School of Economics and Finance

Preference Symmetries, Partial Differential Equations, and Functional Forms for Utility

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Working Paper No. 702 April 2013 ISSN 1473-0278

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University of London

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April 18, 2013

Abstract

A discrete symmetry of a preference relation is a mapping from the domain of choice to itself under which preference comparisons are invariant; a continuous symmetry is a one-parameter family of such transformations that includes the identity; and a symmetry field is a vector field whose trajectories generate a continuous symmetry. Any continuous symmetry of a preference relation implies that its representations satisfy a system of PDEs. Conversely the system implies the continuous symmetry if the latter is generated by a field. Moreover, solving the PDEs yields the functional form for utility equivalent to the symmetry. This framework is shown to encompass a variety of representation theorems related to univariate separability, multivariate separability, and homogeneity, including the cases of Cobb-Douglas and CES utility.

J.E.L. classification codes: C60, D01, D81.

Keywords: continuous symmetry, separability, smooth preferences, utility representation.

1 Introduction

A representation theorem asserts the equivalence between specified properties of a preference relation and the existence of a utility function with a particular structure. Examples include the familiar results connecting quasilinear preferences to additive utility functions and homothetic preferences to homogeneous representations.

This paper investigates representation theorems in the context of smooth preferences, as defined by Debreu [2]. More specifically, we take as given a preference relation $\succsim$ over $X \subset \Re^K$ that admits a utility representation $u : X \to \Re$ of class C^2. In this setting, we study how additional assumptions on $\succsim$ impose further structure on the function u.

Our approach is based on the notion of a *preference symmetry*; that is, a manipulation of the domain of alternatives under which preference comparisons are invariant. This idea

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is formalized in three interrelated definitions that will be used to state our results. Firstly, we define a "discrete symmetry" of the relation $\succsim$ to be a transformation $\tau : X \to X$ with the property that preference rankings are identical before and after the transformation is applied. We then define a "continuous symmetry" of $\succsim$ to be a one-parameter family of discrete symmetries $\sigma : X \times [0, 1) \to X$ such that $\sigma(\cdot, 0)$ is the identity map. And finally, we define a "symmetry field" of $\succsim$ to be a vector field $S : X \to \Re^K$ whose trajectories trace out a continuous symmetry.

These three definitions can be illustrated in the simple case of quasilinear preferences and $K = 2$. If the utility function has the form $u(x) = f(x_1 + h(x_2))$ for some strictly increasing f, then the transformation $\tau(x) = \langle x_1 + 1/2, x_2\rangle$ is a discrete symmetry of $\succsim$. This is because

$$\tau(x) \succsim \tau(y) \iff u(x_1 + 1/2, x_2) \geq u(y_1 + 1/2, y_2) \iff u(x) \geq u(y) \iff x \succsim y, \tag{1}$$

and so preferences are the same before and after τ is applied. Indeed, the one-parameter family of transformations $\sigma(x, \alpha) = \langle x_1 + \alpha, x_2\rangle$ is a continuous symmetry of $\succsim$, since

$$\sigma(x, \alpha) \succsim \sigma(y, \alpha) \iff u(x_1 + \alpha, x_2) \geq u(y_1 + \alpha, y_2) \iff u(x) \geq u(y) \iff x \succsim y \tag{2}$$

and $\sigma(x, 0) = x$. Moreover, the path $\sigma(x, \cdot) : [0, 1) \to X$ is the trajectory starting from x of the vector field $S(x) = \langle 1, 0\rangle$, which is thus a symmetry field of $\succsim$.

The final component of our theory is a system of partial differential equations that links a given preference symmetry to the corresponding functional form for utility, and can therefore be used to prove representation theorems.[1] This system can be constructed for any continuous symmetry σ of $\succsim$, and in particular for any σ generated by a symmetry field. The PDEs have the representation u as their unknown, and so solving the system determines the structure imposed on utility by the preference symmetry in question.

In the two-dimensional quasilinear case, the system of PDEs associated with the continuous symmetry $\sigma(x, \alpha) = \langle x_1 + \alpha, x_2\rangle$ mentioned above consists of the single equation

$$\frac{\partial \textsc{mrs}[u]^1_2(x)}{\partial x_1} = 0, \tag{3}$$

where $\textsc{mrs}[u]^1_2(x)$ denotes the marginal rate of substitution of the function u between x_1 and x_2. Here the intuition is apparent: Quasilinearity in x_1 implies that preferences, and hence tradeoffs between the two variables, will not change when we shift the first component of x. And the general solution of Equation 3 is precisely the functional form $u(x) = f(x_1 + h(x_2))$ that demonstrates the existence of an additive representation.

In summary, our theory will take a continuous symmetry or symmetry field of a smooth preference relation $\succsim$ and use it to obtain a system of PDEs in the corresponding utility function u, the solution of which has the structure imposed by the symmetry. Specifically, the first of our two main results (Theorem 2.7) will derive PDEs that are *necessary* for a given continuous symmetry. Our second main result (Theorem 2.10) will then specialize these equations to the context of a continuous symmetry generated by a vector field, and

[1] The idea of characterizing functional forms by means of partial differential equations has precursors in the work of Leontief [7] and Samuelson [11], mentioned below in Section 3.

show that here they are both *necessary and sufficient* for the symmetry. These results are developed in Section 2 below.

As a second illustration, consider the case of homothetic preferences, once again with $K = 2$. A continuous symmetry of $\succsim$ that captures homotheticity is $\sigma(x, \alpha) = e^{\alpha}x$, which follows the trajectories of the vector field $S(x) = x$. This symmetry will lead to the PDE

$$\frac{\partial \text{MRS}[u]^1_2(x)}{\partial x_1} x_1 + \frac{\partial \text{MRS}[u]^1_2(x)}{\partial x_2} x_2 = 0. \tag{4}$$

And the general solution of Equation 4 is $u(x) = f(x_1 h(x_2/x_1))$, establishing the existence of a homogeneous representation.

From the pair of examples provided thus far it may not be clear why the concept of a symmetry field is needed at all, and why our two-way result cannot be phrased directly in terms of continuous symmetries. The reason is that not every continuous symmetry is generated by a vector field, and moreover two distinct continuous symmetries can yield the same system of PDEs. Indeed, Equation 4 is also implied by the continuous symmetry $\sigma(x, \alpha) = [1 + \alpha]x$, an equivalent way of expressing the homotheticity hypothesis and one that has no associated field. (In this connection, see also Example 2.11 below.) Hence it is the correspondence between symmetry fields and PDEs that is exact, with continuous symmetries comprising a larger class of properties.

While our main results will supply the system of PDEs that follows from an arbitrary preference symmetry, they will not tell us how to solve these equations. This must be done case by case to determine the structure imposed on the utility representation. Whenever the relevant functional form can be guessed, checking that it solves the PDEs is typically straightforward. Showing that no other solutions exist could be more difficult, but here also there are some factors that make the task relatively tractable.

Observe that since $\text{MRS}[u]^1_2(x)$ involves partial derivatives of the utility function, Equations 3 and 4 are both second-order PDEs in u. This will be true also in the general case, and thus our system of equations will need to be integrated twice to obtain solutions. The first integration will be aided by the fact that the equations involve partial derivatives of marginal rates of substitution. For example, Equation 3 is manifestly equivalent to

$$\text{MRS}[u]^1_2(x) = \eta(x_2), \tag{5}$$

where η is an arbitrary function. And likewise (though less transparently), Equation 4 is equivalent to

$$\text{MRS}[u]^1_2(x) = [x_2/x_1]\hat{\eta}(x_2/x_1) \tag{6}$$

with $\hat{\eta}$ arbitrary.

In order to carry out the second required integration of our system of PDEs, we will make use of the ordinal nature of utility: Two functions represent the same preferences if and only if each is a monotone transformation of the other. As recorded in Proposition 2.1, two alternative characterizations of ordinal equivalence are (pointwise) proportionality of gradient vectors and equality of all marginal rates of substitution.[2] Therefore, if we can

[2]See, for example, the discussion of utility representations in Debreu [2, p. 606].

show that a function v has the same marginal rates of substitution as our representation u, then we can immediately conclude that u is a monotone transformation of v.

To see how this works in practice, let us return to the case of quasilinear preferences and define both $h(x_2) = \int_1^{x_2} \eta(t)^{-1}\mathrm{d}t$ and $v(x) = x_1 + h(x_2)$. It follows that

$$\textsc{mrs}[v]^1_2(x) = [h'(x_2)]^{-1} = \eta(x_2) = \textsc{mrs}[u]^1_2(x), \tag{7}$$

establishing that the functions v and u are ordinally equivalent. Hence we can conclude that there exists a monotone f such that $u(x) = f(v(x)) = f(x_1 + h(x_2))$, and thus that the solutions of Equation 3 have the desired structure. This argument can be adapted to the homothetic case in Equation 4, and will be used repeatedly to solve specific instances of our PDE system.

Several applications of the theory are provided in Section 3. We study first "univariate separable" utility representations of the form $u(x) = f(g(x_1) + h(x_2, \ldots, x_K))$, where g is a prespecified function, yielding the additive case when $g(x_1) = x_1$. We then consider the "multivariate separable" form $u(x) = f(\sum_{k=1}^K \lambda_k g_k(x_k))$, where each g_k is prespecified, which yields Cobb-Douglas utility when $g_k(x_k) = \log x_k$ and (other) CES representations when $g_k(x_k) = x_k^p$. Thirdly, we discuss "joint separability" of variables in the context of the form $u(x) = f(\lambda h(x_1, x_2) + g(x_3))$, with both g and h known; an example that can be generalized to more complicated specifications. And finally we examine homogeneity and related functional forms, showing in particular that preferences admit both homogeneous and additively separable representations if and only if they admit a CES representation. In each application we supply an exact characterization of the structured utility function in question in terms of symmetry fields, and of course by finding the trajectories of these fields we can always express the same result in terms of continuous symmetries.[3]

2 Theory

2.1 Smooth preferences and their representations

Fix an integer $K \geq 2$ and let $X \subset \Re^K$ be open and path-connected.[4] Let $\succsim$ be a weak preference relation on X represented by a utility function $u : X \to \Re$ in the sense that $\forall x, y \in X$ we have $u(x) \geq u(y) \iff x \succsim y$. As usual, we partition $\succsim$ into its asymmetric part $\succ$ indicating strict preference and its symmetric part $\sim$ indicating indifference.

We assume both that u is of class C^2 and that $\forall x \in X$ we have $\nabla_x u(x) \gg \vec{0}$.[5] These assumptions can be transferred to $\succsim$ using the work of Debreu [2], who has shown that a preference relation admits a utility representation with the desired properties if and only if it is both "smooth" and strictly monotone. Of course the content of this result lies in Debreu's definition of smoothness, but we need not be concerned here with this aspect of

[3]To demonstrate that our analytical approach is tractable in each instance, in Appendix A we provide complete proofs of all of the results in Section 3.

[4]Among other possibilities, the points in X could represent consumption bundles, lotteries, physical or temporal locations, or attribute vectors in a hedonic model.

[5]We denote the gradient of u with respect to the vector x by $\nabla_x u(x) = \langle \partial u(x)/\partial x_k \rangle_{k=1}^K$, and the zero vector by $\vec{0} \in \Re^K$.

his contribution.[6] For our purposes the result is important because it obtains the desired features of u independently of any structural properties, and also because the implication is two-way. We can thus consider the issue of differentiability to have been conclusively settled by Debreu, and can focus on the incremental assumptions on preferences needed to obtain particular functional forms for utility.

In addition to the function u taken as given throughout our analysis, many other C^2 maps will represent the same preference relation $\succsim$. These alternate representations can be described in various ways, four of which appear in the following familiar result.

Proposition 2.1. *If $v : X \to \Re$ is of class C^2 and $\forall x \in X$ we have $\nabla_x v(x) \gg \vec{0}$, then the following are equivalent:*

- **(i)** *The function v represents $\succsim$.*
- **(ii)** *There exists a C^1 function $\rho : X \to \Re_{++}$ such that $\forall x \in X$ we have*

$$\nabla_x v(x) = \rho(x) \nabla_x u(x). \tag{8}$$

- **(iii)** *For each $1 \leq k < K$ and $x \in X$ we have $\text{MRS}[u]_K^k(x) = \text{MRS}[v]_K^k(x)$.*[7]
- **(iv)** *There exists a strictly increasing, C^1 function $f : v[X] \to u[X]$ such that $u = f(v)$.*

Proof (Sketch). It is immediate that (iv) $\Longrightarrow$ (iii) $\Longrightarrow$ (ii). Moreover, Debreu [2, p. 610] shows that the preferences represented by a C^2 v with no critical points are characterized by the normalized gradient map $x \mapsto \|\nabla_x v(x)\|^{-1} \nabla_x v(x)$, and it follows that (ii) $\Longrightarrow$ (i).[8]

To confirm that (i) $\Longrightarrow$ (iv), note first that since X is path-connected, the continuous images $u[X], v[X] \subset \Re$ are also path-connected and are therefore intervals. Now, for each $\xi \in v[X]$, take any $z_\xi \in X$ such that $v(z_\xi) = \xi$ and let $f(\xi) = u(z_\xi)$. In view of (i), this leads to a well-defined function $f : v[X] \to u[X]$.[9] Furthermore, for each $x \in X$ we have $v(z_{v(x)}) = v(x)$ and hence $z_{v(x)} \sim x$ by (i), implying that $f(v(x)) = u(z_{v(x)}) = u(x)$. This establishes that $u = f(v)$.

For any $\xi_1, \xi_2 \in v[X]$ we have

$$\xi_1 \geq \xi_2 \Longleftrightarrow v(z_{\xi_1}) \geq v(z_{\xi_2}) \Longleftrightarrow z_{\xi_1} \succsim z_{\xi_2} \Longleftrightarrow u(z_{\xi_1}) \geq u(z_{\xi_2}) \Longleftrightarrow f(\xi_1) \geq f(\xi_2), \tag{9}$$

using (i), and it follows that f is both one-to-one and strictly increasing. Moreover, for any $\zeta \in u[X]$ there exists a $y_\zeta \in X$ with $u(y_\zeta) = \zeta$, so that $f(v(y_\zeta)) = u(y_\zeta) = \zeta$ and $\zeta \in f[v[X]]$. This shows that f is onto, and as a monotone bijection between intervals it must then be a homeomorphism.

Finally, it is straightforward to demonstrate that f is continuously differentiable, with $f'(\xi) = [[\partial u(x)/\partial x_1]/[\partial v(x)/\partial x_1]]_{x=z_\xi} > 0$ for each $\xi \in v[X]$. □

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[6] Roughly speaking, preferences over X are smooth if they are continuous and the indifference relation is a differentiable manifold when viewed as a subset of $\Re^{2K}$.

[7] We denote the marginal rates of substitution of u by $\text{MRS}[u]_k^j(x) = [\partial u(x)/\partial x_j][\partial u(x)/\partial x_k]^{-1}$.

[8] See also Debreu [3] and Mas-Colell [9, p. 1389].

[9] Indeed, for any $x \in X$ such that $v(x) = \xi = v(z_\xi)$, we have $x \sim z_\xi$ by (i) and so $u(x) = u(z_\xi) = f(\xi)$.

2.2 Discrete symmetries

A preference symmetry is a mapping from the domain of choice to itself that preserves preference comparisons. This concept is formalized in the following definition, which will serve as the starting point for our analysis.

Definition 2.2. A function $\tau : X \to X$ is a *discrete symmetry* of $\succsim$ if $\forall x, y \in X$ we have $\tau(x) \succsim \tau(y) \iff x \succsim y$.

Example 2.3. Let $X = \Re^2$ and $u(x) = 2[x_1+x_2]+\sin[x_1-x_2]$. Then the transformations $\tau(x) = \langle x_1 + \pi, x_2 + \pi \rangle$ and $\bar{\tau}(x) = \langle x_1 + \pi, x_2 - \pi \rangle$ are both discrete symmetries of $\succsim$.[10]

While Definition 2.2 expresses the idea of a discrete symmetry most directly, a different characterization of these transformations will at times be more useful.

Proposition 2.4. *A C^2 function $\tau : X \to X$ is a discrete symmetry of $\succsim$ if and only if there exists a $\rho : X \to \Re_{++}$ such that $\forall x \in X$ we have $\nabla_x[u(\tau(x))] = \rho(x)\nabla_x u(x)$.*[11]

Proof. By Proposition 2.1 a suitable ρ exists if and only if $u(\tau)$ represents $\succsim$, which is to say that $\forall x, y \in X$ we have $u(\tau(x)) \geq u(\tau(y)) \iff x \succsim y$. But since u also represents $\succsim$ this is equivalent to $\tau(x) \succsim \tau(y) \iff x \succsim y$, which is the discrete symmetry property. $\square$

In other words, the discrete symmetries of $\succsim$ are those and only those transformations τ for which the normalized gradients of $u(\tau)$ and u are identical (for the reason that these two functions represent the same preferences).

Example 2.5. Let $X = \Re^2_{++}$ and $u(x) = \|x\|$. Then the transformations $\tau(x) = 2\langle x_2, x_1 \rangle$ and $\bar{\tau}(x) = \left\langle [x_1^2 + 1]^{1/2}, x_2 \right\rangle$ are both discrete symmetries of $\succsim$. Indeed, in this case we have that $\nabla_x[u(\tau(x))] = 2\nabla_x u(x)$ and $\nabla_x[u(\bar{\tau}(x))] = \|x\| \, [x_1^2+x_2^2+1]^{-1/2}\nabla_x u(x)$, consistent with Proposition 2.4.

2.3 Continuous symmetries

In Example 2.3, the mappings $\tau(x) = \langle x_1 + \pi, x_2 + \pi \rangle$ and $\bar{\tau}(x) = \langle x_1 + \pi, x_2 - \pi \rangle$ are both discrete symmetries of the preferences represented by $u(x) = 2[x_1 + x_2] + \sin[x_1 - x_2]$. There is, however, an important difference between these two symmetries. Suppose that we define a family of transformations by $\sigma(x, \alpha) = \langle x_1 + 2\pi\alpha, x_2 + 2\pi\alpha \rangle$. Since for each $\alpha \in [0, 1)$ we have that $\sigma(\cdot, \alpha)$ is a discrete symmetry of $\succsim$, the transformation τ (realized by $\alpha = 1/2$) and the identity mapping (realized by $\alpha = 0$) together belong to a one-parameter class of such symmetries. In contrast, if we define the family $\bar{\sigma}(x, \alpha) = \langle x_1 + 2\pi\alpha, x_2 - 2\pi\alpha \rangle$ so as to include $\bar{\tau}$, then it is *not* true that each $\bar{\sigma}(\cdot, \alpha)$ is a discrete symmetry of $\succsim$. (For instance, $\bar{\sigma}(\cdot, 1/4)$ does not have this property.)

The notion of a one-parameter family of discrete symmetries that includes the identity mapping can be formalized as follows.

[10]Note that here $u(\tau(x)) = u(x) + 4\pi$ and hence $\tau(x) \succ x$, while $u(\bar{\tau}(x)) = u(x)$ and hence $\bar{\tau}(x) \sim x$.

[11]Observe that the gradient of $u(\tau)$ at x in general differs from the gradient of u at $\tau(x)$. For instance, in Example 2.5 below we have $\nabla_x[u(\tau(x))] = 2\|x\|^{-1}\langle x_1, x_2 \rangle \neq \|x\|^{-1}\langle x_2, x_1 \rangle = [\nabla_y u(y)]_{y=\tau(x)}$.

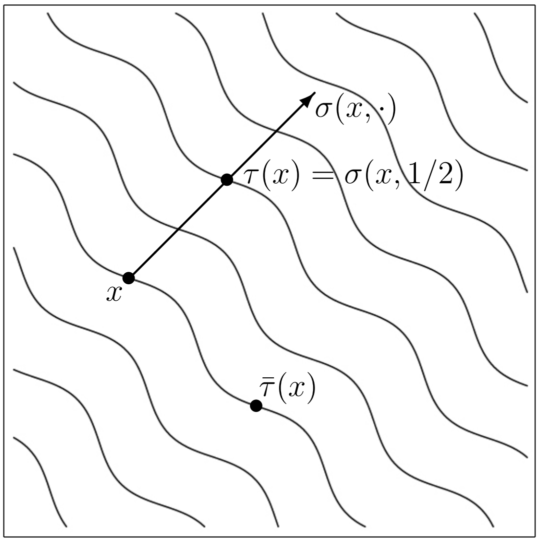

Figure 1: Discrete and continuous preference symmetries. The indifference curves shown are generated by preferences $\succsim$ with representation $u(x) = 2[x_1 + x_2] + \sin[x_1 - x_2]$. Both $\tau(x) = \langle x_1 + \pi, x_2 + \pi \rangle$ and $\bar{\tau}(x) = \langle x_1 + \pi, x_2 - \pi \rangle$ are discrete symmetries of $\succsim$; that is, transformations of the domain that preserve preference comparisons. Moreover, τ is part of a one-parameter family of such transformations that includes the identity mapping, the continuous symmetry $\sigma(x, \alpha) = \langle x_1 + 2\pi\alpha, x_2 + 2\pi\alpha \rangle$.

Definition 2.6. A C^2 function $\sigma : X \times [0, 1) \to X$ is a *continuous symmetry* of $\succsim$ if both $\sigma(\cdot, 0)$ is the identity mapping and $\forall \alpha \in [0, 1)$ the function $\sigma(\cdot, \alpha)$ is a discrete symmetry of $\succsim$.[12]

Our illustrative example, with discrete symmetries τ and $\bar{\tau}$ and continuous symmetry σ, is depicted in Figure 1.

The continuous symmetries of a preference relation impose structure on its utility representations. This fact is established by our first main result, which exhibits a system of partial differential equations in u implied by a given continuous symmetry. Its proof leverages the fact that, as we vary the parameter α locally near 0, the marginal rates of substitution (or, equivalently, the normalized gradient) of u must transform in order to maintain the preference symmetry.

Theorem 2.7. *If σ is a continuous symmetry of $\succsim$, then for each $1 \leq j < k \leq K$ and $\forall x \in X$ we have*

$$\sum_{i=1}^{K} \frac{\partial \mathrm{MRS}[u]_k^j(x)}{\partial x_i} \frac{\partial \sigma_i(x,0)}{\partial \alpha} = \sum_{i=1}^{K} \mathrm{MRS}[u]_k^i(x) \left[\mathrm{MRS}[u]_k^j(x) \frac{\partial^2 \sigma_i(x,0)}{\partial \alpha \partial x_k} - \frac{\partial^2 \sigma_i(x,0)}{\partial \alpha \partial x_j} \right]. \quad (10)$$

[12]Since differentiability of $\sigma(x, \alpha)$ with respect to α will be central to our theory, it would be more precise to refer to this concept as a "differentiable symmetry." However, the term "continuous symmetry" is well established in, e.g., modern expositions of Noether's [10] famous results connecting the idea to physical conservation laws, and so we conform to this usage.

Proof. Let σ be a continuous symmetry of $\succsim$. For each $\alpha \in [0,1)$ we then have that $\sigma(\cdot,\alpha)$ is a discrete symmetry of $\succsim$, and so by Proposition 2.4 there exists a $\rho(\cdot,\alpha) : X \to \Re_{++}$ such that $\forall x \in X$ we have $\nabla_x[u(\sigma(x,\alpha))] = \rho(x,\alpha)\nabla_x u(x)$. Using the chain rule, we can write the mth component of the latter equation as

$$\sum_{i=1}^{K} \left[\frac{\partial u(y)}{\partial y_i} \frac{\partial \sigma_i(x,\alpha)}{\partial x_m} \right]_{y=\sigma(x,\alpha)} = \rho(x,\alpha) \frac{\partial u(x)}{\partial x_m}. \tag{11}$$

Since u and σ are both of class C^2, the LHS of this equation is differentiable with respect to α. Hence the RHS too (and in particular the function ρ) is differentiable with respect to α, and for each $1 \le m \le K$ we obtain

$$\sum_{i=1}^{K} \left[\frac{\partial u(y)}{\partial y_i} \frac{\partial^2 \sigma_i(x,\alpha)}{\partial \alpha \partial x_m} + \frac{\partial \sigma_i(x,\alpha)}{\partial x_m} \sum_{l=1}^{K} \frac{\partial^2 u(y)}{\partial y_l \partial y_i} \frac{\partial \sigma_l(x,\alpha)}{\partial \alpha} \right]_{y=\sigma(x,\alpha)} = \frac{\partial \rho(x,\alpha)}{\partial \alpha} \frac{\partial u(x)}{\partial x_m}. \tag{12}$$

Recalling that $\sigma(x,0) = x$ and therefore

$$\frac{\partial \sigma_i(x,0)}{\partial x_m} = \begin{cases} 1 & \text{for } i = m, \\ 0 & \text{for } i \neq m; \end{cases} \tag{13}$$

we can evaluate Equation 12 at $\alpha = 0$, simplify, and rearrange terms to yield

$$\left[\frac{\partial u(x)}{\partial x_m} \right]^{-1} \sum_{i=1}^{K} \left[\frac{\partial u(x)}{\partial x_i} \frac{\partial^2 \sigma_i(x,0)}{\partial \alpha \partial x_m} + \frac{\partial^2 u(x)}{\partial x_i \partial x_m} \frac{\partial \sigma_i(x,0)}{\partial \alpha} \right] = \frac{\partial \rho(x,0)}{\partial \alpha}. \tag{14}$$

Now, since the RHS of Equation 14 is independent of the component m, we can equate the LHS for $m = j,k$ to establish that

$$\begin{aligned} \sum_{i=1}^{K} \left[\frac{\partial u(x)}{\partial x_k} \frac{\partial^2 u(x)}{\partial x_i \partial x_j} - \frac{\partial u(x)}{\partial x_j} \frac{\partial^2 u(x)}{\partial x_i \partial x_k} \right] \frac{\partial \sigma_i(x,0)}{\partial \alpha} \\ = \sum_{i=1}^{K} \frac{\partial u(x)}{\partial x_i} \left[\frac{\partial u(x)}{\partial x_j} \frac{\partial^2 \sigma_i(x,0)}{\partial \alpha \partial x_k} - \frac{\partial u(x)}{\partial x_k} \frac{\partial^2 \sigma_i(x,0)}{\partial \alpha \partial x_j} \right]. \end{aligned} \tag{15}$$

And dividing both sides of Equation 15 by $[\partial u(x)/\partial x_k]^2$ then confirms Equation 10. □

Note that whenever for each $1 \le i,m \le K$ and $x \in X$ we have $\partial^2 \sigma_i(x,0)/\partial\alpha\partial x_m = 0$, the RHS of Equation 10 vanishes and we obtain

$$\sum_{i=1}^{K} \frac{\partial \mathrm{MRS}[u]_k^j(x)}{\partial x_i} \frac{\partial \sigma_i(x,0)}{\partial \alpha} = 0. \tag{16}$$

This requires simply that as we force $\sigma(\cdot,\alpha)$ away from the identity mapping (realized at $\alpha = 0$), no net change can be induced in any of the marginal rates of substitution.

Example 2.8. Let $X = \Re^2$. If the function $\sigma : X \times [0, 1) \to X$ defined by

$$\sigma(x, \alpha) = \langle x_1 \cos\alpha - x_2 \sin\alpha, x_1 \sin\alpha + x_2 \cos\alpha \rangle \tag{17}$$

is a continuous symmetry of $\succsim$, then by Theorem 2.7 we have that u is a solution of

$$\frac{\partial \mathrm{MRS}[u]_2^1(x)}{\partial x_1} x_2 - \frac{\partial \mathrm{MRS}[u]_2^1(x)}{\partial x_2} x_1 = [\mathrm{MRS}[u]_2^1(x)]^2 + 1. \tag{18}$$

(Here the transformation $\sigma(\cdot, \alpha)$ is a rotation of α radians around the origin.)

2.4 Symmetry fields

We have seen that if σ is a continuous symmetry of $\succsim$, then u must solve the system of PDEs in Equation 10. The converse could not possibly hold, since our derivation of this system uses only local (i.e., $\alpha \searrow 0$) information about σ. If, however, we take this local information as our starting point, constructing both the continuous symmetry and the associated PDEs from the "symmetry field" that records the direction and speed each point is to be locally transformed, then Theorem 2.7 can be made into a two-way result.

Given a Lipschitz-continuous, class C^2 vector field S on X, we denote by $\zeta^S(x, \alpha)$ the trajectory of S from initial point $x \in X$ after time $\alpha \in \Re_+$.[13] We then have that

$$\frac{\partial \zeta^S(x, 0)}{\partial \alpha} = S(\zeta^S(x, 0)) = S(x), \tag{19}$$

which is to say that the one-parameter family of mappings $\zeta^S : X \times \Re_+ \to X$ transforms points locally according to the field S.

Definition 2.9. A vector field $S : X \to \Re^K$ is a *symmetry field* of $\succsim$ if the associated ζ^S yields a continuous symmetry of $\succsim$.

Our strengthened version of Theorem 2.7 can now be stated as follows.

Theorem 2.10. *A vector field $S : X \to \Re^K$ is a symmetry field of $\succsim$ if and only if for each $1 \le j < k \le K$ and $\forall x \in X$ we have*

$$\sum_{i=1}^{K} \frac{\partial \mathrm{MRS}[u]_k^j(x)}{\partial x_i} S_i(x) = \sum_{i=1}^{K} \mathrm{MRS}[u]_k^i(x) \left[\mathrm{MRS}[u]_k^j(x) \frac{\partial S_i(x)}{\partial x_k} - \frac{\partial S_i(x)}{\partial x_j} \right]. \tag{20}$$

Proof. In view of Equation 19, if S is a symmetry field of $\succsim$ then Equation 20 is a direct consequence of Theorem 2.7.

[13] The vector field S maps each $x \in X \subset \Re^K$ to an $S(x) \in \Re^K$. This mapping is Lipschitz continuous if $\exists M \in \Re_+$ such that $\forall x, y \in X$ we have $\|S(x) - S(y)\| \le M \|x - y\|$. When Lipschitz continuity holds, the Picard-Lindelöf Theorem guarantees $\forall x \in X$ the existence of a unique solution $\zeta^S(x, \cdot) : \Re_+ \to X$ to the differential equations $\partial \zeta^S(x, \alpha)/\partial \alpha = S(\zeta^S(x, \alpha))$ and initial conditions $\zeta^S(x, 0) = x$. This solution is the trajectory of S from x, and when S is of class C^2 the function ζ^S will be C^2 as well.

Conversely, if Equation 20 holds then for each $1 \leq j < k \leq K$ and $\forall x \in X$ we have

$$\sum_{i=1}^{K} \left[\frac{\partial u(x)}{\partial x_k} \frac{\partial^2 u(x)}{\partial x_i \partial x_j} - \frac{\partial u(x)}{\partial x_j} \frac{\partial^2 u(x)}{\partial x_i \partial x_k} \right] S_i(x) = \sum_{i=1}^{K} \frac{\partial u(x)}{\partial x_i} \left[\frac{\partial u(x)}{\partial x_j} \frac{\partial S_i(x)}{\partial x_k} - \frac{\partial u(x)}{\partial x_k} \frac{\partial S_i(x)}{\partial x_j} \right]. \quad (21)$$

This is equivalent to

$$\left[\frac{\partial u(x)}{\partial x_j} \right]^{-1} \sum_{i=1}^{K} \left[\frac{\partial u(x)}{\partial x_i} \frac{\partial S_i(x)}{\partial x_j} + \frac{\partial^2 u(x)}{\partial x_i \partial x_j} S_i(x) \right] = \left[\frac{\partial u(x)}{\partial x_k} \right]^{-1} \sum_{i=1}^{K} \left[\frac{\partial u(x)}{\partial x_i} \frac{\partial S_i(x)}{\partial x_k} + \frac{\partial^2 u(x)}{\partial x_i \partial x_k} S_i(x) \right], \quad (22)$$

and it follows that there exists a $\phi(x) \in \Re$ such that for each $1 \leq m \leq K$ we have

$$\sum_{i=1}^{K} \left[\frac{\partial u(x)}{\partial x_i} \frac{\partial S_i(x)}{\partial x_m} + \frac{\partial^2 u(x)}{\partial x_i \partial x_m} S_i(x) \right] = \phi(x) \frac{\partial u(x)}{\partial x_m}. \quad (23)$$

Given $y \in X$, $\alpha \in [0, 1)$, and $1 \leq l \leq K$, we now set $x = \zeta^S(y, \alpha)$ in Equation 23, multiply by $\partial \zeta_m^S(y, \alpha)/\partial y_l$, and sum over m to yield

$$\sum_{m=1}^{K} \frac{\partial \zeta_m^S(y, \alpha)}{\partial y_l} \sum_{i=1}^{K} \left[\frac{\partial u(x)}{\partial x_i} \frac{\partial S_i(x)}{\partial x_m} + \frac{\partial^2 u(x)}{\partial x_i \partial x_m} S_i(x) \right]_{x=\zeta^S(y,\alpha)} = \sum_{m=1}^{K} \frac{\partial \zeta_m^S(y, \alpha)}{\partial y_l} \left[\phi(x) \frac{\partial u(x)}{\partial x_m} \right]_{x=\zeta^S(y,\alpha)}. \quad (24)$$

The LHS of the latter equation can be expressed as

$$\begin{aligned}
&\left[\sum_{i=1}^{K} \frac{\partial u(x)}{\partial x_i} \sum_{m=1}^{K} \frac{\partial S_i(x)}{\partial x_m} \frac{\partial \zeta_m^S(y, \alpha)}{\partial y_l} + \sum_{h=1}^{K} \sum_{n=1}^{K} \frac{\partial \zeta_n^S(y, \alpha)}{\partial y_l} \frac{\partial^2 u(x)}{\partial x_h \partial x_n} S_h(x) \right]_{x=\zeta^S(y,\alpha)} \\
&= \sum_{i=1}^{K} \left[\frac{\partial u(x)}{\partial x_i} \frac{\partial S_i(\zeta^S(y, \alpha))}{\partial y_l} + \sum_{h=1}^{K} \frac{\partial \zeta_i^S(y, \alpha)}{\partial y_l} \frac{\partial^2 u(x)}{\partial x_h \partial x_i} S_h(x) \right]_{x=\zeta^S(y,\alpha)} \\
&= \sum_{i=1}^{K} \left[\frac{\partial u(x)}{\partial x_i} \frac{\partial^2 \zeta_i^S(y, \alpha)}{\partial \alpha \partial y_l} + \frac{\partial \zeta_i^S(y, \alpha)}{\partial y_l} \sum_{h=1}^{K} \frac{\partial^2 u(x)}{\partial x_h \partial x_i} \frac{\partial \zeta_h^S(y, \alpha)}{\partial \alpha} \right]_{x=\zeta^S(y,\alpha)} \\
&= \frac{\partial}{\partial \alpha} \sum_{i=1}^{K} \left[\frac{\partial u(x)}{\partial x_i} \frac{\partial \zeta_i^S(y, \alpha)}{\partial y_l} \right]_{x=\zeta^S(y,\alpha)} = \frac{\partial^2 u(\zeta^S(y, \alpha))}{\partial \alpha \partial y_l}; \quad (25)
\end{aligned}$$

using the chain rule, the product rule, and the equality $\partial\zeta^S(y,\alpha)/\partial\alpha = S(\zeta^S(y,\alpha))$. After applying the chain rule to its RHS as well, Equation 24 can then be simplified to

$$\frac{\partial^2 u(\zeta^S(y,\alpha))}{\partial\alpha\partial y_l} = \phi(\zeta^S(y,\alpha))\frac{\partial u(\zeta^S(y,\alpha))}{\partial y_l}, \tag{26}$$

or equivalently

$$\frac{\partial}{\partial\alpha}\log\frac{\partial u(\zeta^S(y,\alpha))}{\partial y_l} = \phi(\zeta^S(y,\alpha)). \tag{27}$$

Since $\zeta^S(y,0) = y$, integrating Equation 27 yields

$$\log\frac{\partial u(\zeta^S(y,\alpha))}{\partial y_l} - \log\frac{\partial u(y)}{\partial y_l} = \int_0^\alpha \phi(\zeta^S(y,t))\mathrm{d}t. \tag{28}$$

Recalling that the component l is arbitrary, and defining

$$\rho(y,\alpha) = \exp\int_0^\alpha \phi(\zeta^S(y,t))\mathrm{d}t > 0, \tag{29}$$

we then obtain $\nabla_y[u(\zeta^S(y,\alpha))] = \rho(y,\alpha)\nabla_y u(y)$. By Proposition 2.4 it follows that $\zeta^S(\cdot,\alpha)$ is a discrete symmetry of $\succsim$, and since $\zeta^S(\cdot,0)$ is the identity mapping we can conclude that ζ^S is a continuous symmetry of $\succsim$. Hence S is a symmetry field of $\succsim$, as desired. $\square$

Example 2.11. Let $X = \Re^2$ and define $S : X \to \Re^2$ by $S(x) = \langle x_1 - x_2, x_1 + x_2\rangle$. In this case the trajectory of S from x is given by

$$\zeta^S(x,\alpha) = e^\alpha\left\langle x_1\cos\alpha - x_2\sin\alpha, x_1\sin\alpha + x_2\cos\alpha\right\rangle, \tag{30}$$

since we then have both

$$\frac{\partial\zeta^S(x,\alpha)}{\partial\alpha} = e^\alpha\begin{bmatrix} -x_1 - x_2 & x_1 - x_2 \\ x_1 - x_2 & x_1 + x_2\end{bmatrix}\begin{bmatrix}\sin\alpha \\ \cos\alpha\end{bmatrix} = S(\zeta^S(x,\alpha)) \tag{31}$$

and $\zeta^S(x,0) = x$. Note that the family of transformations $\sigma : X \times [0,1) \to X$ defined by

$$\sigma(x,\alpha) = [1+\alpha]\langle x_1\cos\alpha - x_2\sin\alpha, x_1\sin\alpha + x_2\cos\alpha\rangle \tag{32}$$

has $\sigma(x,0) = x$,

$$\frac{\partial\sigma(x,\alpha)}{\partial\alpha} = \begin{bmatrix} -x_1 - x_2 - x_1\alpha & x_1 - x_2 - x_2\alpha \\ x_1 - x_2 - x_2\alpha & x_1 + x_2 + x_1\alpha\end{bmatrix}\begin{bmatrix}\sin\alpha \\ \cos\alpha\end{bmatrix}, \tag{33}$$

and hence $\partial\sigma(x,0)/\partial\alpha = S(x)$, but does *not* satisfy $\partial\sigma(x,\alpha)/\partial\alpha = S(\sigma(x,\alpha))$ for $\alpha > 0$ except at $x = \vec{0}$. This illustrates how two distinct families of transformations can generate the same field $\partial\zeta^S(x,0)/\partial\alpha = S(x) = \partial\sigma(x,0)/\partial\alpha$ locally, and therefore impose the same restrictions on u via Equation 20. However, at most one of the two families will trace out the trajectories of the field.

Whenever a continuous symmetry σ traces out the trajectories of a field, this field is always $\partial\sigma(\cdot, 0)/\partial\alpha = S : X \to \Re^K$. The symmetries with this property are those such that $\forall x \in X$, $\beta \in [0, 1)$, and $\gamma \in [0, 1 - \beta)$ we have

$$\sigma(x, \beta + \gamma) = \sigma(\sigma(x, \beta), \gamma). \tag{34}$$

On the one hand, the trajectories of a vector field will satisfy this identity. And conversely, differentiating Equation 34 with respect to γ and evaluating at $\gamma = 0$ yields

$$\left[\frac{\partial\sigma(x, \alpha)}{\partial\alpha}\right]_{\alpha=\beta} = \left[\frac{\partial\sigma(\sigma(x, \beta), \alpha)}{\partial\alpha}\right]_{\alpha=0} = S(\sigma(x, \beta)), \tag{35}$$

which is the condition for $\sigma(x, \cdot)$ to be the trajectory of S from x.

One advantage of describing preference symmetries in terms of vector fields is that this gives them a natural algebraic structure.

Proposition 2.12. *The set of symmetry fields of $\succsim$ is a convex cone in the space $[\Re^K]^X$ of vector fields over X.*

Proof. Given $S, T : X \to \Re^K$ and $a_1, a_2 \in \Re_+$, suppose that S and T are both symmetry fields of $\succsim$. We aim to show that $a_1S + a_2T$ is also a symmetry field of $\succsim$.

Since S and T are symmetry fields, ζ^S and ζ^T are continuous symmetries of $\succsim$. Now consider the function $\sigma : X \times [0, 1) \to X$ defined by $\sigma(x, \alpha) = \zeta^T(\zeta^S(x, a_1\alpha), a_2\alpha)$. Note first that

$$\sigma(x, 0) = \zeta^T(\zeta^S(x, 0), 0) = \zeta^T(x, 0) = x. \tag{36}$$

Given $x, y \in X$ and $\alpha \in [0, 1)$, we have also

$$\begin{aligned}\sigma(x, \alpha) \succsim \sigma(y, \alpha) &\iff \zeta^T(\zeta^S(x, a_1\alpha), a_2\alpha) \succsim \zeta^T(\zeta^S(y, a_1\alpha), a_2\alpha) \\ &\iff \zeta^S(x, a_1\alpha) \succsim \zeta^S(y, a_1\alpha) \iff x \succsim y, \end{aligned} \tag{37}$$

since $\zeta^S(\cdot, a_1\alpha)$ and $\zeta^T(\cdot, a_2\alpha)$ are both discrete symmetries of $\succsim$. It follows that σ is a continuous symmetry of $\succsim$, and so Equation 10 holds by Theorem 2.7. Moreover, for each $1 \leq i \leq K$ we can compute

$$\begin{aligned}\frac{\partial\sigma_i(x, 0)}{\partial\alpha} &= \left[\frac{\partial[\zeta_i^T(\zeta^S(x, a_1\beta), a_2\beta)]}{\partial\beta}\right]_{\beta=0} = a_1\left[\sum_{j=1}^K \frac{\partial\zeta_i^T(x, 0)}{\partial x_j}\frac{\partial\zeta_j^S(x, 0)}{\partial\alpha}\right] + a_2\frac{\partial\zeta_i^T(x, 0)}{\partial\alpha} \\ &= a_1\frac{\partial\zeta_i^S(x, 0)}{\partial\alpha} + a_2\frac{\partial\zeta_i^T(x, 0)}{\partial\alpha} = a_1S_i(x) + a_2T_i(x); \end{aligned} \tag{38}$$

using the chain rule and the facts that $\zeta^S(x, 0) = x = \zeta^T(x, 0)$, $\partial\zeta^S(x, 0)/\partial\alpha = S(x)$, and $\partial\zeta^T(x, 0)/\partial\alpha = T(x)$. But then by Theorem 2.10 we have that $a_1S + a_2T$ is a symmetry field of $\succsim$. □

Proposition 2.12 is important because it shows that the limitation to one-parameter families of symmetries in Definition 2.6 is not essential to our theory: The symmetry field $a_1S + a_2T$ has two degrees of freedom, as does the corresponding continuous symmetry. Moreover, this freedom to identify different symmetries separately and then combine them algebraically has substantial practical value. For this reason, and because of the two-way nature of Theorem 2.10, symmetry fields will be our primary tool of analysis below.

3 Applications

3.1 Univariate separability

As a first application of the theory outlined in Section 2, we now characterize all utility functions that are additively separable in one variable.

Proposition 3.1. *Let $X = \Re^K_{++}$. Given a strictly increasing, C^2 function $g : \Re_{++} \to \Re$, the following are equivalent:*

(i) *The vector field S defined by $S(x) = [g'(x_1)]^{-1}\vec{\imath}_1$ is a symmetry field of $\succsim$.*[14]

(ii) *There exist strictly increasing, C^1 functions $h : \Re^{K-1}_{++} \to \Re$ and $f : v[X] \to \Re$, where $v(x) = g(x_1) + h(x_{\neg 1})$, such that $u = f(v)$.*

This and all other results in Section 3 are proved in Appendix A.

Observe the structure of Proposition 3.1: The function g governing the effect of x_1 on u is taken as given, and the symmetry field S is expressed in terms of this function. In contrast, both h and f remain unknown, their existence merely being asserted by the result.

One consequence of this characterization of univariate separability in general is the standard characterization of additive utility in particular. Indeed, this is the special case in which $g(x_1) = x_1$ and hence $v(x) = x_1 + h(x_{\neg 1})$.[15]

Corollary 3.2. *Let $X = \Re^K_{++}$. Then the vector field S defined by $S(x) = \vec{\imath}_1$ is a symmetry field of $\succsim$ if and only if there exist strictly increasing, C^1 functions $h : \Re^{K-1}_{++} \to \Re$ and $f : v[X] \to \Re$, where $v(x) = x_1 + h(x_{\neg 1})$, such that $u = f(v)$.*

Similar corollaries link the functional form $v(x) = x_1^p + h(x_{\neg 1})$ to $S(x) = p^{-1}x_1^{1-p}\vec{\imath}_1$; the form $v(x) = e^{px_1} + h(x_{\neg 1})$ to $S(x) = p^{-1}e^{-px_1}\vec{\imath}_1$; and the form $v(x) = \log x_1 + h(x_{\neg 1})$ to $S(x) = x_1\vec{\imath}_1$. Of course, by integrating these fields to obtain their trajectories we can express the same results in terms of continuous symmetries. For example, if $S(x) = \vec{\imath}_1$ is a symmetry field then $\sigma(x, \alpha) = \langle x_1 + \alpha, x_{\neg 1}\rangle$ is a continuous symmetry of $\succsim$ (a way of describing quasilinearity with respect to x_1). And likewise, if $S(x) = x_1\vec{\imath}_1$ is a symmetry field then $\sigma(x, \alpha) = \langle e^\alpha x_1, x_{\neg 1}\rangle$ is a continuous symmetry.

To sketch the argument for Proposition 3.1, specialize Equation 20 in Theorem 2.10 to the vector field in (i) to yield

$$\frac{\partial \mathrm{MRS}[u]^1_K(x)}{\partial x_1}\frac{1}{g'(x_1)} = -\mathrm{MRS}[u]^1_K(x)\left[\frac{\partial}{\partial x_1}\frac{1}{g'(x_1)}\right] \tag{39}$$

[14] We denote the kth unit vector by $\vec{\imath}_k$ and write $x_{\neg k} = \langle x_1, \ldots, x_{k-1}, x_{k+1}, \ldots, x_K\rangle$.

[15] Our approach to proving the equivalence between quasilinear preferences and additive utility may be contrasted with the conventional method, which assigns utilities to the points on a special path through the domain and then maps all other points into counterparts on this path to which they are indifferent. While this strategy does not rely on differentiability, it will break down if part of the domain intersecting the special path is removed. In contrast, our theory is based entirely on local analysis and imposes only mild topological conditions on the domain.

and for each $2 \leq j < K$

$$\frac{\partial \text{MRS}[u]^j_K(x)}{\partial x_1} \frac{1}{g'(x_1)} = 0. \tag{40}$$

Integrating Equations 39 and 40 leads to expressions for the marginal rates of substitution of u, and it can be shown that for some h these rates are shared by the function v defined by $v(x) = g(x_1) + h(x_{\neg 1})$. Applying Proposition 2.1, we then have that there exists an f such that $u = f(v)$, as desired. This shows that (i) implies (ii), and for the converse we need only check that Equation 20 holds for the vector field and functional form specified.[16]

3.2 Multivariate separability

We proceed now to characterize functional forms for u with additive separability in all variables simultaneously.

Proposition 3.3. *Let $X = \Re^K_{++}$. Given K strictly increasing, C^2 functions $g_1, \ldots, g_K : \Re_{++} \to \Re$, the following are equivalent:*

- **(i)** *For each $1 \leq k \leq K$ the vector field S^k defined by $S^k(x) = [g'_k(x_k)]^{-1}\vec{\imath}_k$ is a symmetry field of $\succsim$.*
- **(ii)** *There exist a $\lambda \in \Re^K_{++}$ and a strictly increasing, C^1 function $f : v[X] \to \Re$, where $v(x) = \sum_{k=1}^K \lambda_k g_k(x_k)$, such that $u = f(v)$.*

Here each function g_k is taken as given, with the vector λ and the function f remaining unknown.[17]

Special cases of Proposition 3.3 include characterizations of Cobb-Douglas and other CES utility functions.

Corollary 3.4. *Let $X = \Re^K_{++}$ and fix $p > 0$. Then:*

- **(A)** *For each $1 \leq k \leq K$ the vector field S^k defined by $S^k(x) = x_k\vec{\imath}_k$ is a symmetry field of $\succsim$ if and only if there exist $\lambda \in \Re^K_{++}$ and a strictly increasing, C^1 function $f : v[X] \to \Re$, where $v(x) = \sum_{k=1}^K \lambda_k \log x_k$, such that $u = f(v)$.*
- **(B)** *For each $1 \leq k \leq K$ the vector field S^k defined by $S^k(x) = p^{-1}x_k^{1-p}\vec{\imath}_k$ is a symmetry field of $\succsim$ if and only if there exist $\lambda \in \Re^K_{++}$ and a strictly increasing, C^1 function $f : v[X] \to \Re$, where $v(x) = \sum_{k=1}^K \lambda_k x_k^p$, such that $u = f(v)$.*

For the case of Cobb-Douglas utility in Corollary 3.4A, note that by Proposition 2.12 we have for each $a \in \Re^K_+$ the symmetry field $\sum_{k=1}^K a_k S^k(x) = \sum_{k=1}^K a_k x_k \vec{\imath}_k = \langle a_k x_k \rangle_{k=1}^K$, with corresponding continuous symmetry $\sigma(x, \alpha) = \langle e^{\alpha a_k} x_k \rangle_{k=1}^K$.[18] In the CES context of Corollary 3.4B, setting $p = 1$ links the linear specification $v(x) = \sum_{k=1}^K \lambda_k x_k$ to the

[16]Cf. Samuelson [11, pp. 176–177], who for $K = 2$ obtains a version of Equation 39.

[17]In contrast, the multivariate separability results of Debreu [1, pp. 20–25], Fishburn [4, pp. 346–349], and Leontief [7] involve unknown g_k functions.

[18]Regarding the behavioral characterization of Cobb-Douglas preferences, note also the comment of Maccheroni et al. [8, p. 1472].

collection of symmetry fields $S^k(x) = \vec{\imath}_k$ for each $1 \le k \le K$. It follows that for each $a \in \Re^K_+$ the vector field $\sum_{k=1}^K a_k S^k(x) = \sum_{k=1}^K a_k \vec{\imath}_k = a$ is also a symmetry field, and the associated continuous symmetry is $\sigma(x,\alpha) = x + \alpha a$.[19]

3.3 Joint separability

Now let $K = 3$ for simplicity and consider again Proposition 3.1. Given $g : \Re_{++} \to \Re$, this result implies that $S^3(x) = [g'(x_3)]^{-1}\vec{\imath}_3$ is a symmetry field of $\succsim$ if and only if it has a representation equal to $g(x_3)$ plus an unspecified function of the variables $\langle x_1, x_2\rangle$. Suppose, however, that we wish the latter dependence also to have a particular form, rather than remaining unknown. Our next result determines the additional restrictions that this imposes on the preference relation.

Proposition 3.5. *Let $X = \Re^3_{++}$. Given strictly increasing, C^2 functions $g : \Re_{++} \to \Re$ and $h : \Re^2_{++} \to \Re$, suppose that $\exists\, \langle x_1^*, x_2^*\rangle \in \Re^2_{++}$ with $[\partial^2 h(x_1^*, x_2^*)/\partial x_1 \partial x_2] \neq 0$. Then the following are equivalent:*

(i) *The vector fields S^1, S^2, and S^3 defined by*

$$S^1(x) = \langle [\partial h(x_1,x_2)/\partial x_1]^{-1}, 0, 0\rangle, \tag{41}$$

$$S^2(x) = \langle 0, [\partial h(x_1,x_2)/\partial x_2]^{-1}, 0\rangle, \tag{42}$$

$$S^3(x) = \langle 0, 0, [g'(x_3)]^{-1}\rangle, \tag{43}$$

are all symmetry fields of $\succsim$.

(ii) *There exist a $\lambda \in \Re_{++}$ and a strictly increasing, C^1 function $f : v[X] \to \Re$, where $v(x) = \lambda h(x_1,x_2) + g(x_3)$, such that $u = f(v)$.*

Here the condition that $\exists\, \langle x_1^*, x_2^*\rangle \in \Re^2_{++}$ with $[\partial^2 h(x_1^*, x_2^*)/\partial x_1 \partial x_2] \neq 0$ rules out the case of $h(x_1,x_2) = h_1(x_1) + h_2(x_2)$, which is already covered by Proposition 3.3 above.

To see how Proposition 3.5 can be used, suppose we wish to characterize the functional form $v(x) = \lambda x_1 x_2 + x_3$. The result says that $S^1(x) = \langle 1/x_2, 0, 0\rangle$, $S^2(x) = \langle 0, 1/x_1, 0\rangle$, and $S^3(x) = \langle 0,0,1\rangle$ are all symmetry fields of $\succsim$. And integrating these fields determines the corresponding trajectories $\sigma^1(x,\alpha) = \langle x_1 + \alpha/x_2, x_2, x_3\rangle$, $\sigma^2(x,\alpha) = \langle x_1, x_2 + \alpha/x_1, x_3\rangle$, and $\sigma^3(x,\alpha) = \langle x_1, x_2, x_3 + \alpha\rangle$; each a continuous symmetry of $\succsim$.

3.4 Homogeneity and related forms

As our last set of applications, we develop results relating to homogeneity of degree one. For simplicity, we limit attention here to the case of $K = 2$.

The basic characterization of homogeneity appears as follows.

[19] Alternatively, the linear utility specification can be characterized by the symmetry field $S(x) = x - b$ for each $b \in \Re^K_{++}$, with corresponding continuous symmetry $\sigma(x,\alpha) = e^\alpha x + [1 - e^\alpha]b$. This is in effect a restatement of the expected utility theorem, with the parameterized continuous symmetry recognizable as the standard independence axiom. Moreover, related "certainty-independence" axioms (see, e.g., Gilboa and Schmeidler [6] and Ghirardato et al. [5]) can likewise be expressed as continuous symmetries.

Proposition 3.6. *Let $X = \Re^2_{++}$. The following are equivalent:*

(i) *The vector field S defined by $S(x) = x$ is a symmetry field of $\succsim$.*

(ii) *There exist a homogeneous of degree one, C^1 function v and a strictly increasing, C^1 function $f : v[X] \to \Re$ such that $u = f(v)$.*

Here the continuous symmetry of $\succsim$ associated with S is easily seen to be $\sigma(x, \alpha) = e^\alpha x$. (Note also that while we state and prove Proposition 3.6 only for the two-dimensional case, the result in fact holds for arbitrary K.)

Proving necessity of the symmetry field S for a representation that is homogeneous of degree one amounts to verifying Equation 20 in this instance. Sufficiency, on the other hand, is most easily established as a corollary of a more general characterization.

Proposition 3.7. *Let $X = \Re^2_{++}$. Given strictly increasing, C^2 functions $g_1, g_2 : \Re_{++} \to \Re$, the following are equivalent:*

(i) *The vector field S defined by $S(x) = \langle [g_1'(x_1)]^{-1}, [g_2'(x_2)]^{-1} \rangle$ is a symmetry field of $\succsim$.*

(ii) *There exist a C^1 function $h : \Re \to \Re_{++}$ and a strictly increasing, C^1 function $f : v[X] \to \Re$, where*

$$v(x) = g_1(x_1) + \int_1^{g_2(x_2) - g_1(x_1)} \frac{\mathrm{d}t}{h(t) + 1}, \tag{44}$$

such that $u = f(v)$.

Here Proposition 3.6 is the special case in which each $g_k(x_k) = \log x_k$, and therefore the marginal rate of substitution

$$\text{MRS}[v]^1_2(x) = \frac{x_2}{x_1} h(\log[x_2/x_1]) \tag{45}$$

depends on x only through the ratio x_2/x_1. More generally in Proposition 3.7 we have

$$\text{MRS}[v]^1_2(x) = \frac{g_1'(x_1)}{g_2'(x_2)} h(g_2(x_2) - g_1(x_1)). \tag{46}$$

For example, when each $g_k(x_k) = b_k x_k$ for some $b_k \in \Re$, the rate of substitution $\text{MRS}[v]^1_2(x)$ depends on x only through the quantity $b_2 x_2 - b_1 x_1$.[20]

Our final characterization relates to preferences that admit both additively separable and homogeneous of degree one representations. We establish that these are precisely the preferences that admit CES utility (including the Cobb-Douglas case), and we describe their symmetry fields.

[20]Note that Proposition 3.7(i) yields the class of symmetry fields $\langle a/g_1'(x_1), a/g_2'(x_2) \rangle$ for $a \geq 0$, while Proposition 3.3(i) yields $\langle a_1/g_1'(x_1), a_2/g_2'(x_2) \rangle$ for $a_1, a_2 \geq 0$. The latter class is more general in that it allows independent scaling of the two components of the vector $\langle [g_1'(x_1)]^{-1}, [g_2'(x_2)]^{-1} \rangle$.

Proposition 3.8. *Let $X = \Re^2_{++}$. The following are equivalent:*

(i) *There exist C^2 functions $s_1, s_2 : \Re_{++} \to \Re_{++}$ such that the vector fields S^1 and S^2 defined by $S^1(x) = \langle s_1(x_1), 0\rangle$ and $S^2(x) = \langle 0, s_2(x_2)\rangle$ are both symmetry fields of $\succsim$; and such that for each $1 \leq k \leq 2$ and $x_k \in \Re_{++}$ we have $s'_k(x_k) \leq s_k(x_k)/x_k$. Moreover, the vector field S^3 defined by $S^3(x) = x$ is a symmetry field of $\succsim$.*

(ii) *There exist $p \geq 0$, $a \in \Re^2_{++}$, and a strictly increasing, C^1 function $f : v[X] \to \Re$, where*

$$v(x) = \begin{cases} a_1 \log x_1 + a_2 \log x_2 & \text{for } p = 0, \\ a_1 x_1^p + a_2 x_2^p & \text{for } p > 0; \end{cases} \tag{47}$$

such that $u = f(v)$.

(iii) *There exist strictly increasing, C^2 functions $g_1, g_2 : \Re_{++} \to \Re$ and a strictly increasing, C^1 function $\chi : w[X] \to \Re$, where $w(x) = g_1(x_1) + g_2(x_2)$, such that $u = \chi(w)$; and such that for each $1 \leq k \leq 2$ and $x_k \in \Re_{++}$ we have $g''_k(x_k) \geq -g'_k(x_k)/x_k$. Moreover, there exist a homogeneous of degree one, C^1 function $\hat{w}$ and a strictly increasing, C^1 function $\hat{\chi} : \hat{w}[X] \to \Re$ such that $u = \hat{\chi}(\hat{w})$.*

Here in (iii) the g_k functions are not taken as given (in contrast to Proposition 3.3), though their form is deduced in (ii). The requirements that $s'_k(x_k) \leq s_k(x_k)/x_k$ in (i) and that $g''_k(x_k) \geq -g'_k(x_k)/x_k$ in (iii) correspond to the requirement that $p \geq 0$ in (ii), which is needed to ensure that $\forall x \in X$ we have $\nabla_x u(x) \gg 0$. Note also that the equivalence of (ii) and (iii) is a useful result in its own right, and for its statement does not require any of our notions of preference symmetry.

A Appendix

Proof of Proposition 3.1. If (i) holds, then $\forall x \in X$ we have

$$\frac{\partial \text{MRS}[u]^1_K(x)}{\partial x_1} \frac{1}{g'(x_1)} = -\text{MRS}[u]^1_K(x) \left[\frac{\partial}{\partial x_1} \frac{1}{g'(x_1)}\right] \tag{48}$$

and for each $2 \leq j < K$

$$\frac{\partial \text{MRS}[u]^j_K(x)}{\partial x_1} \frac{1}{g'(x_1)} = 0, \tag{49}$$

by Theorem 2.10. Integrating Equations 48 and 49 with respect to x_1, we obtain

$$\text{MRS}[u]^1_K(x) = g'(x_1)\eta_1(x_{\neg 1}) \tag{50}$$

and for each $2 \leq j < K$

$$\text{MRS}[u]^j_K(x) = \eta_j(x_{\neg 1}), \tag{51}$$

where $\log \eta_1(x_{\neg 1})$ and each $\eta_j(x_{\neg 1})$ are constants of integration. Now for each $x_{\neg 1} \in \Re^{K-1}_{++}$ let $\eta_K(x_{\neg 1}) = 1$, and define a vector field $H : \Re^{K-1}_{++} \to \Re^{K-1}_{++}$ by $H(y) = \langle \eta_k(y)/\eta_1(y)\rangle^K_{k=2}$. For each $2 \leq j < k \leq K$ and $\forall x \in X$ we then have

$$\frac{\partial \text{MRS}[u]^j_k(x)}{\partial x_1} = \frac{\partial}{\partial x_1} \frac{\eta_j(x_{\neg 1})}{\eta_k(x_{\neg 1})} = 0, \tag{52}$$

so that

$$\frac{\partial u(x)}{\partial x_k}\frac{\partial^2 u(x)}{\partial x_1 \partial x_j} = \frac{\partial u(x)}{\partial x_j}\frac{\partial^2 u(x)}{\partial x_1 \partial x_k} \tag{53}$$

and therefore

$$\begin{aligned}
\frac{\partial H_j(x_{\neg 1})}{\partial x_k} &= \frac{\partial}{\partial x_k}\frac{\eta_j(x_{\neg 1})}{\eta_1(x_{\neg 1})} = g'(x_1)\frac{\partial \text{MRS}[u]_1^j(x)}{\partial x_k} \\
&= g'(x_1)\left[\frac{\partial u(x)}{\partial x_1}\frac{\partial^2 u(x)}{\partial x_k \partial x_j} - \frac{\partial u(x)}{\partial x_j}\frac{\partial^2 u(x)}{\partial x_k \partial x_1}\right]\left[\frac{\partial u(x)}{\partial x_1}\right]^{-2} \\
&= g'(x_1)\left[\frac{\partial u(x)}{\partial x_1}\frac{\partial^2 u(x)}{\partial x_j \partial x_k} - \frac{\partial u(x)}{\partial x_k}\frac{\partial^2 u(x)}{\partial x_j \partial x_1}\right]\left[\frac{\partial u(x)}{\partial x_1}\right]^{-2} \\
&= g'(x_1)\frac{\partial \text{MRS}[u]_1^k(x)}{\partial x_j} = \frac{\partial}{\partial x_j}\frac{\eta_k(x_{\neg 1})}{\eta_1(x_{\neg 1})} = \frac{\partial H_k(x_{\neg 1})}{\partial x_j}.
\end{aligned} \tag{54}$$

This shows that the vector field H is conservative and hence admits a strictly increasing, C^1 potential function $h : \Re^{K-1}_{++} \to \Re$. We then have

$$\text{MRS}[v]_K^1(x) = \frac{g'(x_1)}{\partial h(x_{\neg 1})/\partial x_K} = \frac{g'(x_1)}{\eta_K(x_{\neg 1})/\eta_1(x_{\neg 1})} = \text{MRS}[u]_K^1(x) \tag{55}$$

and for each $2 \leq j < K$

$$\text{MRS}[v]_K^j(x) = \frac{\partial h(x_{\neg 1})/\partial x_j}{\partial h(x_{\neg 1})/\partial x_K} = \frac{\eta_j(x_{\neg 1})/\eta_1(x_{\neg 1})}{\eta_K(x_{\neg 1})/\eta_1(x_{\neg 1})} = \text{MRS}[u]_K^j(x). \tag{56}$$

And by Proposition 2.1 we can conclude that there exists a strictly increasing, C^1 function $f : v[X] \to \Re$ such that $u = f(v)$. Thus (ii) holds.

Conversely, if (ii) holds then for each $1 \leq j < k \leq K$ and $x \in X$ we have

$$\begin{aligned}
\sum_{i=1}^{K}\frac{\partial \text{MRS}[u]_k^j(x)}{\partial x_i}S_i(x) &= \begin{cases} [g''(x_1)/g'(x_1)][\partial h(x_{\neg 1})/\partial x_k]^{-1} & \text{for } j = 1, \\ 0 & \text{for } j > 1; \end{cases} \\
&= \sum_{i=1}^{K}\text{MRS}[u]_k^i(x)\left[\text{MRS}[u]_k^j(x)\frac{\partial S_i(x)}{\partial x_k} - \frac{\partial S_i(x)}{\partial x_j}\right].
\end{aligned} \tag{57}$$

Hence by Theorem 2.10 we have that S is a symmetry field of $\succsim$, and (i) holds. $\square$

Proof of Proposition 3.3. If (i) holds, then for each $1 \leq j < K$ and $x \in X$ we have

$$\frac{\partial \text{MRS}[u]_K^j(x)}{\partial x_j}\frac{1}{g_j'(x_j)} = -\text{MRS}[u]_K^j(x)\left[\frac{\partial}{\partial x_j}\frac{1}{g_j'(x_j)}\right] \tag{58}$$

and

$$\frac{\partial \text{MRS}[u]_K^j(x)}{\partial x_K}\frac{1}{g_K'(x_K)} = \text{MRS}[u]_K^j(x)\left[\frac{\partial}{\partial x_K}\frac{1}{g_K'(x_K)}\right] \tag{59}$$

by Theorem 2.10. Similarly, for each $1 \leq k < K$ such that $k \neq j$ and $\forall x \in X$ we have

$$\frac{\partial \text{MRS}[u]^j_K(x)}{\partial x_k} \frac{1}{g'_k(x_k)} = 0. \tag{60}$$

Integrating Equations 58–60 now yields

$$\text{MRS}[u]^j_K(x) = \eta_j(x_{\neg j}) g'_j(x_j), \tag{61}$$
$$\text{MRS}[u]^j_K(x) = \eta_K(x_{\neg K})/g'_K(x_K), \tag{62}$$

and for each $k \neq j$

$$\text{MRS}[u]^j_K(x) = \eta_k(x_{\neg k}); \tag{63}$$

where $\log \eta_j(x_{\neg j})$, $\log \eta_K(x_{\neg K})$, and each $\eta_k(x_{\neg k})$ are all constants of integration. From Equations 61–63 we can deduce that there exists a $\lambda_j \in \Re_{++}$ such that

$$\text{MRS}[u]^j_K(x) = \frac{\lambda_j g'_j(x_j)}{g'_K(x_K)}. \tag{64}$$

Letting $\lambda_K = 1 \in \Re_{++}$, we then have

$$\text{MRS}[v]^j_K(x) = \frac{\lambda_j g'_j(x_j)}{\lambda_K g'_K(x_K)} = \text{MRS}[u]^j_K(x). \tag{65}$$

And by Proposition 2.1 we can conclude that there exists a strictly increasing, C^1 function $f : v[X] \to \Re$ such that $u = f(v)$. Thus (ii) holds.

Conversely, if (ii) holds then for each $1 \leq k \leq K$ and for each $1 \leq j < l \leq K$ and $x \in X$ we have

$$\begin{aligned}\sum_{i=1}^{K} \frac{\partial \text{MRS}[u]^j_l(x)}{\partial x_i} S^k_i(x) &= \begin{cases} [\lambda_j/\lambda_l g'_l(x_l)][g''_j(x_j)/g'_j(x_j)] & \text{for } k = j, \\ -[\lambda_j g'_j(x_j)/\lambda_l] g''_l(x_l)[g'_l(x_l)]^{-3} & \text{for } k = l, \\ 0 & \text{for } j \neq k \neq l; \end{cases} \\ &= \sum_{i=1}^{K} \text{MRS}[u]^i_l(x) \left[\text{MRS}[u]^j_l(x) \frac{\partial S^k_i(x)}{\partial x_l} - \frac{\partial S^k_i(x)}{\partial x_j} \right]. \end{aligned} \tag{66}$$

Hence by Theorem 2.10 we have that S^k is a symmetry field of $\succsim$, and (i) holds. □

Proof of Proposition 3.5. If (i) holds, then S^3 is a symmetry field of $\succsim$ and thus for each $1 \leq j \leq 2$ and $x \in X$ we have

$$\frac{\partial \text{MRS}[u]^j_3(x)}{\partial x_3} \frac{1}{g'(x_3)} = \text{MRS}[u]^j_3(x) \left[\frac{\partial}{\partial x_3} \frac{1}{g'(x_3)} \right] \tag{67}$$

by Theorem 2.10. Integrating then yields

$$\text{MRS}[u]^j_3(x) = \frac{\eta_j(x_1, x_2)}{g'(x_3)}, \tag{68}$$

where $\log \eta_j(x_1,x_2)$ is a constant of integration. From the identity

$$\frac{\partial \text{MRS}[u]_3^1(x)}{\partial x_2} - \frac{\partial \text{MRS}[u]_3^2(x)}{\partial x_1} = [\text{MRS}[u]_3^2(x)]^2 \frac{\partial \text{MRS}[u]_2^1(x)}{\partial x_3} \tag{69}$$

it follows that

$$\frac{1}{g'(x_3)}\left[\frac{\partial \eta_1(x_1,x_2)}{\partial x_2} - \frac{\partial \eta_2(x_1,x_2)}{\partial x_1}\right] = \left[\frac{\eta_2(x_1,x_2)}{g'(x_3)}\right]^2 \left[\frac{\partial}{\partial x_3}\frac{\eta_1(x_1,x_2)}{\eta_2(x_1,x_2)}\right] = 0, \tag{70}$$

and therefore

$$\frac{\partial \eta_1(x_1,x_2)}{\partial x_2} = \frac{\partial \eta_2(x_1,x_2)}{\partial x_1}. \tag{71}$$

Hence the vector field $G : \Re^2_{++} \to \Re^2_{++}$ defined by $G(x_1,x_2) = \langle \eta_1(x_1,x_2), \eta_2(x_1,x_2)\rangle$ is conservative and admits a strictly increasing, C^1 potential function $\theta : \Re^2_{++} \to \Re$. And we can then rewrite Equation 68 as

$$\text{MRS}[u]_3^j(x) = \frac{\partial \theta(x_1,x_2)/\partial x_j}{g'(x_3)}. \tag{72}$$

Turning to the symmetry fields S^1 and S^2, by Theorem 2.10 we have

$$\frac{\partial \text{MRS}[u]_3^1(x)/\partial x_1}{\partial h(x_1,x_2)/\partial x_1} = -\text{MRS}[u]_3^1(x)\left[\frac{\partial}{\partial x_1}\frac{1}{\partial h(x_1,x_2)/\partial x_1}\right], \tag{73}$$

$$\frac{\partial \text{MRS}[u]_3^2(x)/\partial x_1}{\partial h(x_1,x_2)/\partial x_1} = -\text{MRS}[u]_3^1(x)\left[\frac{\partial}{\partial x_2}\frac{1}{\partial h(x_1,x_2)/\partial x_1}\right], \tag{74}$$

$$\frac{\partial \text{MRS}[u]_3^1(x)/\partial x_2}{\partial h(x_1,x_2)/\partial x_2} = -\text{MRS}[u]_3^2(x)\left[\frac{\partial}{\partial x_1}\frac{1}{\partial h(x_1,x_2)/\partial x_2}\right], \tag{75}$$

$$\frac{\partial \text{MRS}[u]_3^2(x)/\partial x_2}{\partial h(x_1,x_2)/\partial x_2} = -\text{MRS}[u]_3^2(x)\left[\frac{\partial}{\partial x_2}\frac{1}{\partial h(x_1,x_2)/\partial x_2}\right]; \tag{76}$$

and substituting Equation 72 into Equations 74 and 75 yields

$$\frac{\partial^2 h(x_1,x_2)}{\partial x_1 \partial x_2}\frac{\partial \theta(x_1,x_2)/\partial x_1}{\partial h(x_1,x_2)/\partial x_1} = \frac{\partial^2 \theta(x_1,x_2)}{\partial x_1 \partial x_2} = \frac{\partial^2 h(x_1,x_2)}{\partial x_1 \partial x_2}\frac{\partial \theta(x_1,x_2)/\partial x_2}{\partial h(x_1,x_2)/\partial x_2}, \tag{77}$$

so that

$$\Psi(x_1,x_2) = \frac{\partial^2 h(x_1,x_2)}{\partial x_1 \partial x_2}\left[\frac{\partial \theta(x_1,x_2)/\partial x_1}{\partial h(x_1,x_2)/\partial x_1} - \frac{\partial \theta(x_1,x_2)/\partial x_2}{\partial h(x_1,x_2)/\partial x_2}\right] = 0. \tag{78}$$

Equations 73–76 now imply, respectively, that

$$\frac{1}{g'(x_3)}\left[\frac{\partial}{\partial x_1}\frac{\partial \theta(x_1,x_2)/\partial x_1}{\partial h(x_1,x_2)/\partial x_1}\right] = \frac{\partial}{\partial x_1}\frac{\text{MRS}[u]_3^1(x)}{\partial h(x_1,x_2)/\partial x_1} = 0, \tag{79}$$

$$\frac{1}{g'(x_3)}\left[\frac{\partial}{\partial x_1}\frac{\partial \theta(x_1,x_2)/\partial x_2}{\partial h(x_1,x_2)/\partial x_2}\right] = \frac{\partial}{\partial x_1}\frac{\text{MRS}[u]_3^2(x)}{\partial h(x_1,x_2)/\partial x_2} = \frac{\Psi(x_1,x_2)/g'(x_3)}{\partial h(x_1,x_2)/\partial x_2} = 0, \tag{80}$$

$$\frac{1}{g'(x_3)}\left[\frac{\partial}{\partial x_2}\frac{\partial \theta(x_1,x_2)/\partial x_1}{\partial h(x_1,x_2)/\partial x_1}\right] = \frac{\partial}{\partial x_2}\frac{\text{MRS}[u]_3^1(x)}{\partial h(x_1,x_2)/\partial x_1} = \frac{-\Psi(x_1,x_2)/g'(x_3)}{\partial h(x_1,x_2)/\partial x_1} = 0, \tag{81}$$

$$\frac{1}{g'(x_3)}\left[\frac{\partial}{\partial x_2}\frac{\partial \theta(x_1,x_2)/\partial x_2}{\partial h(x_1,x_2)/\partial x_2}\right] = \frac{\partial}{\partial x_2}\frac{\text{MRS}[u]_3^2(x)}{\partial h(x_1,x_2)/\partial x_2} = 0; \tag{82}$$

and it follows that for each $1 \leq j \leq 2$ there must exist a $\lambda_j \in \Re_{++}$ such that

$$\frac{\partial \theta(x_1, x_2)/\partial x_j}{\partial h(x_1, x_2)/\partial x_j} = \lambda_j. \tag{83}$$

Moreover, we have

$$\lambda_1 \frac{\partial^2 h(x_1^*, x_2^*)}{\partial x_1 \partial x_2} = \frac{\partial \theta^2(x_1^*, x_2^*)}{\partial x_1 \partial x_2} = \lambda_2 \frac{\partial^2 h(x_1^*, x_2^*)}{\partial x_1 \partial x_2} \tag{84}$$

and can therefore write $\lambda_1 = \lambda_2 = \lambda \in \Re_{++}$. But then for each $1 \leq j \leq 2$ we have

$$\textsc{mrs}[v]_3^j(x) = \lambda \frac{\partial h(x_1, x_2)/\partial x_j}{g'(x_3)} = \frac{\partial \theta(x_1, x_2)/\partial x_j}{g'(x_3)} = \textsc{mrs}[u]_3^j(x). \tag{85}$$

And by Proposition 2.1 we can conclude that there exists a strictly increasing, C^1 function $f : v[X] \to \Re$ such that $u = f(v)$. Thus (ii) holds.

Conversely, if (ii) holds then for each $1 \leq j, k \leq 2$ and $x \in X$ we have

$$\begin{aligned}\sum_{i=1}^{K} \frac{\partial \textsc{mrs}[u]_3^j(x)}{\partial x_i} S_i^k(x) &= \frac{\lambda}{g'(x_3)} \frac{\partial^2 h(x_1, x_2)/\partial x_j \partial x_k}{\partial h(x_1, x_2)/\partial x_k} \\ &= \sum_{i=1}^{K} \textsc{mrs}[u]_3^i(x) \left[\textsc{mrs}[u]_3^j(x) \frac{\partial S_i^k(x)}{\partial x_3} - \frac{\partial S_i^k(x)}{\partial x_j} \right], \end{aligned} \tag{86}$$

as well as

$$\begin{aligned}\sum_{i=1}^{K} \frac{\partial \textsc{mrs}[u]_2^1(x)}{\partial x_i} S_i^k(x) &= \left[\frac{\partial}{\partial x_k} \frac{\partial h(x_1, x_2)/\partial x_1}{\partial h(x_1, x_2)/\partial x_2} \right] \frac{1}{\partial h(x_1, x_2)/\partial x_k} \\ &= \sum_{i=1}^{K} \textsc{mrs}[u]_2^i(x) \left[\textsc{mrs}[u]_2^1(x) \frac{\partial S_i^k(x)}{\partial x_2} - \frac{\partial S_i^k(x)}{\partial x_1} \right], \end{aligned} \tag{87}$$

so that S^k is a symmetry field of $\succsim$ by Theorem 2.10. Moreover, for each $1 \leq j \leq 2$ and $x \in X$ we have both

$$\begin{aligned}\sum_{i=1}^{K} \frac{\partial \textsc{mrs}[u]_3^j(x)}{\partial x_i} S_i^3(x) &= -\frac{\lambda g''(x_3)}{[g'(x_3)]^3} \frac{\partial h(x_1, x_2)}{\partial x_j} \\ &= \sum_{i=1}^{K} \textsc{mrs}[u]_3^i(x) \left[\textsc{mrs}[u]_3^j(x) \frac{\partial S_i^3(x)}{\partial x_3} - \frac{\partial S_i^3(x)}{\partial x_j} \right] \end{aligned} \tag{88}$$

and

$$\sum_{i=1}^{K} \frac{\partial \textsc{mrs}[u]_2^1(x)}{\partial x_i} S_i^3(x) = 0 = \sum_{i=1}^{K} \textsc{mrs}[u]_2^i(x) \left[\textsc{mrs}[u]_2^1(x) \frac{\partial S_i^3(x)}{\partial x_2} - \frac{\partial S_i^3(x)}{\partial x_1} \right], \tag{89}$$

so that S^3 is also a symmetry field of $\succsim$, and (i) holds. $\square$

Proof of Proposition 3.6. If (i) holds, then by Proposition 3.7 there exist a C^1 function $h : \Re \to \Re_{++}$ and a strictly increasing, C^1 function $g : w[X] \to \Re$, where w is defined by

$$w(x) = \log x_1 + \int_1^{\log[x_2/x_1]} \frac{\mathrm{d}t}{h(t)+1}, \tag{90}$$

such that $u = g(w)$. Letting $v = \exp w$ and $f = g(\log[\cdot])$, we have that v is homogeneous of degree one, f is strictly increasing, and $u = g(w) = g(\log v) = f(v)$. Thus (ii) holds.

Conversely, if (ii) holds then there exists a C^1 function $\mu : \Re_{++} \to \Re$ such that $\forall x \in X$ we have $v(x) = x_2\mu(x_1/x_2)$. Therefore

$$\text{MRS}[u]_2^1(x) = \text{MRS}[v]_2^1(x) = \frac{\mu'(x_1/x_2)}{\mu(x_1/x_2) - [x_1/x_2]\mu'(x_1/x_2)}, \tag{91}$$

and since this quantity depends on x only through the ratio x_1/x_2 it follows that

$$\frac{\partial \text{MRS}[u]_2^1(x)}{\partial x_2} x_2 = -\frac{\partial \text{MRS}[u]_2^1(x)}{\partial x_1} x_1. \tag{92}$$

We then have

$$\sum_{i=1}^{K} \frac{\partial \text{MRS}[u]_2^1(x)}{\partial x_i} S_i(x) = 0 = \sum_{i=1}^{K} \text{MRS}[u]_2^i(x) \left[\text{MRS}[u]_2^1(x) \frac{\partial S_i(x)}{\partial x_2} - \frac{\partial S_i(x)}{\partial x_1}\right]. \tag{93}$$

Hence by Theorem 2.10 we have that S is a symmetry field of $\succsim$, and (i) holds. □

Proof of Proposition 3.7. If (i) holds, then $\forall x \in X$ we have

$$\frac{\partial \text{MRS}[u]_2^1(x)}{\partial x_1} \frac{1}{g_1'(x_1)} + \frac{\partial \text{MRS}[u]_2^1(x)}{\partial x_2} \frac{1}{g_2'(x_2)} = \text{MRS}[u]_2^1(x) \left[\frac{g_1''(x_1)}{[g_1'(x_1)]^2} - \frac{g_2''(x_2)}{[g_2'(x_2)]^2}\right] \tag{94}$$

by Theorem 2.10. Defining $\mu(x) = \log \text{MRS}[u]_2^1(x)$ and expressing Equation 94 as

$$\frac{\partial \mu(x)}{\partial x_1} \frac{1}{g_1'(x_1)} + \frac{\partial \mu(x)}{\partial x_2} \frac{1}{g_2'(x_2)} = \frac{g_1''(x_1)}{[g_1'(x_1)]^2} - \frac{g_2''(x_2)}{[g_2'(x_2)]^2}, \tag{95}$$

a particular solution is $\mu(x) = \log g_1'(x_1) - \log g_2'(x_2)$. Furthermore, the general solution of the homogeneous equation

$$\frac{\partial \bar{\mu}(x)}{\partial x_1} \frac{1}{g_1'(x_1)} + \frac{\partial \bar{\mu}(x)}{\partial x_2} \frac{1}{g_2'(x_2)} = 0 \tag{96}$$

is $\bar{\mu}(x) = \log h(g_2(x_2) - g_1(x_1))$, where $h : \Re \to \Re_{++}$ is an arbitrary C^1 function. Hence the general solution of Equation 95 is

$$\mu(x) = \log g_1'(x_1) - \log g_2'(x_2) + \log h(g_2(x_2) - g_1(x_1)), \tag{97}$$

which implies that

$$\text{MRS}[v]_2^1(x) = \frac{g_1'(x_1)}{g_2'(x_2)} h(g_2(x_2) - g_1(x_1)) = \exp \mu(x) = \text{MRS}[u]_2^1(x). \tag{98}$$

And by Proposition 2.1 we can conclude that there exists a strictly increasing, C^1 function $f : v[X] \to \Re$ such that $u = f(v)$. Thus (ii) holds.

Conversely, if (ii) holds then $\forall x \in X$ we have

$$\sum_{i=1}^{K} \frac{\partial \text{MRS}[u]_2^1(x)}{\partial x_i} S_i(x) = \frac{g_1'(x_1)}{g_2'(x_2)} h(g_2(x_2) - g_1(x_1)) \left[\frac{g_1''(x_1)}{[g_1'(x_1)]^2} - \frac{g_2''(x_2)}{[g_2'(x_2)]^2} \right]$$
$$= \sum_{i=1}^{K} \text{MRS}[u]_2^i(x) \left[\text{MRS}[u]_2^1(x) \frac{\partial S_i(x)}{\partial x_2} - \frac{\partial S_i(x)}{\partial x_1} \right]. \quad (99)$$

Hence by Theorem 2.10 we have that S is a symmetry field of $\succsim$, and (i) holds. □

Proof of Proposition 3.8. If (i) holds, then since S^1 and S^2 are both symmetry fields of $\succsim$ there exists a $\lambda > 0$ such that $\forall x \in X$ we have

$$\text{MRS}[u]_2^1(x) = \frac{\lambda s_2(x_2)}{s_1(x_1)}, \quad (100)$$

a special case of Equation 64. Since S^3 too is a symmetry field of $\succsim$, we have also

$$\frac{\partial \text{MRS}[u]_2^1(x)}{\partial x_1} x_1 + \frac{\partial \text{MRS}[u]_2^1(x)}{\partial x_2} x_2 = -\text{MRS}[u]_2^1(x) + \text{MRS}[u]_2^1(x) = 0 \quad (101)$$

by Theorem 2.10. Combining Equations 100 and 101 yields

$$\frac{-\lambda s_2(x_2) s_1'(x_1)}{[s_1(x_1)]^2} x_1 + \frac{\lambda s_2'(x_2)}{s_1(x_1)} x_2 = 0. \quad (102)$$

There must then exist a $p \in \Re$ such that

$$\frac{x_1 s_1'(x_1)}{s_1(x_1)} = 1 - p = \frac{x_2 s_2'(x_2)}{s_2(x_2)}, \quad (103)$$

with $p \geq 0$ since $s_1'(x_1) \leq s_1(x_1)/x_1$. For each $1 \leq k \leq 2$ it follows that $s_k(x_k) = \eta_k x_k^{1-p}$, where $\eta_k \in \Re_{++}$ is a constant of integration, and from Equation 100 we then have

$$\text{MRS}[u]_2^1(x) = \frac{\lambda \eta_2}{\eta_1} \left[\frac{x_2}{x_1} \right]^{1-p}. \quad (104)$$

Letting $a_1 = \lambda \eta_2 > 0$ and $a_2 = \eta_1 > 0$, we now have

$$\text{MRS}[v]_2^1(x) = \frac{a_1}{a_2} \left[\frac{x_2}{x_1} \right]^{1-p} = \frac{\lambda \eta_2}{\eta_1} \left[\frac{x_2}{x_1} \right]^{1-p} = \text{MRS}[u]_2^1(x). \quad (105)$$

And by Proposition 2.1 we can conclude that there exists a strictly increasing, C^1 function $f : v[X] \to \Re$ such that $u = f(v)$. Thus (ii) holds.

If (ii) holds then we can let $\chi = f$ and each

$$g_k(x_k) = \begin{cases} a_k \log x_k & \text{for } p = 0, \\ a_k x_k^p & \text{for } p > 0; \end{cases} \quad (106)$$

whereupon $g_k''(x_k) \geq -g_k'(x_k)/x_k$ and $\forall x \in X$ we have

$$u(x) = f(v(x)) = \begin{cases} f(a_1 \log x_1 + a_2 \log x_2) & \text{for } p = 0, \\ f(a_1 x_1^p + a_2 x_2^p) & \text{for } p > 0; \end{cases}$$
$$= f(g_1(x_1) + g_2(x_2)) = \chi(w(x)). \quad (107)$$

Likewise, we can let

$$\hat{w}(x) = \begin{cases} [x_1^{a_1} x_2^{a_2}]^{\frac{1}{a_1+a_2}} & \text{for } p = 0, \\ [a_1 x_1^p + a_2 x_2^p]^{1/p} & \text{for } p > 0; \end{cases} \quad (108)$$

$$\hat{\chi}(\xi) = \begin{cases} f(\log \xi^{a_1+a_2}) & \text{for } p = 0, \\ f(\xi^p) & \text{for } p > 0; \end{cases} \quad (109)$$

whereupon $\hat{w}$ is homogeneous of degree one, $\hat{\chi}$ is strictly increasing, and $\forall x \in X$ we have

$$u(x) = f(v(x)) = \begin{cases} f(a_1 \log x_1 + a_2 \log x_2) & \text{for } p = 0, \\ f(a_1 x_1^p + a_2 x_2^p) & \text{for } p > 0; \end{cases}$$
$$= \left\{ \begin{array}{ll} f(\log[\hat{w}(x)]^{a_1+a_2}) & \text{for } p = 0, \\ f([\hat{w}(x)]^p) & \text{for } p > 0; \end{array} \right\} = \hat{\chi}(\hat{w}(x)). \quad (110)$$

Thus (iii) holds.

If (iii) holds then S^3 is a symmetry field of $\succsim$ by Proposition 3.6. For each $1 \leq k \leq 2$ and $x_k \in \Re_{++}$ let $s_k(x_k) = [g_k'(x_k)]^{-1} > 0$, so that

$$s_k'(x_k) = \frac{-g_k''(x_k)}{[g_k'(x_k)]^2} \leq \frac{1}{x_k g_k'(x_k)} = \frac{s_k(x_k)}{x_k}. \quad (111)$$

Then S^1 and S^2 are also symmetry fields of $\succsim$, by Proposition 3.3, and (i) holds. □

References

[1] Gerard Debreu. Topological methods in cardinal utility theory. In Kenneth J. Arrow, Samuel Karlin, and Patrick C. Suppes, editors, *Mathematical Methods in the Social Sciences*, chapter 2, pages 16–26. Stanford University Press, 1959.

[2] Gerard Debreu. Smooth preferences. *Econometrica*, 40(4):603–615, July 1972.

[3] Gerard Debreu. Smooth preferences: A corrigendum. *Econometrica*, 44(4):831–832, July 1976.

[4] Peter C. Fishburn. Utility theory. *Management Science*, 14(5):335–378, January 1968.

[5] Paolo Ghirardato, Fabio Maccheroni, and Massimo Marinacci. Certainty independence and the separation of utility and beliefs. *Journal of Economic Theory*, 120(1):129–136, January 2005.

[6] Itzhak Gilboa and David Schmeidler. Maxmin expected utility with non-unique prior. *Journal of Mathematical Economics*, 18(2):141–153, 1989.

[7] Wassily W. Leontief. Introduction to a theory of the internal structure of functional relationships. *Econometrica*, 15(4):361–373, October 1947.

[8] Fabio Maccheroni, Massimo Marinacci, and Aldo Rustichini. Ambiguity aversion, robustness, and the variational representation of preferences. *Econometrica*, 74(6):1447–1498, November 2006.

[9] Andreu Mas-Colell. Regular, nonconvex economies. *Econometrica*, 45(6):1387–1407, September 1977.

[10] A. Emmy Noether. Invariante Variationsprobleme. *Nachrichten von der Königlichen Gesellschaft der Wissenschaften zu Göttingen: Mathematisch-physikalische Klasse*, 1918.

[11] Paul A. Samuelson. *Foundations of Economic Analysis*. Atheneum, New York, 1965.

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