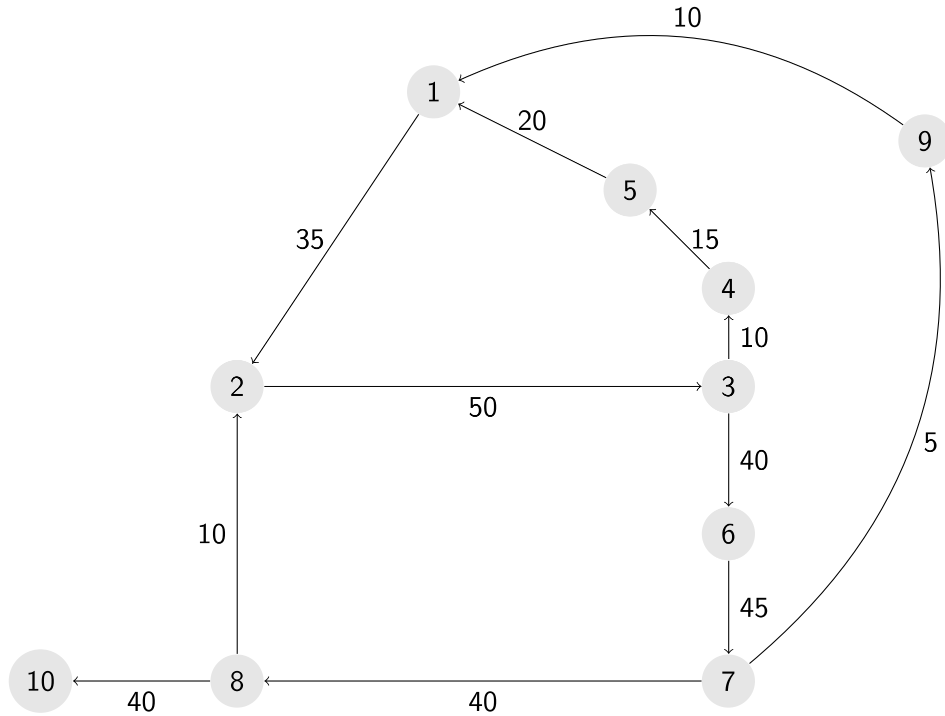

Figure 5. The worst equilibrium

4.2. **Arbitrary financial network.** Now we investigate the case of an arbitrary financial network. Recall that an arbitrary financial network can be transformed to a *directed acyclic graph* (henceforth, DAG)—that is, a network with no cycles–by contracting each SCC into a single large node.

In the following, we will rely on *transitive reduction*, which is a uniquely defined operation on a DAG, to compute the pure–strategy Nash equilibria of a financial network with multiple SCCs. A transitive reduction of a DAG is the network with the fewest possible links that preserves the chains of default of the original financial network. That is, it removes all the links that are unnecessary for the chain of default to be realized and only nodes which were connected by a path in the original network remain connected in the transitively reduced network. For instance, if A links to B, and B links to C, then the transitive reduction removes the link A to C, if it exists.

Observe that, from the minimality of links in the transitive reduction, there exists a unique partition of the set of agents $\mathcal{W} = \{W_1, \dots, W_k\}$ such that W_1 corresponds to the

SCCs with no incoming links, W_2 corresponds to the SCCs with only incoming links from W_1, W_3 corresponds to the SCCs with only incoming links from $W_1 \cup W_2$, and so on.

Then, the algorithm USCCNE can be easily extended to compute the Nash equilibria with multiple SCCs. The algorithm, which we call MSCCNE, goes as follows:

(1) Apply USCCNE to find all Nash equilibria for each SCC in W_1.

(2) For each product of Nash equilibria of SCCs in W_1, apply USCCNE to find all Nash equilibria for each SCC in W_2.

(3) For each product of Nash equilibria of SCCs in $W_1 \cup W_2$, apply USCCNE to find all Nash equilibria for each SCC in W_3.

(4) Repeat the procedure until visiting all the elements of the partition $\mathcal{W}$.

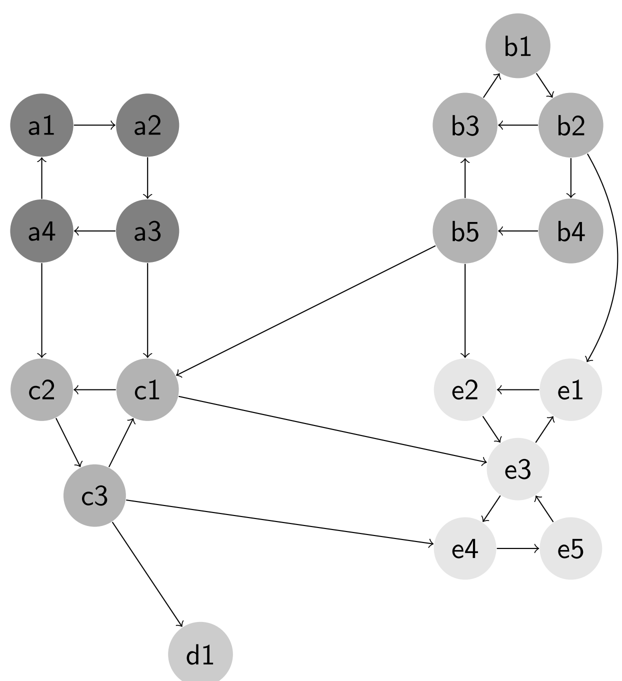

Figure 6. Example of a DAG

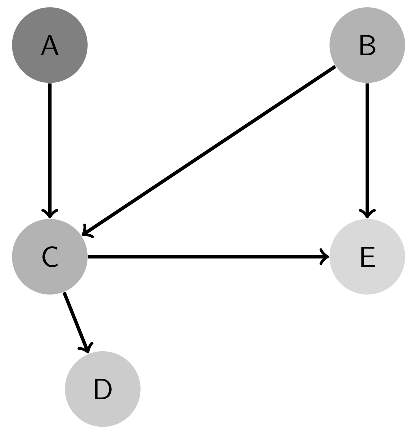

Figure 7. Condensation of the DAG

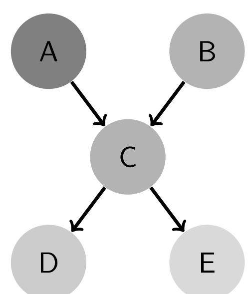

Figure 8. Transitive reduction of the DAG

The MSCCNE algorithm is a simple algorithm that exploits a network decomposition technique to find all the pure–strategy Nash equilibria of a financial network. It is worth noting that the MSCCNE algorithm can be easily adapted to compute the clearing payment vector of Eisenberg and Noe (2001).

Corollary 2. *Assume that the first-period endowment of each agent i is zero—that is, $z_i^1 = 0$. Then the MSCCNE algorithm computes the clearing payment vector in Eisenberg and Noe (2001).*

Recall that the clearing payment vector of Eisenberg and Noe (2001) is unique under mild conditions. Hence the existence of cyclical financial interconnections, while necessary for multiple equilibria, is not sufficient.

At the heart of the seminal contribution of Eisenberg and Noe (2001) lies the elegant *fictitious default algorithm* that computes the unique clearing payment vector. The fictitious default algorithm goes as follows. First, determine the set of agents who cannot fulfill their obligation, even when we assume that all agents receive their due payments. These agents will be called the *first wave of default*. Then, assume that the agents in the first wave of default pay their liabilities pro rata and the new defaulting agents will be called the *second wave of default* and so on until the algorithm terminates. In this way, the fictitious default algorithm produces a natural measure of systemic risk, which is the number of waves required to induce a given agent to default.

Echenique (2007) provides the most efficient algorithm for computing all pure–strategy Nash equilibria in the class of games of strategic complements, of which the default game is a special case. The algorithm elegantly checks whether there is another Nash equilibrium once the smallest and largest pure–strategy Nash equilibria are computed from classical algorithms (for example, Topkis (1979)).

While each of the above algorithms is clearly interesting in many aspects, arguably, the advantage of the MSCCNE algorithm developed in this paper is that it relies on the financial network architecture to compute the Nash equilibria. Generally, algorithms that exploit the financial network structure such as the algorithm developed in this paper, as well as having a clear computational advantage, provide valuable information on the strategic interactions among agents, as we will show below.

5. Policy Implications

From a policy perspective, in view of the multiplicity of Nash equilibria of the default game, there is the central policy question of equilibrium selection. In particular, it may be desirable to implement the best equilibrium in order to achieve financial stability and minimize the cost of default.

Given the best and worst equilibria, agents in the network can be classified into three types:[5]

(1) agents that choose 0 in the worst equilibrium and 1 in the best equilibrium;

(2) agents that choose 0 in the worst equilibrium and 0 in the best equilibrium;

(3) agents that choose 1 in the worst equilibrium and 1 in the best equilibrium.

From now on, without loss of generality, we may assume that all agents are of type (1). Note that agents of type (2) and (3) are not strategically relevant since they play the same action in the worst and the best equilibrium. Actually, we could construct a *reduced* financial network containing only agents of type (1). To do so, we first eliminate all outgoing links emanating from agents of type (3) and, since none of them defaults, add their liabilities pro rata to the cash flow of the agents intercepting their outgoing links. As for agents of type (2), given that they default and pay their inflows—i.e. their cash flow and the payments they receive from their debtors—they can be eliminated from the network by adding their cash flow to the cash flow of their creditors pro rata and by extending their ingoing liabilities links to their creditors pro rata so that the new liabilities directly link between their debtors and their creditors.

Recently, central clearing counterparty has become increasingly the cornerstone of policy reform in financial markets. Introducing a CCP modifies the structure of the financial network: the original liability between a debtor and a creditor is erased and replaced by two new liabilities—one liability between the debtor and the CCP, and another one between the CCP and the creditor. Hence, if the CCP is able to honour its liabilities, it eliminates the risk borne by the creditor that the debtor defaults. As a consequence, one of the key benefits of central clearing is that, by breaking down the cyclical connections of financial liabilities, it reduces the aggregate level of default exposure, which in turn reduces default contagion.

The following proposition points out another potential benefit of introducing central clearing in financial markets.

Proposition 3. *Introducing a CCP in each SCC of the reduced financial network achieves the best equilibrium in the default game at no additional cost.*

[5]Obviously, it is not possible for an agent to choose 1 in the worst equilibrium and 0 in the best.

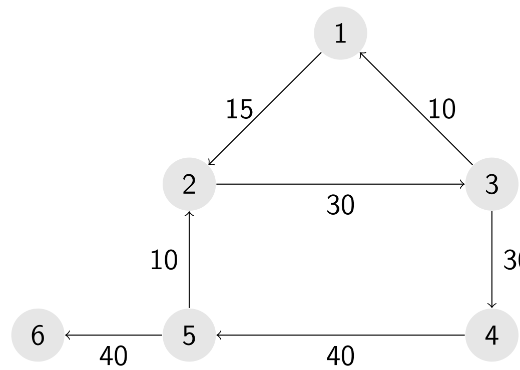

Figure 9. A financial network with five agents

Proposition 3 shows that when a CCP intermediates the liabilities of each SCC of the reduced financial network, the best equilibrium is achieved and the CCP is budget neutral. As a consequence, in addition to reducing default contagion by eliminating the cyclical financial interconnections, central clearing can also serve as a coordination device that achieves the best equilibrium of the default game.

The following example illustrates this point.

Example 2 Consider an economy of six agents connected through their ownership of each other's liabilities, among which only the first five agents are strategically relevant. Agents' endowments in the first period are $\mathbf{z}^1 = (22, 22, 75, 170, 100)$ and in the second period are $\mathbf{z}^2 = (3, 3, 3, 3, 3)$ and the interest rate is $r = 0.1$. All agents have the same utility function $U_i(e_i^1, e_i^2) = e_i^1 + e_i^2$. The financial liabilities of agents to each other are illustrated in the network in Figure 9.

This financial network contains a unique SCC $\{1, 2, 3, 4, 5\}$. To compute the Nash equilibria, we apply the USCCNE algorithm described above. We find three Nash equilibria—the best equilibrium $1, 1, 1, 1, 1$, the intermediate equilibrium $0, 0, 0, 1, 1$, and the worst equilibrium $0, 0, 0, 0, 0$—which we illustrate in Figures 10-12.

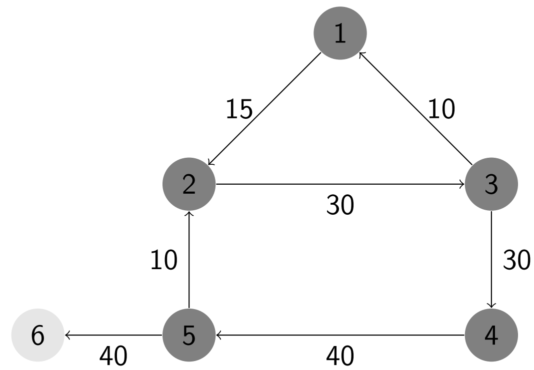

Figure 10. The best equilibrium

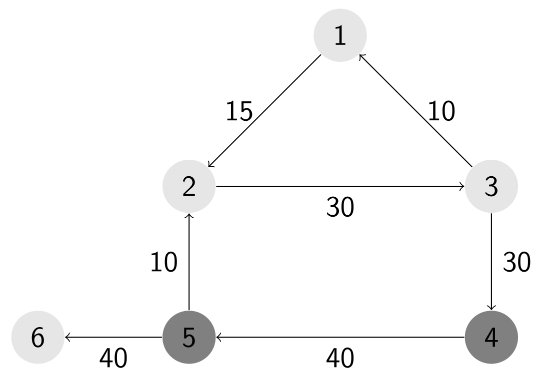

Figure 11. The intermediate equilibrium

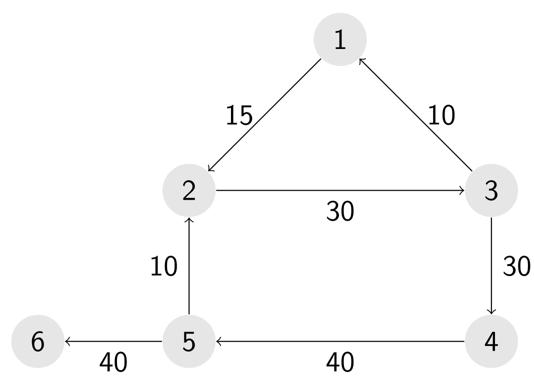

Figure 12. The worst Equilibrium

Adding a CCP will result in a new financial network as shown in Figure 13, with the following liabilities vector:

$$\tilde{\mathbf{L}} = (5, 5, 10, 10, 10, -40) \,.$$

Given that there are no feedback effects in the presence of the CCP, the minimum cash flow for an agent i to escape default is equal to the new liability $\tilde{L}_i$. Therefore, after the introduction of a CCP, it is easy to check that the best equilibrium is implemented at no additional cost since the inflows and outflows of CCP are equal.

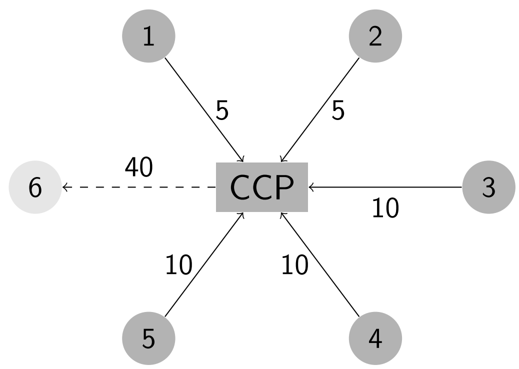

Figure 13. Adding a CCP

6. Conclusion

This paper shows that the CCP allows agents to achieve the best equilibrium at no additional cost. As a consequence, central clearing can serve as a coordination device in financial markets. While our result reinforces the key role CCP plays in financial markets, which is quite desirable, it remains to be seen whether other policies can be designed to minimize the number of defaults, such as identifying key agents and targeting them through either cash injection or minimum endowment requirement.

7. Appendix

Proof of Proposition 1. Observe that agent i will choose to play 1 whenever

$$U_i(a_i = 1, \mathbf{a}_{-i}) \geq U_i(a_i = 0, \mathbf{a}_{-i}),$$

which is equivalent to

$$U_i(0, (1+r)z_i^1 - T_i(\mathbf{a_{-i}})) \geq U_i(0, \hat{z}_i^1),$$

which is also equivalent to

$$(1+r)z_i^1 - T_i(\mathbf{a_{-i}}) \geq \hat{z}_i^1.\square$$

Proof of Theorem 1. A profile of actions $\mathbf{a}^* \in \{0,1\}^N$ is a Nash equilibrium if $a_i^* = \Psi_i\left(\mathbf{a}^*_{-\mathbf{i}}\right)$. Clearly, $T_i\left(\mathbf{a_{-i}}\right)$ is decreasing in $\mathbf{a_{-i}}$ since, as the number of agents playing 1 increases the minimum cash flow and consequently the threshold get smaller. Hence, Ψ_i is increasing in $\mathbf{a_{-i}}$. By the Knaster–Tarski Theorem, there exists a fixed point of the following map:

$$\Psi : \{0,1\}^N \longrightarrow \{0,1\}^N$$

$$\Psi\left(\mathbf{a}\right) = \left(\Psi_1\left(\mathbf{a_{-1}}\right), ..., \Psi_n\left(\mathbf{a_{-n}}\right)\right),$$

which will be a Nash equilibrium of the default game.$\square$

Proof of Proposition 2. Suppose not—that is, the default game has multiple equilibria and the financial network does not have cyclical obligations. Let R denote the set of agents who play 0 in the worst Nash equilibrium and 1 in the best Nash equilibrium. Then the subnetwork induced by R contains an agent i that does not have any ingoing link. As a consequence, the inflow of agent i does not change between the worst equilibrium and the best equilibrium, and as a result agent i will not change his choice in the worst equilibrium and the best equilibrium. This is a contradiction.$\square$

Proof of Proposition 3. Adding a CCP in the middle of the financial network will net out the liabilities and will sort agents into two types: debtors and creditors to the CCP. Let node 0 represent the CCP, and $\tilde{L}_{i0}$ the liabilities to/from the CCP such that

$$\tilde{L}_{i0} = \sum_{j\in N} L_{ij} - \sum_{j\in N} L_{ji}.$$

Hence, if $\tilde{L}_{i0}$ is positive (resp. negative), agent i is a debtor (resp. creditor) to the CCP.

Since the best equilibrium can be reached, it follows that whenever agent i receives all the liabilities from his debtors, he will choose not default. Therefore, it holds that

$$z_i^2 + (1+r)\, z_i^1 + \sum_{j\in N} L_{ji} \geq \sum_{j\in N} L_{ij},$$

which implies

$$z_i^2 + (1 + r)\, z_i^1 \geq \tilde{L}_{i0}.$$

Hence, the non-default condition is satisfied for each agent in the network with liabilities intermediated by the CCP and the best equilibrium is reached.□

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