School of Economics and Finance

Spectral Approach to Parameter-Free Unit Root Testing

Natalia Bailey and Liudas Giraitis

Working Paper No. 746 May 2015 ISSN 1473-0278

Queen Mary
University of London

Spectral Approach to Parameter-Free Unit Root Testing

Natalia Bailey Liudas Giraitis
Queen Mary, University of London

May 12, 2015

Abstract

A relatively simple frequency-type testing procedure for unit root potentially contaminated by an additive stationary noise is introduced, which encompasses general settings and allows for linear trends. The proposed test for unit root versus stationarity is based on a finite number of periodograms computed at low Fourier frequencies. It is not sensitive to the selection of tuning parameters defining the range of frequencies so long as they are in the vicinity of zero. The test does not require augmentation, has parameter-free non-standard asymptotic distribution and is correctly sized. The consistency rate under the alternative of stationarity reveals the relation between the power of the test and the long-run variance of the process. The finite sample performance of the test is explored in a Monte Carlo simulation study, and its empirical application suggests rejection of the unit root hypothesis for some of the Nelson-Plosser time series.

Keywords: Unit root test; Additive noise; Parameter-free distribution

JEL Codes: C21, C23

1 Introduction

A common starting point in time series analysis is the assessment of whether a stationary or non-stationary type of statistical/econometric model best characterises the properties of the data under investigation. This has strong methodological and statistical implications. There exists a broad and still evolving literature aimed at determining the presence of a unit root in economic data sets.

One of the most widely employed unit root testing procedures is the classical Dickey Fuller (DF) test (Dickey and Fuller (1979)), and its augmented (ADF) version (Said and Dickey (1984)). The DF method has undergone numerous refinements, enabling its deep theoretical understanding and practical use. The most prominent ones include Phillips (1987), Phillips and Perron (1988), Elliot et al. (1996) and Ng and Perron (2001) while useful surveys on issues associated with unit root testing can be found in Stock (1994), Maddala and Kim (1998) and Phillips and Xiao (1998). For more recent developments on unit root testing see e.g. Westerlund (2014) and Shelef (2015).

At the core of these tests lie assumptions about the structural form of the times series studied, say y_j, $j = 1, ..., n$. For example, the simplest form of the DF test examines the null hypothesis that $\{y_j\}$ has pure unit root against an AR(1) stationary alternative. Other settings allow for inclusion of an intercept or intercept and time trend which yield different, complex asymptotic distributional representations, and hence altered critical values (MacKinnon (1991)). In turn, the Augmented Dickey Fuller (ADF) test centres on the null hypothesis of an ARIMA$(p, 1, 0)$ process against the stationary ARMA$(p + 1, 0, q)$ alternative, see e.g. Cheung and Lai (1995) or Lopez (1997). Augmentation or selection of the appropriate lag order p is needed to absorb the additional dependence structure as well as computing adjusted critical values, see e.g. Cheung and Lai (1995), Ng and Perron (1995), Elliot et al. (1996), Ng and Perron (2001), Perron and Qu (2007) where these issues are studied thoroughly. Breitung (2002), on the other hand, suggests a different type of unit root test which is based on the variance-ratio statistic. The key advantage of this test is that it does not require specification of the short-run dynamics (augmentation).

Alternative approaches that focus on testing for stationarity versus unit root are inherently adaptable to a wide range of dependence structures of an underlying stationary time series (residuals), from short to long or negative memory. The most prominent of these is the KPSS test by Kwiatkowski et al. (1992), and its subsequent developments, see e.g. Giraitis et al. (2006). The non-standard asymptotic distributions of the corresponding test statistics are again complex and rather intractable, and require estimation of the long-run variance of the associated series. The need for augmentation arises again, this time in the context of estimation of the long-run variance, which may complicate the practical implementation of these tests.

This paper suggests a new and relatively simple frequency-type method for testing for a unit root (potentially contaminated by an additive stationary noise) versus stationarity, which makes use of fundamental properties of the spectrum and periodogram in the vicinity of zero frequency. More precisely, under unit root the periodogram has a sharp peak at zero frequency, and therefore testing can be based on a finite number of periodograms computed at low Fourier frequencies, $u_l, ..., u_k$; $u_p, ..., u_q$. Theory points out the need for k, q to be small, but it does not require data based selection of tuning parameters l, k, p, q, different values of which yield similar size/power performance of the test. Hence, the range of frequencies can be selected a priori. Furthermore, the frequency-type method allows testing for a unit root contaminated by an additive stationary noise, which constitutes the main structural novelty of the paper. The method is easily implemented and does not require augmentation. Under the null it has a parameter-free, tractable asymptotic distribution with critical values that do not require finite sample adjustment and yield correct size for sample sizes $n = 64, 128, 256, 1024$. Monte Carlo simulation results show that the test is well-sized and has satisfactory power under different data specifications.

The rest of the paper is organised as follows. In Section 2 we introduce the low-frequency-type testing procedure for a unit root (Q test) and derive its theoretical properties. In its current format, the differenced unit root ∇x_t is required to be a stationary linear process but an extension to a non-linear framework can be considered as well. The consistency rate under the alternative of stationarity reveals the relation between the power of the test and the long-run variance of ∇x_t. In Section 3 we analyze the finite sample properties of the Q test for a number of data generating models. Of particular interest is the case of a pure unit root augmented by an additive stationary noise, where we compare performance the Q test with the ADF and Breitung tests. Finally, Section 4 contains the empirical application of the Q test to the popular set of time series studied in Nelson and Plosser (1982) and Schotman and van Dijk (1991). Results indicate that for some time series the null hypothesis of unit root can be rejected. Proofs of the main results are contained in the Appendix.

2 Low-Frequency-Periodogram-Type Test

In this section we present a new frequency domain procedure for testing for a unit root in a time series potentially contaminated by an additive stationary noise. The idea behind such an approach is based on the observation that the periodogram ('spectrum') of a unit root process has high-order singularity at zero frequency. Construction of the test takes into account the fundamental asymptotic properties of a vector of the periodograms $\big(I(u_1), ..., I(u_q))\big)$ and the discrete Fourier transforms (DFT) computed at low Fourier frequencies $u_1, ..., u_q$ for a fixed q. The main advantages of frequency-type methods are well documented in Choi and Phillips (1993): (i) no explicit structural form of the error terms is required, (ii) the resulting limiting distributions are parameter-free, (iii) a strong peak of the periodogram at zero frequency under unit root is taken into account, (iv) such tests are predominantly correctly sized. Our objective in this section is to devise a test with parameter-free asymptotic distribution, the tuning parameters of which do not require data based selection. Compared to spectral-type testing procedures by Choi and Phillips (1993) or Fan and Gençay (2010), this is a very different type of test with a different limit distribution. It does not require estimation of the spectral density, uses a preselected finite number of Fourier frequencies and is easy to compute.

To proceed with the definition of the test and its theoretical properties, we set up the null and alternative hypotheses. A process $\{\xi_j, j \in \mathbb{Z}\}$ is said to be a short memory process if it has absolutely summable autocovariances $\gamma_\xi(k) = \mathrm{Cov}(\xi_k, \xi_0)$,

$$\sum_{k=-\infty}^{\infty} |\gamma_\xi(k)| < \infty, \qquad \sum_{k=-\infty}^{\infty} \gamma_\xi(k) > 0. \tag{1}$$

We describe first the hypotheses of unit root with no trend.

Hypothesis $\boldsymbol{H}_0$ (unit root). We say that the random variables y_j, $j = 1, \dots, n$ satisfy the null hypothesis H_0 of unit root with an additive noise if

$$y_j = x_j + \varepsilon_j, \quad j \geq 1, \qquad \text{where } x_j = x_{j-1} + \xi_j, \tag{2}$$

and $\{\xi_j\}$ and $\{\varepsilon_j\}$ are zero mean short memory processes, as defined above in (1).

The alternative hypothesis to H_0 includes a stationary process with unknown mean.

Alternative $\boldsymbol{H}_S$ (stationarity). We say that the random variables y_j, $j = 1, \dots, n$ satisfy the alternative hypothesis H_S of stationarity if

$$y_j = \mu + \xi_j, \tag{3}$$

where $\{\xi_j\}$ is a zero mean short memory stationary ergodic process, and $\mu = Ey_j$ is unknown.

Note that the increments of unit root process (2) have zero mean which may be a restrictive assumption in applications. To relax this restriction, we introduce next the H_0^T hypothesis of a unit root with a drift.

Hypothesis $\boldsymbol{H}_0^T$ (trending unit root). We say that the random variables y_j, $j = 1, \dots, n$ satisfy the null hypothesis H_0^T if

$$y_j = \mu + \beta j + x_j + \varepsilon_j, \quad j \geq 1, \qquad \text{where } x_j = x_{j-1} + \xi_j, \tag{4}$$

μ is an unknown constant, β is unknown drift parameter, and $\{\xi_j\}$ and $\{\varepsilon_j\}$ are zero mean short memory processes.

We test it against the hypothesis of trend stationarity.

Alternative $\boldsymbol{H}_S^T$ (stationarity). We say that the random variables y_j, $j = 1, \dots, n$ satisfy the alternative hypothesis H_S^T if

$$y_j = \mu + \beta j + \xi_j, \tag{5}$$

where μ and β are unknown parameters of the linear trend, and $\{\xi_j\}$ is a zero mean short memory stationary ergodic process.

In the latter case when the unit root with a drift hypothesis H_0^T is tested against trend-stationarity H_S^T, we compute the test using de-trended data

$$\hat{y}_j = y_j - \hat{\beta} j, \qquad j = 1, \dots, n,$$

where

$$\hat{\beta} = \frac{\sum_{j=1}^{n}(y_j - \bar{y})j}{\sum_{j=1}^{n}(j - \bar{j})^2}, \qquad \bar{j} = n^{-1}\sum_{s=1}^{n} s = \frac{(n+1)}{2}. \tag{6}$$

By Theorem 2.1 of Abadir et al. (2011), the OLS estimator $\hat{\beta}$ under H_S^T has the following consistency rate,

$$\begin{aligned} E(\hat{\beta} - \beta)^2 &= O(n^{-1/2}), \quad \text{under } H_0^T, \\ &= O(n^{-3/2}), \quad \text{under } H_S^T. \end{aligned} \tag{7}$$

To derive the asymptotic distribution of the test statistic under H_0 we have to impose additional conditions.

Assumption A. Processes $\{\xi_j\}$ and $\{\varepsilon_j\}$ are linear short memory processes,

$$\xi_j = \sum_{k=0}^{\infty} \alpha_k \zeta_{j-k}, \qquad \varepsilon_j = \sum_{k=0}^{\infty} \alpha'_k \zeta'_{j-k}, \quad j \in \mathbb{Z}, \tag{8}$$

such that $\sum_{k=0}^{\infty} |\alpha_k| < \infty$, $\sum_{k=0}^{\infty} |\alpha'_k| < \infty$, with *i.i.d.* innovations $\{\zeta_j\}$, $\{\zeta'_j\}$ having zero mean and unit variance. Noises $\{\zeta_j\}$, $\{\zeta'_j\}$ can be mutually dependent. Assumption of positive long-run variance of $\{\xi_j\}$ in (1) requires $\sum_{k=0}^{\infty} \alpha_k > 0$.

We write $a_n \sim b_n$ to denote that $a_n/b_n \to 1$ as $n \to \infty$, and $\to_D$ denotes convergence in distribution.

Testing under no trend. To proceed with the testing of H_0 against H_S, first we discuss the basic properties of the periodogram and the Discrete Fourier Transform (DFT), which clarify the idea of the test and allow to establish its asymptotic null distribution and consistency.

Basic properties of periodogram and DFT. Let there be a sample $x_1, \cdots, x_n$. Denote by

$$w_X(u_j) = (2\pi n)^{-1/2} \sum_{k=1}^{n} e^{iku_j} x_k, \quad j = 0, 1, \cdots, n$$

the discrete Fourier transform computed at Fourier frequencies $u_j = 2\pi j/n$, $j = 0, 1, \cdots, n$, and by

$$I_X(u_j) = |w_X(u_j)|^2, \quad j = 0, 1, \cdots, n$$

the periodogram at frequency u_j.

The concepts of periodogram and DFT are usually related to a stationary time series $\{x_j\}$. If $\{x_j\}$ is a stationary sequence with spectral density f_X, then the periodogram $I_X(u_j)$ is a sample version of the spectral density $f_X(u_j)$ at the frequency u_j. In addition, if $\{x_j\}$ has short memory then its spectral density f_X is continuous, and $f_X(u_j) \to f_X(0)$, $u_j \to 0$. Consequently, at low frequencies, $u_1, u_2, \cdots, u_k$, the spectral density $f_X(u_j)$ is approximately equal to $f_X(0)$, in the following sense:

$$f_X(u_j) \sim f_X(0) > 0, \quad n \to \infty. \tag{9}$$

Since periodogram $I_X(u_j)$ is only a mean consistent estimator of f_X, i.e. $EI_X(u_j) \sim f(u_j)$, as $n \to \infty$, the same pattern will be observed in the mean $EI_X(u_j)$ and the periodogram itself. In the case of a unit root, as we will see below, for low frequencies $j = 1, 2, \cdots, k$ the mean $EI_X(u_j)$ peaks at zero frequency u_0, and then sharply decreases when j rises. We use this fact to discriminate between a unit root and a stationary process.

The periodogram and DFT have the following tractable distributional properties.

Let $x_j = \xi_j$, $j \in \mathbb{Z}$ be a stationary process as in (8). Below we denote $f_{\xi,0} := f_\xi(0)$ and $f_{\zeta,0} := f_\zeta(0)$. Recall that the standardized i.i.d. sequence $\{\zeta_j\}$ has the spectral density $f_\zeta(u) = 1/2\pi$. Then, by Lemma 3, see Appendix II,

$$\{w_\xi(u_0), w_\xi(u_1), \cdots, w_\xi(u_p)\} = (f_{\xi,0}/f_{\zeta,0})^{1/2}\{w_\zeta(u_0), w_\zeta(u_1), \cdots, w_\zeta(u_p)\} + o_p(1), \tag{10}$$
$$\{I_\xi(u_1), \cdots, I_\xi(u_p)\} = (f_{\xi,0}/f_{\zeta,0})\{I_\zeta(u_1), \cdots, I_\zeta(u_p)\}) + o_p(1).$$

By (40) of Lemma 4,

$$f_{\zeta,0}^{-1/2}\{w_\zeta(u_0), w_\zeta(u_1), \cdots, w_\zeta(u_p)\} \to_D \left\{Z_0, \frac{Z_1 + iZ_2}{\sqrt{2}}, \cdots, \frac{Z_{2p-1} + iZ_{2p}}{\sqrt{2}}\right\},$$
$$f_{\zeta,0}^{-1}\{I_\zeta(u_1), \cdots, I_\zeta(u_p)\} \to_D \left\{\frac{Z_1^2 + Z_2^2}{2}, \cdots, \frac{Z_{2p-1}^2 + Z_{2p}^2}{2}\right\} =: \{\tau_1, \cdots, \tau_p\}, \tag{11}$$

where $Z_0, \cdots, Z_{2p}$ is a vector of independent standardized normal r.v.'s. This implies

$$\{I_\xi(u_1), \cdots, I_\xi(u_p)\} \to_D f_{\xi,0}\{\tau_1, \cdots, \tau_p\}.$$

Suppose now that the sample $x_0, x_1, \cdots, x_n$ comes from a unit root model

$$x_j = x_{j-1} + \xi_j, \qquad j = 1, \cdots, n, \tag{12}$$

where $\{\xi_j\}$ is a stationary short memory process as shown in (8) with the spectral density f_ξ. Firstly, recall the algebraic relation which first was observed in Phillips (1999).

Lemma 1 *(Phillips 1999) For $j = 1, \cdots, n-1$, with $\nabla x_j = x_j - x_{j-1}$,*

$$w_X(u_j) = (1 - e^{iu_j})^{-1}\{w_{\nabla X}(u_j) - e^{iu_j} w_{\nabla X}(0)\}, \tag{13}$$
$$I_X(u_j) = |1 - e^{iu_j}|^{-2}|w_{\nabla X}(u_j) - e^{iu_j} w_{\nabla X}(0)|^2.$$

Note that in (13), for any fixed $j \geq 1$, as $n \to \infty$,

$$e^{iu_j} = 1 + O(n^{-1}), \qquad e^{iu_j} - 1 = iu_j(1 + O(n^{-1})). \tag{14}$$

Hence, for x_j as in (12), by (13)-(14) , we can write

$$w_X(u_j) = (1 - e^{iu_j})^{-1}\{w_\xi(u_j) - e^{iu_j} w_\xi(0)\} = -(iu_j)^{-1}(f_{\xi,0}/f_{\zeta,0})^{1/2}\{w_\zeta(u_j) - w_\zeta(0)\}(1 + o_p(1)).$$

Thus, by (13)-(14) and (11), for a fixed number p of low frequencies,

$$u_1^2\{I_X(u_1), \cdots, I_X(u_p)\} = f_{\xi,0}f_{\zeta,0}^{-1}\{1^{-2}|w_\zeta(u_1) - w_\zeta(0)|^2, \cdots, p^{-2}|w_\zeta(u_p) - w_\zeta(0)|^2\} + o_p(1)$$
$$\to_D f_{\xi,0}\left\{1^{-2}\frac{(Z_1 - Z_0)^2 + Z_2^2}{2}, \cdots, p^{-2}\frac{(Z_{2p-1} - Z_0)^2 + Z_{2p}^2}{2}\right\}$$
$$=: f_{\xi,0}\left\{1^{-2}U_1, \cdots, p^{-2}U_p\right\}, \tag{15}$$

where $Z_0, Z_1, \cdots, Z_{2p}$ is a vector of independent standardized normal r.v.'s.

Testing H_0 vs H_S. From the above it becomes clear that by using the periodogram one can test the hypothesis both for existence of unit root and stationarity. In this respect, we introduce a test, the power of which tends to 1 as the sample size increases. We define the test statistic as

$$\hat{Q}_{Y;\, l\cdots k;p\cdots q} := \frac{n^2}{(2\pi)^2}\frac{I_{\nabla Y}(u_l) + \cdots + I_{\nabla Y}(u_k)}{I_Y(u_p) + \cdots + I_Y(u_q)},$$

where the integers $1 \leq l \leq k$ and $1 \leq p \leq q$ specify the range of frequencies, and the periodograms in the numerator are computed using the differenced series $\nabla y_t = y_t - y_{t-1}$. Note that in our test l, k, p and q are fixed *a priori* and the test based on $\hat{Q}_{Y;\, l\cdots k;p\cdots q}$ is well sized so long as u_k and u_q are close enough to zero.

Theorem 1 *(H_0) Suppose that $y_j = x_j + \varepsilon_j$, $j = 1, \cdots, n$ is as in (2) and $\{\xi_j\}$ and $\{\varepsilon_j\}$ satisfy Assumption A. Let τ_j, U_j be defined as in (11) and (15).*

Then for fixed $k \geq l \geq 1$ and $q \geq p \geq 1$, as $n \to \infty$,

$$\hat{Q}_{Y;\, l\cdots k;p\cdots q} \to_D Q^U_{l\cdots k;p\cdots q} := \frac{\tau_l + \cdots + \tau_k}{p^{-2}U_p + \cdots + q^{-2}U_q}, \tag{16}$$

where the limit has parameter-free distribution.

(H_S) Suppose that $y_j = \mu + \xi_j$, $j = 1, \cdots, n$ is as in (3) and ξ_j's satisfy Assumption A. Then for fixed $k \geq l \geq 1$ and $q \geq p \geq 1$, as $n \to \infty$,

$$\{n(y_n - y_0)^2\}^{-1}\hat{Q}_{Y;\, l\cdots k;p\cdots q} \to_D \frac{k}{(2\pi)^3 f_{\xi,0}} \frac{1}{\tau_p + \cdots + \tau_q}. \tag{17}$$

Testing leads to a parameter-free null distribution $Q^U_{l\cdots k;p\cdots q}$ that depends only on the range of Fourier frequencies (integers l, k, p, q) used in the test. For example,

$$Q^U_{1;1\ldots 2} := \frac{\tau_1}{U_1 + 2^{-2}U_2}, \qquad Q^U_{2;1\ldots 3} := \frac{\tau_2}{U_1 + 2^{-2}U_2 + 3^{-2}U_3},$$
$$Q^U_{3\ldots 5;2\ldots 3} := \frac{\tau_3 + \tau_4 + \tau_5}{2^{-2}U_2 + 3^{-2}U_3}.$$

The consistency rate under alternative H_S equals n.

The test $Q^U_{l\ldots k;p\ldots q}$ is not sensitive to selection of tuning parameters l, k, p, q which define the Fourier frequencies used in the test as long they are relatively small. In our simulation study we tried a number of frequency combinations in the vicinity of zero and the testing results were little changed.

The test for a unit root is conducted as follows. For a given significance level $\alpha \in (0, 1)$, find the critical value $\bar{c}^U_{lk;pq}$ defined by

$$P(Q^U_{l\cdots k;p\cdots q} \geq \bar{c}^U_{lk;pq}) = \alpha.$$

Rule: reject H_0 (unit root, no drift) in favor of H_S (stationarity, no drift), if $\hat{Q}_{Y;\, l\cdots k;p\cdots ,q} \geq \bar{c}^U_{lk;pq}$. Such a test has asymptotic size α, and asymptotic power 1 with consistency rate n. Critical values $\bar{c}^U_{lk;pq}$ are parameter-free and for specific l, k, p, q can be computed using Monte Carlo simulations. For the different choices of frequencies of the Q test used in our simulation study, the respective critical values based on 10,000 replications and $n = 2014$ are given in Table 1.

Table 1: Critical values at 5% and 10% significance level for unit root Q testing

H_0: unit root, no trend	C.V. (5%)	C.V. (10%)	H_0: unit root, with trend	C.V. (5%)	C.V. (10%)
$\hat{Q}_{Y;3\ldots 10;1\ldots 2}$	27.80	19.01	$\hat{Q}_{\hat{Y};3\ldots 10;1\ldots 2}$	78.53	51.61
$\hat{Q}_{Y;3\ldots 8;1\ldots 3}$	14.63	10.83	$\hat{Q}_{\hat{Y};3\ldots 8;1\ldots 3}$	33.37	23.99
$\hat{Q}_{Y;3\ldots 7;1\ldots 4}$	10.55	8.01	$\hat{Q}_{\hat{Y};3\ldots 7;1\ldots 4}$	20.84	15.97

Testing with detrending. Next, we discuss testing of hypothesis H_0^T against H_S^T. Under H_0^T we assume that a sample $y_0, \cdots, y_n$ is generated by a unit root model with a drift (4). Such a test is based on statistic $\hat{Q}_{\hat{Y};\, l\cdots k;p\cdots q}$ computed using residuals $\hat{y}_j = y_j - \hat{\beta}j$, $j = 0, \cdots, n$ where $\hat{\beta}$ is the OLS estimator (6). Detrending will affect the limit distribution of the test statistic $\hat{Q}_{\hat{Y};l\cdots k;p\cdots q}$ under the null: instead of (16) it is now described by distribution of a random variable

$$Q^U_{T,l\cdots k;p\cdots q} := \frac{\tau_l + \cdots + \tau_k}{p^{-2}U'_p + \cdots + q^{-2}U'_q}. \tag{18}$$

The variables τ_j, U'_j are defined as follows. Let $Z'_0, Z_1, Z_2 \cdots$, be a vector of zero mean normal r.v.'s, such that $Z_1, Z_2 \cdots$, are independent standardized normal r.v.'s, while Z'_0 has variance $EZ_0'^2 = 1/5$, and is uncorrelated with Z_{2j} but correlated with Z_{2j-1}: $EZ_{2j}Z'_0 = 0$ and $EZ_{2j-1}Z'_0 = 3/(\pi j)^2$ for $j \geq 1$. We set $\tau_j := (Z^2_{2j} + Z^2_{2j-1})/2$ and $U'_j := ((Z_{2j} - Z'_0)^2 + Z^2_{2j-1})/2$, $j \geq 1$.

To define asymptotic distribution of the test statistic under alternative H^T_S, introduce a normal zero mean random variable Z''_0 with variance $EZ_0''^2 = 3$, such that $EZ_{2j}Z''_0 = 3(\pi j)^{-1}$ and $EZ_{2j-1}Z''_0 = 0$ for $j \geq 1$, and set $U''_j := \big(Z_{2j} + (Z_{2j-1} - (\pi j)^{-1}Z''_0)^2\big)/2$, $j \geq 1$.

Theorem 2 *(H^T_0) Suppose that $y_j = \mu + \beta j + x_j + \varepsilon_j$, $j = 1, \cdots, n$ is as in (4) and $\{\xi_j\}$ and $\{\varepsilon_j\}$ satisfy Assumption A. Then, for fixed $k \geq l \geq 1$ and $q \geq p \geq 1$, as $n \to \infty$,*

$$\hat{Q}_{\hat{Y};l\cdots k;p\cdots q} \to_D Q^U_{T;l\cdots k;p\cdots q}, \tag{19}$$

where $Q^U_{T,l\cdots k;p\cdots q}$ is as in (18) and has parameter-free distribution.

(H^T_S) Suppose that $y_j = \mu + \beta j + \xi_j$, $j = 1, \cdots, n$ is as in (5) and $\{\xi_j\}$ satisfy Assumption A. Then for fixed $k \geq l \geq 1$ and $q \geq p \geq 1$, as $n \to \infty$,

$$\{n(y_n - y_0)^2\}^{-1}\hat{Q}_{\hat{Y};l\cdots k;p\cdots q} \to_D \frac{k}{(2\pi)^3 f_{\xi,0}} \frac{1}{U''_p + \cdots + U''_q}. \tag{20}$$

Testing H^T_0 vs H^T_S. To construct a test for H^T_0 against H^T_S, for a given significance level $\alpha \in (0,1)$ define the critical value $\bar{c}^U_{T,lk;pq}$ by

$$P(Q^U_{T;l\cdots k;p\cdots q} \geq \bar{c}^U_{T;lk;pq}) = \alpha.$$

Since the limit distribution is parameter-free, the critical value for different choices of l, k, p, q can be found using Monte Carlo simulations.

Rule: reject H^T_0 (unit root with a drift) in favor of H_S (stationarity with a drift), if $\hat{Q}_{\hat{Y};l\cdots k;p\cdots q} \geq \bar{c}^U_{T;lk;pq}$. Such a test has asymptotic size α, and asymptotic power 1. For the different choices of l, k, p, q used in our simulation study, the respective critical values based on $10,000$ replications are given in Table 1.

3 Monte Carlo study

This section contains Monte Carlo simulations illustrating the finite sample performance of the unit root frequency-type Q test. Under the null hypothesis we consider the following data generating process

$$y_j = \mu + \beta j + x_j + \sqrt{\varsigma}\varepsilon_j, \text{ for } j = 1, ..., n, \tag{21}$$

where a unit root process $x_j = \sum_{k=1}^{j} \xi_j$ is contaminated by the noise $\sqrt{\varsigma}\varepsilon_j$, and $\{\xi_j\}$, $\{\varepsilon_j\}$ are two independent stationary short memory processes. We consider three specifications for $\{\xi_j\}$:

(s1) AR(1) $\xi_j = \rho\xi_{j-1} + \eta_j$,

(s2) MA(1) $\xi_j = \eta_j - \theta\eta_{j-1}$,

(s3) ARMA(1,1) $\xi_j = \rho\xi_{j-1} + \eta_j - \theta\eta_{j-1}$, and assume that $\{\varepsilon_j\}$ is a stationary AR(1) process $\varepsilon_j = r\varepsilon_{j-1} + u_j$, where $\{\eta_j\}$ and $\{u_j\}$ are two independent standard normal iid$(0,1)$ processes, with a burn-in period of 500 observations. Parameters ρ, θ and r control the strength of dependence the AR or MA processes $\{\xi_j\}$ and $\{\varepsilon_j\}$, β and μ define a linear trend and the scaling parameter ς controls contamination of the unit root by the noise.

We consider two broad settings of process (21).

Experiment A (No additive noise, $\varsigma = 0$). This setup corresponds to the classical specification (21) of the unit root (non-drift and with a drift) hypothesis H_0 and H_0^T. Under the null we set $\rho = 0.0, 0.3, 0.5, 0.8$ in (s1), $\theta = 0.0, 0.3, 0.5, 0.8$ in (s2), $\rho = 0.3$ and $\theta = 0.0, 0.4, 0.5, 0.8$ in (s3).

Under trend stationary alternatives H_S and H_S^T, $y_j = \mu + \beta j + \xi_j$, we set $\rho = 0.8, 0.9, 0.95, 0.99$ in (s1), $\theta = 0.0, 0.5, 0.8, 0.9$ in (s2), and $\rho = 0.3$ and $\theta = 0.0, 0.5, 0.8, 0.9$ in (s3). Selection of trend parameters $\beta = 0$, $\mu = 0$ and $\beta = 0.5$, $\mu = 1$ covers non-trended and trended alternatives. Notice that the Q test statistic in invariant with respect to the values of parameters β and μ.

Experiment B (Additive noise, $\varsigma > 0$). This setup corresponds to a unit root process x_t contaminated by noise $\{\varepsilon_j\}$. It covers the unit root hypothesis H_0, $y_j = x_j + \sqrt{\varsigma}\varepsilon_j$, where $x_j = x_{j-1} + \eta_j$ is a random walk, $\{\eta_j\}$ is the standardized normal iid$(0,1)$ noise, $\{\varepsilon_j\}$ follows an AR(1) model with $r = 0.5$, and the scale parameter takes values $\varsigma = 0.0, 1.0, 3.0, 4.0$. It also covers the trend stationary alternative H_S^T, $y_j = 1 + 0.5j + \xi_j + \sqrt{\varsigma}\varepsilon_j$ where $\{\xi_j\}$ is AR(1) process (s1) with parameter $\rho = 0.5$ and $\{\varepsilon_j\}$ are the same as under the null.

Cochrane (1991) raised the issue of susceptibility of unit root testing procedures to size and power distortions when a unit root is contaminated by an additive stationary noise. We use Experiment B to analyze such distortions for the Q test, conventional Dickey Fuller test with 0 and 4 lags and the variance-type unit root test of Breitung (2002) which does not impose any structural form assumptions on the unit root process. In addition, we have conducted ADF test with 0,1,2,3,4,5 of lags to cover a sufficiently large range of lags that potentially could be used by lag selection criteria. These testing results are available upon request.

In both experiments we apply the Q test $\hat{Q}_{Y;3...10;1...2}$ based on frequencies 3-10 in the numerator and 1-2 in the denominator, and asymptotic critical values given in Table 1. Different combinations of frequencies, have been tried, e.g. $\hat{Q}_{Y;3...7;1...4}$ or $\hat{Q}_{Y;3...8;1...3}$ and results are little changed. The results are available upon request. The relevant critical values are given in Table 1. Testing for a unit root with a drift, in line with Theorem 2, statistic $\hat{Q}_{\hat{Y};3...10;1...2}$ is applied to de-trended data $\hat{y}_j = y_j - \hat{\beta}j$. We consider sample sizes $n = 64, 128, 256, 1024$ and $2,000$ replications.

The 5% critical values (no drift) for $\hat{Q}_{Y;3...10;1...2}$, ADF and Breitung tests are 27.8, 1.94 and 0.02, respectively. The 5% critical values (with drift) for these tests are 78.5, 3.41 and 0.0035.

3.1 Simulation results

Table 2 reports size and power results for the $\hat{Q}_{Y;3...10;1...2}$ test for Experiment A. We consider the non-drift unit root hypothesis $y_j = x_j$ and its stationary alternative, $y_j = \xi_j$, as well as a unit root with a drift hypothesis $y_j = 1 + 0.5j + x_j$ with the trend-stationary alternative $y_j = 1 + 0.5j + \xi_j$, where $x_j - x_{j-1} = \xi_j$ is a stationary process as in (s1)-(s3).

Under AR(1) specification (s1) of $\{\xi_j\}$, the Q test applied to a unit root process $y_j = x_j$ has correct size for all sample sizes n studied when $\{\xi_j\}$ is a random walk ($\rho = 0$), but when persistence in ξ_j increases (ρ approaches to 0.8) the test becomes slightly under-sized for small n. Under the stationary alternative, $y_j = \xi_j$, the power of the Q test increases with n, and it starts loosing power in the near unit root region as ρ approaches to 1 which is in line with the existing literature on unit root testing. For a unit root with a drift, $y_j = 1 + 0.5j + x_j$, Table 2 reports similar size results as for a unit root with no drift, while under the trend-stationary alternative the test suffers some loss of power for small n but it recovers as the sample size increases.

Next, we look at size and power under MA(1) specification (s2) of $\{\xi_j\}$. In the non-drift case $\hat{Q}_{Y;3...10;1...2}$ test is correctly sized for small θ and for all n, and only becomes significantly over-sized when ξ_j approaches non-invertibility, e.g. for $\theta = 0.8$. Under stationary MA(1) alternative the power is universally strong and close to 1, reflecting the fact that the spectral density of an MA(1) process $\{\xi_j\}$ is relatively flat at zero frequency. Size and power of the Q test allowing for a drift are similar as in the non-drift case.

Under ARMA(1,1) specification (s3) of $\{\xi_j\}$, the Q test has similar size properties as those obtained for MA(1) process $\{\xi_j\}$, while power remains strong for both specifications of the Q test, excluding and allowing for a drift.

Table 2: Rejection rates of $Q_{3...10;1...2}$ unit root test (5%)

	Model: $y_j = x_j$, no trend								Model: $y_j = 1 + 0.5j + x_j$, with trend							
	Size: $x_j = x_{j-1} + \xi_j$				Power: $x_j = \xi_j$				Size: $x_j = x_{j-1} + \xi_j$				Power: $x_j = \xi_j$			
								$\xi_j \sim$ AR(1)								
$\rho =$	0.00	0.30	0.50	0.80	0.80	0.90	0.95	0.99	0.00	0.30	0.50	0.80	0.80	0.90	0.95	0.99
$n = 64$	0.05	0.04	0.02	0.01	0.53	0.20	0.11	0.06	0.04	0.03	0.02	0.01	0.25	0.11	0.07	0.05
$n = 128$	0.05	0.04	0.03	0.01	0.87	0.48	0.20	0.07	0.05	0.05	0.04	0.01	0.55	0.23	0.11	0.05
$n = 256$	0.04	0.04	0.04	0.03	0.98	0.85	0.47	0.09	0.05	0.05	0.05	0.03	0.80	0.53	0.23	0.06
$n = 1024$	0.05	0.05	0.05	0.05	1.00	0.99	0.97	0.36	0.05	0.05	0.05	0.04	0.93	0.90	0.79	0.17
								$\xi_j \sim$ MA(1)								
$\theta =$	0.00	0.30	0.50	0.80	0.00	0.50	0.80	0.90	0.00	0.30	0.50	0.80	0.00	0.50	0.80	0.90
$n = 64$	0.05	0.07	0.13	0.61	1.00	1.00	1.00	1.00	0.04	0.06	0.12	0.54	0.93	1.00	1.00	1.00
$n = 128$	0.06	0.06	0.08	0.28	1.00	1.00	1.00	1.00	0.05	0.06	0.07	0.26	0.94	0.99	1.00	1.00
$n = 256$	0.05	0.06	0.06	0.11	1.00	1.00	1.00	1.00	0.05	0.05	0.06	0.12	0.95	0.98	1.00	1.00
$n = 1024$	0.05	0.05	0.05	0.06	1.00	1.00	1.00	1.00	0.05	0.05	0.05	0.06	0.97	0.98	1.00	1.00
								$\xi_j \sim$ ARMA(1,1)								
$\rho =$	0.30	0.30	0.30	0.30	0.30	0.30	0.30	0.30	0.30	0.30	0.30	0.30	0.30	0.30	0.30	0.30
$\theta =$	0.00	0.40	0.50	0.80	0.00	0.50	0.80	0.90	0.00	0.40	0.50	0.80	0.00	0.50	0.80	0.90
$n = 64$	0.03	0.06	0.09	0.47	0.99	1.00	1.00	1.00	0.03	0.06	0.08	0.41	0.86	0.98	1.00	1.00
$n = 128$	0.04	0.05	0.07	0.24	0.99	1.00	1.00	1.00	0.05	0.06	0.07	0.21	0.92	0.96	1.00	1.00
$n = 256$	0.04	0.05	0.05	0.10	1.00	1.00	1.00	1.00	0.05	0.05	0.05	0.09	0.94	0.95	0.99	1.00
$n = 1024$	0.05	0.05	0.05	0.05	1.00	1.00	1.00	1.00	0.05	0.05	0.05	0.05	0.95	0.97	0.99	1.00

Settings of Experiment B in Table 3 reflect more closely real unit root data which can be contaminated by additional stationary noise. They cover unit root model with no drift, $y_j = x_j + \sqrt{\varsigma}\varepsilon_j$, and with a drift, $y_j = 1 + 0.5j + x_j + \sqrt{\varsigma}\varepsilon_j$, and corresponding stationary, $y_j = \xi_j + \sqrt{\varsigma}\varepsilon_j$, and trend-stationary, $y_j = 1 + 0.5j + x_j + \sqrt{\varsigma}\varepsilon_j$, alternatives, where x_j is a random walk, $\{\varepsilon_j\}$ is an AR(1), $r = 0.5$, noise, and $\{\xi_j\}$ is an AR(1), $\rho = 0.5$, process. Table 3 shows that for a random walk free of additional noise ($\varsigma = 0.0$), the Q test, the ADF test based on the lags 0 and 4 and Breitung test all have correct size (both non-drift and drift versions). All tests become over-sized, when the dominance of the noise $\{\varepsilon_j\}$ over the unit root process x_j increases, especially for smaller n. Overall, the variance-ratio test of Breitung shares similar properties with the Q test. In the non-drift case size of the Breitung test remains more stable for small n when ς changes, however, for those n its power is compromised considerably, while the power of the Q test is universally high across ς and n. For a unit root model with a drift, the Breitung and Q tests suffer similar size distortions in small samples when

ς rises, but the size of the Q test improves somewhat faster when n increases. In turn, testing for a unit root contaminated by a noise using ADF test, it is not clear how to justify the additional lags in the augmentation of the test given that the observed process is a pure random walk plus a stationary AR(1) noise. The DF test (ADF with 0 lags) has incorrect size which improves when the number of lags increases to 4. However, the ADF test with 4 lags still records incorrect size as ς rises even as n increases. Further, while under the stationary alternative the standard DF test has universally high power, the augmented ADF test with 4 lags suffers more gravely than either the Q or Breitung tests when n is small.

Overall, the Monte Carlo results support the theoretical findings of Section 2 about asymptotic properties of the Q test. They also highlight an important problem of noise contamination that potentially arises in empirical work and suggest that those tests that do not impose structural form assumptions on the data generating process seem to perform better asymptotically.

Table 3: Size and power of Q, ADF and Breitung tests (5%) for a unit root model $x_j = x_{j-1} + \xi_j$ with an additive noise and a (trend)stationary alternative $x_j = \xi_j$ where $\xi_j \sim$ AR(1), $\rho = 0, 0.5$ and $\varepsilon_j \sim$ AR(1), $r = 0.5$

	Model: $y_j = x_j + \sqrt{\varsigma}\varepsilon_j$, no trend								Model: $y_j = 1 + 0.5j + x_j + \sqrt{\varsigma}\varepsilon_j$, with trend							
	Size: $x_j = x_{j-1} + \xi_j$				Power: $x_j = \xi_j$				Size: $x_j = x_{j-1} + \xi_j$				Power: $x_j = \xi_j$			
	Parameters															
$\rho =$	0.00	0.00	0.00	0.00	0.50	0.50	0.50	0.50	0.00	0.00	0.00	0.00	0.50	0.50	0.50	0.50
$\varsigma =$	0.00	1.00	3.00	4.00	0.00	1.00	3.00	4.00	0.00	1.00	3.00	4.00	0.00	1.00	3.00	4.00
	$Q_{3\ldots10;1\ldots2}$ test															
$n = 64$	0.05	0.12	0.24	0.28	0.96	0.96	0.96	0.96	0.04	0.10	0.20	0.24	0.74	0.74	0.73	0.73
$n = 128$	0.05	0.08	0.15	0.17	0.99	1.00	1.00	1.00	0.05	0.07	0.12	0.15	0.89	0.89	0.89	0.89
$n = 256$	0.04	0.06	0.07	0.09	1.00	1.00	1.00	1.00	0.05	0.06	0.07	0.08	0.93	0.93	0.93	0.93
$n = 1024$	0.05	0.05	0.05	0.05	1.00	1.00	1.00	1.00	0.05	0.05	0.05	0.05	0.96	0.95	0.95	0.95
	ADF test - 0 lags															
$n = 64$	0.07	0.21	0.40	0.48	1.00	1.00	1.00	1.00	0.06	0.28	0.59	0.67	0.99	0.99	0.99	0.99
$n = 128$	0.06	0.21	0.44	0.51	1.00	1.00	1.00	1.00	0.05	0.35	0.75	0.83	1.00	1.00	1.00	1.00
$n = 256$	0.07	0.23	0.47	0.54	1.00	1.00	1.00	1.00	0.06	0.39	0.81	0.89	1.00	1.00	1.00	1.00
$n = 1024$	0.06	0.23	0.46	0.54	1.00	1.00	1.00	1.00	0.05	0.43	0.85	0.92	1.00	1.00	1.00	1.00
	ADF test - 4 lags															
$n = 64$	0.08	0.08	0.13	0.15	0.96	0.96	0.96	0.95	0.08	0.10	0.14	0.16	0.46	0.46	0.46	0.46
$n = 128$	0.06	0.08	0.13	0.15	1.00	1.00	1.00	1.00	0.06	0.09	0.18	0.22	0.96	0.96	0.97	0.97
$n = 256$	0.06	0.07	0.13	0.17	1.00	1.00	1.00	1.00	0.05	0.09	0.20	0.27	1.00	1.00	1.00	1.00
$n = 1024$	0.06	0.08	0.14	0.17	1.00	1.00	1.00	1.00	0.05	0.09	0.22	0.30	1.00	1.00	1.00	1.00
	Breitung Variance-type test															
$n = 64$	0.05	0.06	0.08	0.09	0.63	0.63	0.65	0.64	0.05	0.10	0.22	0.25	0.76	0.76	0.77	0.77
$n = 128$	0.04	0.05	0.06	0.07	0.82	0.82	0.82	0.82	0.05	0.09	0.17	0.20	0.96	0.96	0.97	0.97
$n = 256$	0.05	0.06	0.07	0.07	0.95	0.95	0.95	0.95	0.05	0.08	0.13	0.15	1.00	1.00	1.00	1.00
$n = 1024$	0.05	0.05	0.05	0.05	1.00	1.00	1.00	1.00	0.05	0.06	0.07	0.08	1.00	1.00	1.00	1.00

4 Empirical application

In this section we apply the Q test for unit root to the popular macroeconomic data set analysed by Nelson and Plosser (1982) and its extended version used in Schotman and van Dijk (1991). The original Nelson-Plosser data set contains 14 annual aggregated U.S. macroeconomic time series, recorded over 62 to 111 years, and all ending in 1970, and its extended version includes observations up to 1988. List of the variables and sample sizes is given in Table 4. Nelson and Plosser (1982) found that apart from the unemployment rate the DF test failed to reject the null hypothesis of existence of a unit root in every other case. Since then this data set has been subject to extensive scrutiny. While numerous empirical studies support the notion that the majority of macroeconomic time series do in fact contain a unit root, others suggest that failure to reject a unit root lies in the nature of testing procedure, and that the Nelson and Plosser (1982) results might be questionable, see Perron (1988), Dejong et al. (1992), Phillips (1991), Kwiatkowski et al. (1992), Lucas (1995) and references therein.

To compare the Q testing results for unit root with existing findings in the literature, we apply the $\hat{Q}_{Y;3...10;1...2}$ test (as in the simulation study) to the log-transformation of the 14 Nelson -Plosser series and their extended versions. (Note that money stock (M2) series is not considered in the extended data set.) Table 4 panel I shows that $\hat{Q}_{Y;3...10;1...2}$ falls consistently below the 5% and 10% critical values of 27.8 and 19.01, respectively, failing to reject a unit root in all but the unemployment rate series in the original and extended data sets. This is in line with the findings by Nelson and Plosser (1982) using the DF test, and Kwiatkowski et al. (1992) employing the KPSS test who reject the hypothesis of stationarity for bond yields as well. Interestingly, the value of the test statistic $\hat{Q}_{Y;3...10;1...2}$ for bond yields is the most elevated after the unemployment rate in the original and extended data sets, though well below the 5% and 10% critical values.

There is evidence of potential existence of a trend in all macroeconomic variables of the Nelson-Plosser data set, see Harvey et al. (2009) for a comprehensive analysis of the uncertainty of presence of a linear deterministic trend in data. Therefore, we also test the hypothesis of existence of a unit root with a drift using statistic $\hat{Q}_{\hat{Y};3...10;1...2}$ based on residuals $\hat{y}_j = y_j - \hat{\beta} j$ with 5% and 10% critical values of 78.53 and 51.61 respectively. Table 4 panel II shows that at 5% significance level the hypothesis of unit root with a drift is again rejected only for the unemployment rate while all other series appear to have a unit root, as suggested in Nelson and Plosser (1982). This holds both for the series ending in 1970 and 1988. Closer inspection shows that the value of $\hat{Q}_{\hat{Y};3...10;1...2}$ becomes significantly higher for real per capita GNP and employment when the extended data set is used. At 10% significance level the Q test indicates that real per capita GNP, employment and the unemployment rate are trend-stationary processes. Kwiatkowski et al. (1992) and Phillips (1991) obtain similar results using their respective methods. Unlike the Q test they find trend-stationarity for the GNP deflator, wages and money as well.

In the context of Nelson and Plosser (1982) and Kwiatkowski et al. (1992), Q test results support the notion that the unemployment rate is stationary and that real GNP, nominal GNP, consumer prices, real wages, velocity, bond yields and stock prices contain a unit root, while evidence on real per capita GNP and employment containing a unit root remains inconclusive.

Table 4: $Q_{Y;3...10;1...2}$ test results for unit root in Nelson-Plosser data set (5% and 10% significance levels). All variables are in levels (log transformed).

	Panel I: no trend						Panel II: with trend					
	Data up to 1970			Data up to 1988			Data up to 1970			Data up to 1988		
Macro Variables	n	Q-test	Reject H_0	n	Q-test	Reject H_0	n	Q-test	Reject H_0	n	Q-test	Reject H_0
Real GNP	62	0.99	no	80	0.49	no	62	28.76	no	80	45.76	no
Nominal GNP	62	0.73	no	80	0.25	no	62	9.72	no	80	13.97	no
Real per capita GNP	62	2.69	no	80	1.33	no	62	29.36	no	80	52.36	yes[1]
Industrial production	111	0.36	no	129	0.24	no	111	36.00	no	129	32.89	no
Total employment	81	0.90	no	99	0.52	no	81	25.32	no	99	57.99	yes[1]
Total unemployment rate	81	79.18	yes	99	79.97	yes	81	83.39	yes	99	79.52	yes
GNP deflator	82	0.79	no	100	0.60	no	82	14.39	no	100	9.61	no
Consumer Price Index	111	2.17	no	129	0.96	no	111	5.84	no	129	3.10	no
Nominal wages	71	0.49	no	89	0.32	no	71	12.25	no	89	13.91	no
Real wages	71	0.49	no	89	0.29	no	71	22.29	no	89	11.07	no
Money stock	82	0.26	no	-	-	-	82	24.52	no	-	-	-
Velocity of money	102	1.26	no	120	1.56	no	102	5.45	no	120	3.15	no
Bond yields	72	5.13	no	89	4.46	no	72	4.85	no	89	6.53	no
Stock prices	100	1.90	no	118	0.94	no	100	10.17	no	118	8.50	no

Panel I: 5% and 10% critical values are 27.8 and 19.0 respectively.
Panel II: 5% and 10% critical values are 78.5 and 51.6 respectively.
[1]At 10% significance level.

Appendix I: Proofs of Theorems

This section contains proofs of the theorems.

Proof of Theorem 1. Let H_0 be true. First we verify that for a fixed $j \geq 0$,

$$w_{\nabla\varepsilon}(u_j) = O_p(n^{-1/2}), \qquad j \geq 0, \tag{22}$$
$$I_{\nabla Y}(u_j) = (f_{\xi,0}/f_{\zeta,0})|w_\zeta(u_j)|^2 + o_p(1), \qquad u_1^2 I_Y(u_j) = (f_{\xi,0}/f_{\zeta,0})j^{-2}|w_\zeta(u_j) - w_\zeta(0)|^2 + o_p(1), \quad j \geq 1.$$

To verify (22) for $j \geq 1$, use (13) to write

$$\begin{aligned} w_{\nabla\varepsilon}(u_j) &= (1 - e^{iu_j})w_\varepsilon(u_j) + e^{iu_j} w_{\nabla\varepsilon}(0) \\ &= O_p(n^{-1}) + (1 + O_p(n^{-1/2}))(2\pi n)^{-1/2}(\varepsilon_n - \varepsilon_0) = O_p(n^{-1/2}), \end{aligned} \tag{23}$$

by (10)–(11) and (14). Notice that for $j = 0$, $w_{\nabla\varepsilon}(u_0) = (2\pi n)^{-1/2}(\varepsilon_n - \varepsilon_0) = O_p(n^{-1/2})$.

To prove the second claim in (22), notice that $w_{\nabla Y}(u_j) = w_{\nabla X}(u_j) + w_{\nabla\varepsilon}(u_j) = w_\xi(u_j) + o_p(1) = (f_{\xi,0}/f_{\zeta,0})^{1/2} w_\zeta(u_j) + o_p(1)$, $j \geq 0$, by (22) and (10). To show the third claim, use Lemma 1, (22), (15) and (10), to obtain $u_1^2 I_Y(u_j) = u_1^2 |1 - e^{iu_j}|^{-2} |w_\xi(u_j) - e^{iu_j} w_\xi(0) + o_p(1)|^2 = (f_{\xi,0}/f_{\zeta,0}) j^{-2} |w_\zeta(u_j) - w_\zeta(0)|^2 + o_p(1)$.

Hence, by (22),

$$\begin{aligned} \hat{Q}_{Y;l\cdots k,p\cdots q} &= \frac{I_{\nabla Y}(u_l) + \cdots + I_{\nabla Y}(u_k)}{u_1^2\big(I_Y(u_p) + \cdots + I_Y(u_q)\big)} \\ &= \frac{|w_\zeta(u_l)|^2 + \cdots + |w_\zeta(u_k)|^2 + o_p(1)}{p^{-2}|w_\zeta(u_p) - w_\zeta(0)|^2 + \cdots + q^{-2}|w_\zeta(u_q) - w_\zeta(0)|^2 + o_p(1)} \to_D Q^U_{l\cdots k;p\cdots q}, \end{aligned}$$

by (11), proving (16).

Let H_S be true. Then $y_j = \mu + \xi_j$, where $\{\xi_j\}$ is a stationary short memory process. First we verify that for a fixed $j \geq 1$,

$$I_{\nabla Y}(u_j) = (2\pi n)^{-1}(\xi_n - \xi_0)^2 + O_p(n^{-3/2}), \qquad I_Y(u_j) = (f_{\xi,0}/f_{\zeta,0})|w_\zeta(u_j)|^2 + o_p(1). \tag{24}$$

The first claim follows noting that by (23) $w_{\nabla Y}(u_j) = w_{\nabla\xi}(u_j) = O_p(n^{-1}) + (2\pi n)^{-1/2}(\xi_n - \xi_0)$. The second claim follows from equality $I_Y(u_j) = I_\xi(u_j)$, using (10). Thus, (24) implies

$$\hat{Q}_{Y;l\cdots k,p\cdots q} = \frac{I_{\nabla Y}(u_l) + \cdots + I_{\nabla Y}(u_k)}{u_1^2\big(I_Y(u_p) + \cdots + I_Y(u_q)\big)} = \frac{k\,(2\pi n)^{-1}(\xi_n - \xi_0)^2 + O_p(n^{-3/2})}{(2\pi/n)^2(f_{\xi,0}/f_{\zeta,0})(|w_\zeta(u_p)|^2 + \cdots + |w_\zeta(u_q)|^2 + o_p(1))}$$

which together with (11) proves (17). □

Proof of Theorem 2. Let H_0^T be true. Define approximating variable

$$w'_{\zeta,0} := \sum_{j=1}^{n} \zeta_j z_{nj}, \qquad z_{nj} := (2\pi n)^{-1/2}\big(1 - n\sum_{s=j}^{n} b_{ns}\big), \tag{25}$$

where $b_{nj} := (j-\bar{j})/d_n$ and $d_n := \sum_{j=1}^n (j-\bar{j})^2$. First we show that for a fixed $j \geq 1$,

$$w_{\nabla\hat{Y}}(u_j) = (f_{\xi,0}/f_{\zeta,0})^{1/2} w_\zeta(u_j) + o_p(1), \quad I_{\nabla\hat{Y}}(u_j) = (f_{\xi,0}/f_{\zeta,0})|w_\zeta(u_j)|^2 + o_p(1), \tag{26}$$

$$u_1^2 I_{\hat{Y}}(u_j) = (f_{\xi,0}/f_{\zeta,0}) j^{-2} |w_\zeta(u_j) - w'_{\zeta,0}|^2 + o_p(1). \tag{27}$$

To show (26), observe that $w_{\nabla\hat{Y}}(u_j) = w_{\nabla X}(u_j) + w_{\nabla\varepsilon}(u_j)$, for $j \geq 1$, because $\sum_{s=1}^n e^{iu_j s} = 0$, which in view of (22) and (10) implies (26).

To verify (27), use Lemma 1 and (14), to obtain $I_{\hat{Y}}(u_j) = u_j^{-2}|w_{\nabla\hat{Y}}(u_j) - e^{iu_j} w_{\nabla\hat{Y}}(0)|^2(1+o(1))$, $j \geq 1$. We will show that

$$w_{\nabla\hat{Y}}(0) = (f_{\xi,0}/f_{\zeta,0})^{1/2} w'_{\zeta,0} + o_p(1), \tag{28}$$

which together with (26) and (14) implies (27): $u_1^2 I_{\hat{Y}}(u_j) = (f_{\xi,0}/f_{\zeta,0})^{1/2} j^{-2}|w_\zeta(u_j) - w'_{\zeta,0}|^2 + o_p(1)$.

To verify (28), recall that $\xi_j = x_j - x_{j-1}$. Then $\hat{y}_j = y_j - \hat{\beta} j = \mu + x_j + \varepsilon_j + (\beta - \hat{\beta})j$,

$$w_{\nabla\hat{Y}}(0) = (2\pi n)^{-1/2}(\hat{y}_n - \hat{y}_0) = w_\xi(0) + w_{\nabla\varepsilon}(0) - (2\pi n)^{-1/2} n(\hat{\beta} - \beta).$$

By (22), $w_{\nabla\varepsilon}(0) = o_p(1)$. Moreover, by (6),

$$\hat{\beta} - \beta = r_{X,n} + r_{\varepsilon,n}, \quad r_{X,n} := \sum_{j=1}^n b_{nj} x_j, \quad r_{\varepsilon,n} := \sum_{j=1}^n b_{nj}\varepsilon_j.$$

Because ε_j is a short memory sequence, then by (7), $r_{\varepsilon,n} = O_p(n^{-3/2})$. Finally, since $\sum_{j=1}^n b_{nj} = 0$, we can write

$$r_{X,n} = \sum_{j=1}^n b_{nj}(x_j - x_0) = \sum_{j=1}^n \big(\sum_{k=1}^j \xi_k\big) b_{nj} = \sum_{k=1}^n \xi_k \big(\sum_{j=k}^n b_{nj}\big).$$

Thus,

$$w_{\nabla\hat{Y}}(0) = w_\xi(0) - (2\pi n)^{-1/2} n r_{X,n} + o_p(1) = \sum_{j=1}^n \xi_j z_{nj} + o_p(1).$$

By Lemma 3,

$$\sum_{j=1}^n \xi_j z_{nj} = \left(\frac{f_{\xi,0}}{f_{\zeta,0}}\right)^{1/2} \sum_{j=1}^n \zeta_j z_{nj} + o_p(1),$$

which proves (28). Hence, by (26) and (27),

$$\begin{aligned} \hat{Q}_{\hat{Y};\, l\cdots k,p\cdots q} &= \frac{I_{\nabla\hat{Y}}(u_l) + \cdots + I_{\nabla\hat{Y}}(u_k)}{u_1^2\big(I_{\hat{Y}}(u_p) + \cdots + I_{\hat{Y}}(u_q)\big)} \\ &= \frac{f_{\zeta,0}^{-1}\{|w_\zeta(u_l)|^2 + \cdots + |w_\zeta(u_k)|^2\} + o_p(1)}{f_{\zeta,0}^{-1}\{p^{-2}|w_\zeta(u_p) - w'_{\zeta,0}|^2 + \cdots + q^{-2}|w_\zeta(u_q) - w'_{\zeta,0}|^2\} + o_p(1)} \to_D Q^U_{T,\, l\cdots k;p\cdots q}, \end{aligned}$$

by (41) of Lemma 4, proving (19).

Let H_S^T be true. Then $y_j = \mu + \beta j + \xi_j$, where $\{\xi_j\}$ is a stationary short memory process. Denote

$$w''_{\zeta,0} := \sum_{l=1}^{n} z'_{n,l}\zeta_l, \qquad z'_{n,j} := -(2\pi n)^{-1/2}(n^2/2)b_{nj}. \tag{29}$$

We show that for a fixed $j \geq 1$,

$$I_{\nabla\hat{Y}}(u_j) = (2\pi n)^{-1}(\xi_n - \xi_0)^2 + O_p(n^{-3/2}), \quad I_{\hat{Y}}(u_j) = \frac{f_{\xi,0}}{f_{\zeta,0}}\{|w_\zeta(u_j) - i(\pi j)^{-1}w''_{\zeta,0}|^2\} + o_p(1). \tag{30}$$

The first claim holds by (23), noting that $I_{\nabla\hat{Y}}(u_j) = I_{\nabla\xi}(u_j)$ for $j \geq 1$. To show the second claim, notice that $w_{\hat{Y}}(u_j) = w_\xi(u_j) - r_n^*$ where $r_n^* := (\hat\beta - \beta)(2\pi n)^{-1/2}\sum_{s=1}^n e^{iu_j s}s$. By (10), $w_\xi(u_j) = (f_{\xi,0}/f_{\zeta,0})^{1/2}w_\zeta(u_j) + o_p(1)$. Hence to prove (30) it suffices to show that

$$r_n^* = i(\pi j)^{-1}(f_{\xi,0}/f_{\zeta,0})^{1/2}w''_{\zeta,0} + o_p(1). \tag{31}$$

We have that $\hat\beta - \beta = \sum_{l=1}^n b_{n,l}\xi_l$ which together with (35) implies that $r_n^* \sim -i(2\pi n)^{-1/2}(\hat\beta-\beta)nu_j^{-1} = i(\pi j)^{-1}\sum_{l=1}^n z'_{n,l}\xi_l$. Then, (31) follows from Lemma 3 below. Hence, by (30),

$$\begin{aligned}\hat{Q}_{\hat{Y};l\cdots k,p\cdots q} &= \frac{I_{\nabla\hat{Y}}(u_l) + \cdots + I_{\nabla\hat{Y}}(u_k)}{u_1^2\big(I_{\hat{Y}}(u_p) + \cdots + I_{\hat{Y}}(u_q)\big)} \\ &= \frac{k\,(2\pi n)^{-1}(\xi_n - \xi_0)^2 + O_p(n^{-3/2})}{(2\pi/n)^2(f_{\xi,0}/f_{\zeta,0})\{|w_\zeta(u_p) - i(\pi p)^{-1}w''_{\zeta,0}|^2 + \cdots + |w_\zeta(u_q) - i(\pi q)^{-1}w''_{\zeta,0}|^2 + o_p(1)\}},\end{aligned}$$

which by (42) of Lemma 4, implies (20). □

Appendix II: Auxiliary Lemmas

This section contains auxiliary results.

Lemma 2 *Let $w'_{\zeta,0}$ be as in (25) and $w''_{\zeta,0}$ as in (29). Then, for a fixed $j \geq 1$, as $n \to \infty$,*

$$f_{\zeta,0}^{-1}E(w'_{\zeta,0})^2 \to 1/5, \qquad f_{\zeta,0}^{-1}Ew_\zeta(u_j)w'_{\zeta,0} \to 3(\pi j)^{-2}, \tag{32}$$

$$f_{\zeta,0}^{-1}E(w''_{\zeta,0})^2 \to 3, \qquad f_{\zeta,0}^{-1}Ew_\zeta(u_j)w''_{\zeta,0} \to i\,3(\pi j)^{-1}. \tag{33}$$

Proof. First, note that $n^{-1}\bar{j} \to \int_0^1 x dx = 1/2$, and

$$n^{-3}d_n \to \int_0^1 (x - 1/2)^2 dx = 1/12. \tag{34}$$

In addition, note that $z_{nj} = (2\pi n)^{-1/2}\big(1 - n\sum_{s=j}^n b_{ns}\big) = (2\pi n)^{-1/2}\big(1 + n\sum_{s=1}^{j-1} b_{ns}\big)$, because $\sum_{s=1}^n b_{ns} = 0$. Then,

$$f_\zeta^{-1}E(w'_{\zeta,0})^2 = 2\pi\sum_{j=1}^n z_{nj}^2 \sim n^{-1}\sum_{j=1}^n\left\{1 + \frac{12}{n^2}\sum_{k=1}^{j-1}(k - \bar{k})\right\}^2 \to \int_0^1\{1 + 12\int_0^x (u - 1/2)du\}^2 dx = 1/5,$$

which proves the first claim of (32).

To show the second claim of (32), using the equalities 1.352 from Gradshteyn and Ryzhik (1994),

$$\sum_{t=1}^n t\sin(tu) = \frac{\sin((n+1)u)}{4\sin^2(u/2)} - \frac{(n+1)\cos((2n+1)u/2)}{2\sin(u/2)},$$

$$\sum_{t=1}^n t\cos(tu) = \frac{(n+1)\sin((2n+1)u/2)}{2\sin(u/2)} - \frac{1 - \cos((n+1)u)}{4\sin^2(u/2)},$$

we obtain

$$\sum_{s=1}^{n} s e^{iu_j s} = \sum_{s=1}^{n} s\cos(u_j s) + i\sum_{s=1}^{n} s\sin(u_j s) = \frac{n}{2} - i\frac{n}{2}\frac{\cos(u_j/2)}{\sin(u_j/2)} \sim \frac{n}{2}(1 - i\frac{2}{u_j}). \qquad (35)$$

Set $J_{n,j} := f_{\zeta,0}^{-1} E w_\zeta(u_j) w'_{\zeta,0}$. Since $\sum_{s=1}^{n} e^{iu_j s} = 0$, $j = 1, \cdots, n-1$ and $\sum_{l=1}^{n} b_{nl} = 0$, then,

$$J_{n,j} = f_{\zeta,0}^{-1}(2\pi n)^{-1/2}\sum_{s=1}^{n} e^{iu_j s} z_{ns} = n^{-1}\sum_{s=1}^{n} e^{iu_j s}(1 - n\sum_{l=s}^{n} b_{nl}) = -\sum_{l=1}^{n} b_{nl}\sum_{s=1}^{l} e^{iu_j s}.$$

We have $\sum_{j=1}^{s} e^{iuj} = e^{iu}(e^{ius}-1)/(e^{iu}-1)$, $\sum_{s=1}^{n} b_{ns} = 0$ and $e^{iu_j}(e^{iu_j}-1)^{-1} = (iu_j)^{-1}(1+o(1))$. Thus,

$$J_{n,j} := -e^{iu_j}(e^{iu_j}-1)^{-1}\sum_{l=1}^{n} b_{nl}(e^{iu_j l}-1) = -(iu_j)^{-1}(1+o(1))\sum_{l=1}^{n} b_{nl} e^{iu_j l}.$$

Since $b_{nj} = (j - \bar{j})/d_n$, and $\sum_{l=1}^{n} e^{iu_j l} = 0$, then

$$\sum_{l=1}^{n} b_{nl} e^{iu_j l} = d_n^{-1}\sum_{l=1}^{n} j e^{iu_j l} \sim -12 n^{-2} i u_j^{-1}, \qquad (36)$$

by (34) and (35). Hence, $J_{n,j} \sim -(iu_j)^{-1}\{-12n^{-2}iu_j^{-1}\} \sim 3(\pi j)^{-2}$, proving (32).

To show the first claim of (33), note that $f_{\zeta,0}^{-1}E(w''_{\zeta,0})^2 = (2\pi)\sum_{l=1}^{n} z'^2_{n,l} = n^3\sum_{l=1}^{n} b_{n,l}^2/4 = n^3/4d_n \to 3$, by (34). To show the second claim, note that

$$f_{\zeta,0}^{-1} E w_\zeta(u_j) w''_{\zeta,0} = 2\pi(2\pi n)^{-1}(-n^2/2)\sum_{l=1}^{n} e^{iu_j l} b_{nl} \sim -(n/2)(-12n^{-2}iu_j^{-1}) \sim i\,3(\pi j)^{-1},$$

proving (33). □

Lemma 3 *(i) Suppose that $\{\xi_j\}$ satisfies Assumption A, and $\nu_{n,k}$, $k = 1, \cdots, n$, $n \geq 1$ is an array of (complex) numbers such that*

$$|\nu_{n,1}| + \sum_{k=2}^{n} |\nu_{n,k} - \nu_{n,k-1}| = o(1), \qquad \sum_{k=1}^{n} |\nu_{n,k}|^2 = O(1). \qquad (37)$$

Then, as $n \to \infty$,

$$\sum_{k=1}^{n} \nu_{n,k}\xi_k = (f_{\xi,0}/f_{\zeta,0})^{1/2}\sum_{k=1}^{n} \nu_{n,k}\zeta_k + o_p(1). \qquad (38)$$

(ii) Let $j \geq 1$ be fixed and z_{nk}, z'_{nk} be as in (25) and (29). Then $\{\nu_{n,k}^{(i)}\}$, $i = 1, \cdots, 5$ defined respectively as $\{n^{-1/2}\cos(u_j k)\}$, $\{n^{-1/2}\sin(u_j k)\}$, $\{n^{-1/2}e^{iu_j k}\}$, $\{(2\pi)^{1/2}z_{nk}\}$ and $\{(2\pi)^{1/2}z'_{nk}\}$, satisfy (37).

Proof. (i) By Assumption A, $\xi_j = \sum_{k=0}^{\infty} a_k\zeta_{j-k}$ where $\sum_{k=0}^{\infty}|a_k| < \infty$. Notice that $\eta_j = \xi_j - (f_{\xi,0}/f_{\zeta,0})^{1/2}\zeta_j$ is a stationary process with the spectral density $f_\eta(u) = (2\pi)^{-1}|\sum_{k=0}^{\infty} a_k e^{i\,uk} - \sum_{k=0}^{\infty} a_k|^2$. Under Assumption A, $f_\eta(u)$ is a continuous function and $f_\eta(u) \to 0$, $u \to 0$. This and (37) imply (38) by Proposition 4.3.2 in Giraitis et al. (2012).

(ii) To prove (37), it suffices to verify that for $i = 1, \cdots, 5$,

$$|\nu_{n,1}^{(i)}| + \sum_{k=2}^{n} |\nu_{n,k}^{(i)} - \nu_{n,k-1}^{(i)}| = o(1), \quad \sum_{k=1}^{n} |\nu_{n,k}^{(i)}|^2 \to c_i, \qquad (39)$$

where $c_1 = c_2 = 1/2$, $c_3 = 1$, $c_4 = 1/5$ and $c_5 = 3$.

Let $i = 1$. Then, for a fixed $j \geq 1$, by the mean value theorem, $|\cos(u_j k) - \cos(u_j(k-1))| \leq Cn^{-1}$, and so, $n^{-1/2}\{|\cos(u_j)| + \sum_{k=2}^n |\cos(u_j k) - \cos(u_j(k-1))|\} \leq Cn^{-1/2} \to 0$, proving the first claim in (39). To show the second claim, note that $n^{-1}\sum_{k=1}^n \cos^2(ku_j) = n^{-1}\sum_{k=1}^n (1 + \cos(ku_{2j})/2 = 1/2$, because $\sum_{k=1}^n e^{iku_{2j}} = 0$ and $\sum_{k=1}^n e^{-iku_{2j}} = 0$ imply $\sum_{k=1}^n \cos(ku_{2j}) = 0$ and $\sum_{k=1}^n \sin(ku_j) = 0$.

For $i = 2, 3$, (39) follows using a similar argument as for $i = 1$.

Let $i = 4$. By (25), $z_{n1} = (2\pi n)^{-1/2}$, bearing in mind that $\sum_{k=1}^n b_{nk} = 0$, while for $2 \leq k \leq n$, $|z_{nk} - z_{n,k-1}| = (2\pi n)^{-1/2}|b_{n,k-1}| = (2\pi n)^{-1/2} n|k - 1 - \bar{k}|/d_n \leq Cn^{-3/2}$ by (34). This proves the first claim in (39), while the second one is shown in (32).

Let $i = 5$. The proof is similar as for $i = 4$. From (29) it follows that $|z'_{n1}| = (2\pi n)^{-1/2}(n^2/2)|b_{n1}| \leq Cn^{-1/2}$, while for $2 \leq k \leq n$, $|z'_{nk} - z'_{n,k-1}| = (2\pi n)^{-1/2}(n^2/2)|b_{nk} - b_{n,k-1}| = (2\pi n)^{-1/2}(n^2/2)/d_n \leq Cn^{-3/2}$. Clearly, this proves the first claim in (39), while the second one is shown in (33). □

Denote $S_{n,2j-1} = n^{-1/2}\sum_{k=1}^n \cos(u_j k)\zeta_k$, $S_{n,2j} = n^{-1/2}\sum_{k=1}^n \sin(u_j k)\zeta_k$ for $j \geq 1$, $S_{n,0} = n^{-1/2}\sum_{k=1}^n \zeta_k$, $S'_{n,0} = (2\pi)^{1/2}\sum_{k=1}^n z_{nk}\zeta_k$ and $S''_{n,0} = (2\pi)^{1/2}\sum_{k=1}^n z'_{nk}\zeta_k$.

Let $Z_0, Z_1, Z_2, \cdots$ be standardized i.i.d. normal variables and Z'_0, Z''_0 be the same as in (18) and Theorem 2.

Lemma 4 *Let $\{\zeta_j\}$ be i.i.d. random variables with zero mean and variance* 1. *Then, for any fixed $p \geq 1$, as $n \to \infty$,*

$$(S_{n,0}, S_{n,1}, S_{n,2}, \ldots, S_{n,2p}) \to_D \big(Z_0, Z_1/\sqrt{2}, Z_2/\sqrt{2}, \cdots, Z_{2p}/\sqrt{2}\big), \tag{40}$$

$$(S'_{n,0}, S_{n,1}, S_{n,2}, \ldots, S_{n,2p}) \to_D \big(Z'_0, Z_1/\sqrt{2}, Z_2/\sqrt{2}, \cdots, Z_{2p}/\sqrt{2}\big), \tag{41}$$

$$(S''_{n,0}, S_{n,1}, S_{n,2}, \ldots, S_{n,2p}) \to_D \big(Z''_0, Z_1/\sqrt{2}, Z_2/\sqrt{2}, \cdots, Z_{2p}/\sqrt{2}\big). \tag{42}$$

Proof. Write $S_{n,2i-1} = \sum_{k=1}^n \delta_{nk}^{(2i-1)}\zeta_k$ and $S_{n,2i} = \sum_{k=1}^n \delta_{nk}^{(2i)}\zeta_k$ with $\delta_{nk}^{(2i-1)} := n^{-1/2}\cos(u_j k)$ and $\delta_{nk}^{(2i)} := n^{-1/2}\sin(u_j k)$, $i = 1, \cdots, p$.

Write $S_{n,0} = \sum_{k=1}^n \delta_{nk}^{(0,0)}\zeta_k$, $S'_{n,0} = \sum_{k=1}^n \delta_{nk}^{(0,1)}\zeta_k$ and $S''_{n,0} = \sum_{k=1}^n \delta_{nk}^{(0,2)}\zeta_k$ with $\delta_{nk}^{(0,0)} = n^{-1/2}$, $\delta_{nk}^{(0,1)} = (2\pi)^{1/2}z_{nk}$ and $\delta_{nk}^{(0,2)} = (2\pi)^{1/2}z'_{nk}$. Notice the following properties of the above weights:

$$|\delta_{n1}^{(l)}| + \sum_{k=2}^n |\delta_{nk}^{(l)} - \delta_{n,k-1}^{(l)}| = o(1), \quad \sum_{k=1}^n \Big(\delta_{nk}^{(l)}\Big)^2 \to 1/2, \quad l = 1, \cdots, 2p, \tag{43}$$

$$|\delta_{n1}^{(0,m)}| + \sum_{k=2}^n |\delta_{nk}^{(0,m)} - \delta_{n,k-1}^{(0,m)}| = o(1), \quad \sum_{k=1}^n \Big(\delta_{nk}^{(0,m)}\Big)^2 \to v_m, \quad m = 0, 1, 2, \tag{44}$$

where $v_0 = 1$, $v_1 = 1/5$ and $v_2 = 3$. The facts (43) and (44) are shown in (39).

Thus, by Theorem 4.3.2 of Giraitis et al. (2012) to prove (40)-(42) it suffices to verify that the variance-covariance matrices Σ_n and Σ of the corresponding vectors on the l.h.s. and the r.h.s. of (40)-(42) have property

$$\Sigma_n \to \Sigma, \quad n \to \infty. \tag{45}$$

To prove (45) in (40), note that equality $e^{iu} = \cos(u) + i\sin(u)$ and identity $\sum_{k=1}^n e^{iu_j k} = 0$, $1 \leq j < n$ imply that for any integers $j \geq 0, s \geq 0$, such that $j \neq s$ and $j + s < n$,

$$\sum_{k=1}^n \cos(u_j k)\cos(u_s k) = 0, \quad \sum_{k=1}^n \sin(u_j k)\sin(u_s k) = 0, \quad \sum_{k=1}^n \cos(u_j k)\sin(u_s k) = 0, \tag{46}$$

which implies $\Sigma_n = \Sigma$.

To verify (45) in (41), use (46) and (32). To verify (45) in (42), use (46) and (33). □

References

Abadir, K., Distaso, W., Giraitis, L., 2011. An I(d) model with trend and cycles. Journal of Econometrics 163, 186–199.

Breitung, J., 2002. Nonparametric tests for unit roots and cointegration. Journal of Econometrics 108, 343–363.

Cheung, Y.W., Lai, K.S., 1995. Lag order and critical values of the augmented Dickey-Fuller test. Journal of Business & Economic Statistics 13, 277–280.

Choi, I., Phillips, P.C.B., 1993. Testing for a unit root by frequency domain regression. Journal of Econometrics 59, 263–286.

Cochrane, J.H., 1991. A critique of the application of unit root tests. Journal of Economic Dynamics and Control 15, 275–284.

Dejong, D.N., Nankervis, J.C., Savin, N.E., Whiteman, C.H., 1992. Integration versus trend stationarity in macroeconomic time series. Econometrica 60(2), 423–433.

Dickey, D.A., Fuller, W.A., 1979. Distribution of the estimators for autoregressive time series with a unit root. Journal of the American Statistical Association 74, 427–431.

Elliot, G., Rothenberg, T.J., Stock, J.H., 1996. Efficient tests for an autoregressive unit root. Econometrica 64, 813–836.

Fan, Y., Gençay, R., 2010. Unit root tests with wavelets. Econometric Theory 26, 1305–1331.

Giraitis, L., Koul, H.L., Surgailis, D., 2012. Large Sample Inference for Long Memory Processes, Imperial College Press.

Giraitis, L., Leipus, R., Philippe, A., 2006. A test for stationarity versus trends and unit roots for a wide class of dependent errors. Econometric Theory 22, 989–1029.

Gradshteyn, I.S., Ryzhik, I.M., 1994. Tables of Integrals, Series, and Products, Elsevier Inc.

Harvey, D., Leybourne, S.J., Taylor, A.M.R., 2009. Unit root testing in practice: dealing with uncertainty over the trend and initial condition. Econometric Theory 25, 587–636.

Kwiatkowski, D., Phillips, P.C.B., Schmidt, P., Shin, Y., 1992. Testing the null hypothesis of stationarity against the alternative of a unit root: how sure are we that economic time series have a unit root? Journal of Econometrics 54, 159–178.

Lopez, J.H., 1997. The power of the ADF test. Economics Letters 57, 5–10.

Lucas, A., 1995. An outlier robust unit root test with an application to the extended Nelson-Plosser data. Journal of Econometrics 66, 153–173.

MacKinnon, J.G., 1991. Critical values for cointegration tests. New York: Oxford University Press.

Maddala, G.S., Kim, I-M., 1998. Unit roots, cointegration and structural change. Cambridge University Press.

Nelson, C., Plosser, C., 1982. Trends and random walks in macroeconomic time series. Journal of Monetary Economics 10, 139–162.

Ng, S., Perron, P., 1995. Unit root tests in ARMA models with data dependent methods for the selection of the truncation lag. Journal of the American Statistical Association 90, 268–281.

Ng, S., Perron, P., 2001. Lag length selection and the construction of unit root tests with good size and power. Econometrica 69, 1519–1554.

Perron, P., 1988. Trends and random walks in macroeconomic time series: Further evidence from a new approach. Journal of Economic Dynamics and Control 12, 297–332.

Perron, P., Qu, Z., 2007. A simple modification to improve the finite sample properties of Ng and Perron's unit root tests. Economic Letters 94, 12–19.

Phillips, P.C.B., 1987. Time series regression with unit roots. Econometrica 55, 277–301.

Phillips, P.C.B., 1991. To criticize the critics: an objective Bayesian analysis of stochastic trends. Journal of Applied Econometrics 6, 333–364.

Phillips, P.C.B., 1999. Discrete Fourier transforms of fractional processes. Cowles Foundation Discussion Paper, Yale University 1243.

Phillips, P.C.B., Perron, P., 1988. Testing for a unit root in time series regression. Biometrika 75, 335–346.

Phillips, P.C.B., Xiao, Z., 1998. A primer on unit root testing. Journal of Economic Surveys 12, 423–470.

Said, S., Dickey, D.A., 1984. Testing for unit roots in autoregressive-moving average models of unknown order. Biometrika 71, 599–607.

Shelef, A., 2015. A Gini-based unit root test. Computational Statistics & Data Analysis (In press).

Schotman, P.C., van Dijk, H.K., 1991. On Bayesian routes to unit roots. Journal of Applied Econometrics 6, 387–401.

Stock, J.H., 1994. Unit roots, structural breaks and trends, in R.F. Engle and D.L. McFadden (eds.), Handbook of Econometrics, Volume IV. North Holland, New York.

Westerlund, J., 2014. On the choice of test for a unit root when the errors are conditionally heteroskedastic. Computational Statistics & Data Analysis 69, 40–53.

Queen Mary
University of London

This working paper has been produced by
the School of Economics and Finance at
Queen Mary University of London

School of Economics and Finance
Queen Mary University of London
Mile End Road
London E1 4NS
Tel: +44 (0)20 7882 7356
Fax: +44 (0)20 8983 3580
Web: www.econ.qmul.ac.uk/research/workingpapers/