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Abstract

We consider the issue of Block Bootstrap methods in processes that exhibit strong dependence. The main difficulty is to transform the series in such way that implementation of these techniques can provide an accurate approximation to the true distribution of the test statistic under consideration. The bootstrap algorithm we suggest consists of the following operations: given $x_t \sim I(d_0)$, 1) estimate the long memory parameter and obtain $\widehat{d}$, 2) difference the series $\widehat{d}$ times, 3) apply the block bootstrap on the above and finally, 4) cumulate the bootstrap sample $\widehat{d}$ times. Repetition of steps 3 and 4 for a sufficient number of times, results to a successful estimation of the distribution of the test statistic. Furthermore, we establish the asymptotic validity of this method. Its finite-sample properties are investigated via Monte Carlo experiments and the results indicate that it can be used as an alternative, and in most of the cases to be preferred than the Sieve AR bootstrap for fractional processes.

JEL Code: C15; C22; C63.

Keywords: Block Bootstrap; Long Memory; Resampling; Strong Dependence.

1 Introduction

Since the seminal paper by Efron (1979), the bootstrap has increased rapidly in popularity. The two major factors that contributed to this are: (i) the decreased computational cost and, (ii) the improved theoretical results that stress that under mild conditions the bootstrap provides empirical approximations that are at least as accurate as the approximations implied by the first-order asymptotic distribution theory.

In the nonparametric methodology of Efron (1979), the sampling distribution of iid observations is estimated by resampling. Carlstein (1986) and Kunsch (1989) introduced resampling techniques for weakly dependent stationary observations. In these, the data is divided in blocks which are approximately independent and the joint distribution of the variables in different blocks is about to be the same due to stationarity. In the first one, the blocks do not overlap one another (henceforth

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denoted by NBB), whereas in the second method blocks are overlapping (henceforth mentioned as MBB). Politis and Romano (1992) and Politis and Romano (1994) introduced two other closely related methods: the Circular Block Bootstrap (henceforth denoted by CBB) and the Stationary Bootstrap (henceforth mentioned as SB). In both these methods the observed series are extended periodically and due to this common characteristic, their generalization is called the Dependent Bootstrap (see Politis and White (2004)). It is worth noticing here that the CBB and SB share the same asymptotic properties with the MBB, resulting however in better finite sample estimates.

A crucial issue to all the above techniques is the optimal choice of block length. Hall, Horowitz and Jing (1995) suggested that block length varies given specifications of different models. However, they concluded that three potential choices for the block length could be $b = n^{\lambda}$, $\lambda \in \left\{\frac{1}{3}, \frac{1}{4}, \frac{1}{5}\right\}$ where n denotes the sample size. Buhlmann and Kunsch (1999) introduced a data-dependent algorithm that results to the optimal block choice based on the equivalence of the block bootstrap variance to a lag weight estimator of the spectral density. Politis and White (2004) and Patton, Politis and White (2009) established a data-dependent method that successfully provides the optimal block length for the CBB and the SB methods. The difference among the two bootstraps is that SB uses blocks of random length. In particular, the distribution of the block lengths in the latter method is the Geometric with mean being the optimal block length of CBB.

Despite the fact that significant research has been done concerning the bootstrap and fractionally integrated series, there is little work that addresses the application of bootstrap to long memory processes. Poskitt (2007) introduced the most used bootstrap in $I(d_0)$ data, the Sieve Bootstrap, which is an extension of previous research by Kreiss (1992). Another approach was done by Hidalgo (2003) who suggested a valid resampling scheme in the frequency domain. More details can be found in Park (2002), Kapetanios (2004), Andrews (2006), Kapetanios and Psaradakis (2006) etc.

This paper aims to provide a new block bootstrap procedure that can be applied in long memory processes with consistency. Following Lahiri (2003) and Lahiri (2006), a direct application of any of the block methods described above fails. The general idea of this study is to transform the original data in a weak dependent data, obtain block bootstrap resamples from that data and then re-transform the resamples such that the original dependence is preserved (or "imitated").

At first, we semi parametrically estimate the long memory parameter, d_0, denoting the estimator by $\widehat{d}$. Hence, from this point of view our method is not non parametric. However, neither our competitor, the Sieve AR bootstrap, is non parametric. Then, we apply the difference operator, $(1-L)^{\widehat{d}}$, and we obtain a weak dependent data. We resample from this data and we cumulate the resamples using $(1-L)^{-\widehat{d}}$. Following Hall, Horowitz and Jing (1995) and Politis and White (2004) we use the same methods for the calculation of block sizes. Generic asymptotic results, independent of the exact block method or block size, can be found in the appendix. Furthermore, we conduct a simulation study using different models and sample sizes and we compare them to the Sieve Bootstrap. The outcome is that the out bootstrap algorithm provides better finite results than the Sieve Bootstrap and hence, it should be preferred.

The rest of the paper is divided as follows: section 2 introduces the long memory topic and the Block Bootstrap methods we use, section 3 presents the bootstrap algorithm we propose, section 4 discusses numerical evidence on the distribution of the Normalized Sample Mean and the Gaussian Semi Parametric Estimator and, section 5 summarizes the conclusions.

2 Preliminaries

2.1 Long Memory and Sieve Bootstrap

In general, a fractionally integrated process x_t can be represented as,

$$(1-L)^{d_0} x_t = u_t, \ t = 1, ..., n \tag{1}$$

where L is the lag operator, d_0 is the long memory parameter and $u_t \sim I(0)$. For $|d_0| \leq 0.5$ the series is stationary and invertible, whereas for $0.5 < |d_0| < 1$ the variance is not finite but still has a cumulative impulse response function with finite sum.

The fractional differencing operator is defined by the binomial expansion as,

$$(1-L)^{d_0} = \sum_{j=0}^{\infty} \frac{\Gamma(j-d_0)}{\Gamma(-d_0)\Gamma(j+1)} = 1 - d_0 L + \frac{d_0(d_0-1)}{2} L^2 ..., \tag{2}$$

where $\Gamma(\cdot)$ denotes the gamma function. This expansion can be written as,

$$(1-L)^{d_0} = \sum_{j=0}^{\infty} \alpha_j(d_0) L^j. \tag{3}$$

The most well-known, bootstrap procedure for such data is the sieve bootstrap by Kreiss (1992) and Buhlmann (1997). Following Poskitt (2007), the sieve bootstrap algorithm suggests to:

1. fit an $AR(h)$, $h < \infty$, model in the data, obtain the residuals and create the standardized residuals denoted by $\widehat{u}_t$.
2. Then, create a new randomly resampled series of the above, denoted by $\widehat{u}_t^*$. Apply the desired test statistic on a bootstrap resample x_t^* that is generated by h AR terms and $\widehat{u}_t^*$ error.

3. By repeating the above procedure a number of times B we obtain a bootstrap approximation to the distribution of the test statistic.

The order autoregressive order h can be derived using any information criterion, such as Akaike's or Bayesian. Ng and Perron (1995) prove that under similar setup, AIC and BIC are asymptotically consistent.

2.2 Block Bootstrap

The bootstrap succeeds in the approximation of the underlying distribution due to the assumption that observations are independent. The idea behind block bootstrap is similar. Blocks of observations of stationary processes should be approximately independent and due to stationarity the joint distribution of the variables in different blocks should be almost the same. The main difficulty we confront is the choice of that optimal block size that the above happens. Using data driven methods, like Hall, Horowitz and Jing (1995) and Politis and White (2004), we can find these choices. Suppose we have $x_t \sim I(0)$, $t = 1, 2, ..., n$, a choice of optimal block size, say b, and $M = \left[\frac{n}{b}\right]$ where $[\cdot]$ denotes integer part. Then,

1. x_t:
   (a) (NBB) is divided in M disjoint blocks, the $k-th$ being $x_k^{*NBB} = \left\{x_{(k-1)b+1}, ..., x_{kb}\right\}$ for $1 \leq k \leq M$,
   (b) (MBB) is divided in $(n-b)+1$ overlapping blocks, the $k-th$ being $x_k^{*MBB} = \left\{x_k, ..., x_{(k+b)-1}\right\}$ for $1 \leq k \leq (n-b)+1$,
   (c) (CBB) is wrapped around the beginning (periodically extended) as $x_1, x_2, ..., x_n, x_1, x_2, ..., x_n, ...$. Then blocks are defined as $x_k^{*CBB} = \{x_k, ..., x_{n+k-1}\}$.
   (d) (SB) is wrapped around as before and the blocks defined are of random length. The distribution of the blocks, F_b, is the Geometric with mean equal to b.
2. Given our preferred block method described above, we randomly choose M blocks by resampling with replacement and we obtain $x_t^* = \{x_1^*, x_2^*, ..., x_M^*\}$. Then the test statistic is performed on this bootstrap data.
3. We repeat step 3 a number of times B, in order to obtain a bootstrap approximation to the distribution of the test statistic.

As $B \to \infty$ accuracy of results is guaranteed.

3 A Block Bootstrap Procedure for Long Memory Time Series

The key problem for the application of the Block Bootstrap in long memory time series is the degree of integration. In this part of the paper we propose a methodology that overcomes this difficulty.

Given that the exact value of the degree of long memory is unknown, the main idea is to difference the data using an estimate $\widehat{d}$ in order to obtain weak dependence. Then, we cumulate bootstrap resample $\widehat{d}$ times using the estimate. Any method for the estimation of d that satisfies the following general assumption can be used.

Assumption 1 $\widehat{d} - d_0 = O_p\left(n^{-\delta}\right)$, $\delta > 0$.

d_0 denotes the true degree of integration and n is the sample size

The above is a rather general assumption that does not specify any particular method for estimating d_0. It assumes consistency and some rate of convergence for the estimator. Note that there is complete literature regarding estimators with desirable asymptotic properties even when $|d_0| > 0.5$ (see Velasco (1999) etc.). However, this assumption does not allow for estimators whose rate of convergence depends on d_0.

The algorithm we suggest consists of the following operations:

1. Given $x_t \sim I(d_0)$, $t = 1, 2, ..., n$, we estimate the fractional exponent and we obtain $\widehat{d}$; for more details see Robinson (1995), Baillie and Kapetanios (2008) etc.

2. We difference x_t $\widehat{d}$ times denoting the new series by z_t, $z_t \sim I\left(\widehat{d} - d_0\right)$.

3. Using any of the block bootstrap methods described in the previous section, we create a bootstrap resample of z_t and we cumulate it using $\widehat{d}$. We denote the resulting series by $z_t^* \sim (d_0)$. Then, the desired test statistic is applied.

4. We repeat step 3 B times and we obtain a bootstrap approximation to the distribution of the test statistic.

Theorem 1 *Let $\eta\left(F_X, F_Y\right)$ denote the Mallow's measure of the distance between two probability distributions F_X and F_Y, defined as $\inf\left\{E\left\|X - Y\right\|^2\right\}^{\frac{1}{2}}$, where the infimum is taken over all square integrable random variables X and Y in $\mathbb{R}$ with marginal distributions F_X and F_Y. Then,*

$$\eta\left(F_{S_n^*(\hat{d},\hat{d})}, F_{S_n}\right) = 0 \text{ as } n \to \infty,$$

where S_n denotes the test statistic, F_{S_n} denotes its true distribution and $F_{S_n^*(\hat{d},\hat{d})}$ denotes the bootstrap approximation.

The above theorem is closely related to Theorem 2 in Poskitt (2007) and states that the bootstrap estimate to the true distribution of the test statistic obtained using the previously described algorithm is consistent with the true distribution.

4 Simulation Experiments

This section is concerned with numerical evidence that illustrates the finite-sample performance of the block bootstrap for long range dependent processes. We discuss two sets of experiments, one for the case of the approximation to the distribution of the sample mean and one for the approximation to the distribution of the Gaussian Semi Parametric Estimator, commonly referred to as the Local Whittle.

In all experiments we use $ARFIMA\,(p,d,q)$ models to generate the data. Specifically, we present the following designs: $(\varphi_1,\vartheta_1)=(0,0)$, $(\varphi_1,\vartheta_1)=(0.8,0)$, $(\varphi_1,\vartheta_1)=(0,0.8)$, $(\varphi_1,\vartheta_1)=(-0.3,0.4)$, $(\varphi_1,\vartheta_1)=(0.3,-0.4)$, $(\varphi_1,\varphi_2,\vartheta_1)=(0.2,0.7,0)$ and $(\varphi_1,\varphi_2,\vartheta_1)=(0.7,0.2,0)$. For each case the error term is standard normal, $d\in\{0.2,0.4,0.8\}$ and the sample size is $n\in\{100,400,1000\}$. Tables 1 - 7 present the mean and variance of the estimate of the true distribution of the sample mean and Gaussian Semi Parametric Estimator along with the 95% confidence intervals for the mean. The $\widehat{d}$ used in step 1 of the algorithm is obtained using the Gaussian Semi Parametric Estimator and $h=(\ln n)^2$ for the sieve. The block sizes we use are those suggested by Hall, Horowitz and Jing (1995) and for the CBB and SB those introduced by Politis and White (2004).

4.1 Normalized Sample Mean

The Exact Moments of the distribution of the sample mean are computed from 1000 Monte Carlo values of $n^{\frac{1}{2}-d_0}\left(\overline{x}_n-\mu\right)$, where $\mu=0$. Similarly, the bootstrap estimates are computed as averages of 1000 Monte Carlo trials of 199 bootstrap resamples of $n^{\frac{1}{2}-\widehat{d}}\left(\overline{z_n^*}-\overline{x}_n\right)$, in respect to the notation of the bootstrap algorithm in section 3.

The simulation results suggest that the block bootstrap approximates very well the distribution of the sample mean. Especially when the block method is the CBB or SB, in most of the experiments in large samples ($n>400$), the results are better than those provided by the Sieve AR. This is a very promising result given that it argues and questions why we have to favor the Sieve AR bootstrap in practice, especially when it provides highest mean square error of the moment estimates than any of block bootstrap methods.

4.2 Gaussian Semi Parametric Estimator

It has been a great debate in the literature about the estimation methods of long range dependent time series. A common argument in favor of the semi parametric setup is that specification difficulties of the short-run components can be avoided. Geweke and Porter - Hudak (1983) introduced one of the early semi parametric methods in the frequency domain. However Agiakloglou, Newbold and Wohar (1992) showed its poor performance in relation to the short run dynamics. In more recent years, the Gaussian Semi Parametric Estimator of long range dependence, or *Local Whittle*, of Robinson (1995) is the most preferred method due to its satisfying statistical properties. Extensions of the above in nonlinear and non stationary time series have been developed by Dalla, Giraitis and Hidalgo (2005) and Velasco (1999) respectively.

The Local Whittle estimator of d_0, denoted by $\widehat{d}$, is obtained minimizing the following objective function,

$$R\left(d_0\right) = \log\left[\frac{1}{m}\sum_{j=1}^{m}\omega_j^{2d_0} I\left(\omega_j\right)\right] - 2d_0\frac{1}{m}\sum_{j=1}^{m}\log\omega_j, \tag{4}$$

where $I\left(\omega_j\right)$ is the periodogram of the series in the ω_j Fourier frequencies, $j = 1, 2, ..., m$ where $m < \frac{n}{2}$. Generally, m is chosen so that $\frac{1}{m} + \frac{m}{n} \to 0$ as $n \to \infty$. Obviously, the estimator is sensitive on that matter. Henry (2001) provides data dependent method for the choice of m. Following Robinson (1995),

$$m^{\frac{1}{2}}\left(\widehat{d} - d_0\right) \to_d N\left(0, \frac{1}{4}\right), \tag{5}$$

where d_0 denotes the true value. Generally, in the ignorance of short run dynamics, m is chosen to be $m = n^{1/2}$ and this is also the choice in our experiments.

The Exact Moments of the distribution of the Local Whittle are computed from a 1000 Monte Carlo replications, and the bootstrap estimates are computed from averages of 1000 bootstrap resamples as in step 3 of the bootstrap algorithm.

In the vast majority of the experiments the same result as in the case of normalized sample mean is repeated. The CBB and SB provide better estimates for the moments of the distribution of the Local Whittle making the Sieve to be the next best option. Particularly, the Sieve Bootstrap performs better in the vanilla $ARFIMA\left(0, d, 0\right)$ model. In more complex cases like $ARFIMA\left(1, d, 0\right)$ and $ARFIMA\left(2, d, 0\right)$ for all values of d_0 and in all sample sizes, the SB dominates.

At this point it can be argued that our method heavily relies on the estimation of the long memory parameter and also, from this point of view, our method is semi parametric. This is true. The use of a biased estimate of d_0 when applying the differencing operator might result in a "not so weak" dependent data (z_t in step 2 of our algorithm), especially when the true long memory parameter is high (e.g. $d_0 = 0.9$ and $\widehat{d} = 0.2$). However, this phenomenon is extremely rare in the most common data that is being used in practice. Regarding the setup of our bootstrap methodology, nonparametric econometrics are indeed much favored to semi or full parametric. However, given the nature of the problem, which is the unknown degree of integration, the initial estimation of d cannot be avoided. Furthermore, one must have in mind that neither the Sieve AR Bootstrap is non parametric.

5 Conclusion

In this paper we introduce the block bootstrap in strongly dependent time series. To the best of the authors' knowledge, application of block bootstrap in long memory time series has not been researched before.

In practice, the Sieve AR bootstrap is the most commonly used. Our algorithm tries to avoid the complicated steps of Sieve (i.e. at first the $AR(P)$ order must be obtained and secondly the model has to be estimated in order to retrieve the residuals). The procedure we suggest is described in the following steps: we difference the data using a valid estimate of the long memory parameter, we apply the block bootstrap of our choice and we cumulate this bootstrap resample using the same estimate. Repetition of the above procedure guarantees accurate approximation to the distribution of the test statistic.

The greatest advantage of the block bootstrap is the ease-of-use, making the applied researcher's work less complicated. At her favor, the numerical experiments provide evidence that our method estimates very well the distribution of the sample mean and the Gaussian Semi Parametric Estimator and can provide more accurate approximations in finite samples especially when the SB is used. To conclude with, this paper shows adequate evidence that, at last, the Block Bootstrap can be applied in fractionally integrated processes successfully.

6 Appendix

In this appendix we go through the asymptotic results of the bootstrap procedure described in section 3. Before we proceed with the proof of theorem 1, we first define some additional theory and notation. Using each step of the algorithm and the original long memory series $x_t \sim I(d_0)$, we have already defined $z_t^* \sim I(d_0)$ as the bootstrap resample using $\widehat{d}$ in differencing and cumulating (steps 2 and 3 respectively).

In addition to that, we also define $y_t^* \sim I(d_0)$ as the bootstrap resample in the theoretical case where d_0 is known and is used for differencing and cumulation and, $h_t^* \sim I\left(\widehat{d} - 2d_0\right)$ as the bootstrap resample in the case where we use $\widehat{d}$ for differencing and d_0 for cumulation.

The relevant bootstrap approximations to the true distribution of the test statistic are denoted by $F_{S_n^{*}(\hat{d},\hat{d})}$, $F_{S_n^{*}(d_0,d_0)}$ and $F_{S_n^{*}(\hat{d},d_0)}$ respective to z_t^*, y_t^* and h_t^*.

Following Poskitt (2007), we make use of the following assumption (assumption 4 in his paper).

Assumption 2 *Given a process x_t and $X_n = \{x_1,\ x_2, ..., x_n\}$, and the corresponding block bootstrap sample to be denoted by z_t^* and hence, $Z_n^* = \{z_1^*,\ z_2^*, ..., z_n^*\}$, let $\mathcal{N}$ be a compact subset of $\mathbb{R}^n$. Then for all X_n, $Z_n^*\ \epsilon\mathcal{N}$ there exists a family of Borel measurable functions $B_t : \mathbb{R}^n \times \mathbb{R}^n \to [0, \infty)$, satisfying,*

$$\lim_{n\to\infty} \sup \frac{1}{n} \sum_{t=1}^{n} E\left[E^*\left[B_t\left(X_n, Z_n^*\right)^2\right]\right] < \infty,$$

for which

$$\left\| S_n - S_n^{*\left(\widehat{d},\widehat{d}\right)} \right\|^2 \leq \frac{1}{n} \sum_{t=1}^{n} B_t\left(X_n, Z_n^*\right)^2 \left|x_t - z_t^*\right|.$$

where, $E[\cdot]$ is the expectation taken with respect to $P_{\{x_1,x_2,...,x_n\}}$, $E^*[\cdot]$ is the expectation taken with respect to $P_{\{z_1^*,z_2^*,...,z_n^*\}}$, S_n and S_n^* are the test statistics using the original and bootstrap samples respectively.

Proof. Then we have that,

$$\begin{aligned}
\left\{\eta\left(F_{S_n^*(\hat{d},\hat{d})}, F_{S_n}\right)\right\}^2 &\leq \left\{\eta\left(F_{S_n^*(\hat{d},\hat{d})}, F_{S_n^*(d_0,d_0)}\right)\right\}^2 + \left\{\eta\left(F_{S_n^*(d_0,d_0)}, F_{S_n}\right)\right\}^2 \\
&\leq \left\{\eta\left(F_{S_n^*(\hat{d},\hat{d})}, F_{S_n^*(\hat{d},d_0)}\right)\right\}^2 + \left\{\eta\left(F_{S_n^*(\hat{d},d_0)}, F_{S_n^*(d_0,d_0)}\right)\right\}^2 + \left\{\eta\left(F_{S_n^*(d_0,d_0)}, F_{S_n}\right)\right\}^2
\end{aligned} \tag{6}$$

The last term of the right hand side of the above is straightforward because d_0 is used, see Kunsch (1989) and Politis and White (2004). Thus, we focus our interest on the first and second terms.

From the definition of Mallow's metric, using assumption 2 and applying the Cauchy-Schwartz inequality twice we have,

$$
\begin{aligned}
\left\{\eta\left(F_{S_n^{*(\widehat{d},\widehat{d})}}, F_{S_n^{*(\widehat{d},d_0)}}\right)\right\}^2 &\leq E\left[E^*\left[\left\|S_n^{*(\widehat{d},\widehat{d})} - S_n^{*(\widehat{d},d_0)}\right\|^2\right]\right] \\
&\leq \frac{1}{n}\sum_{t=1}^{n} E\left[E^*\left[\left(B_n\left(H_n^*, Z_n^*\right)\right)^2\right]\right] \cdot \frac{1}{n}\sum_{t=1}^{n} E\left[E^*\left[\left(h_t^* - z_t^*\right)^2\right]\right] \\
&\leq \lim_{n\to\infty}\sup \frac{1}{n}\sum_{t=1}^{n} E\left[E^*\left[\left(B_n\left(H_n^*, Z_n^*\right)\right)^2\right]\right] \cdot \frac{1}{n}\sum_{t=1}^{n} E\left[E^*\left[\left(h_t^* - z_t^*\right)^2\right]\right] \\
&= \lim_{n\to\infty}\sup \frac{1}{n}\sum_{t=1}^{n} E\left[E^*\left[\left(B_n\left(H_n^*, Z_n^*\right)\right)^2\right]\right] \cdot \lim_{n\to\infty}\sup \frac{1}{n}\sum_{t=1}^{n} E\left[E^*\left[\left(h_t^* - z_t^*\right)^2\right]\right]. \qquad (7)
\end{aligned}
$$

Given that there exists such function $B_n\left(H_n^*, Z_n^*\right)$ so that the first sup limit is finite, we are interested in the last term of the right hand side of the above. Hence, we need to prove that,

$$
\lim_{n\to\infty}\sup \frac{1}{n}\sum_{t=1}^{n} E\left[E^*\left[\left(h_t^* - z_t^*\right)^2\right]\right] = 0. \qquad (8)
$$

By the definition of the long memory processes in eq. (1) we have that,

$$
h_t^* = (1-L)^{\widehat{d}-d_0} z_t^*. \qquad (9)
$$

Following the same analysis with Wright (1995) and using eq. (3),

$$
\begin{aligned}
h_t^* - z_t^* &= \sum_{j=1}^{t-1} \alpha_j\left(\widehat{d}-d_0\right) z_{t-j}^* \\
&= \left(\widehat{d}-d_0\right)\sum_{j=1}^{t-1} \alpha_j'\left(\widehat{d}-d_0\right) z_{t-j}^* + \left(\widehat{d}-d_0\right)^2 \sum_{j=1}^{t-1} \alpha_j''\left(\widehat{d}-d_0\right) z_{t-j}^*\,, \qquad (10)
\end{aligned}
$$

since,

$$
\alpha_j\left(\widehat{d}-d_0\right) = \left(\widehat{d}-d_0\right)\alpha_j'\left(\widehat{d}-d_0\right) + \left(\widehat{d}-d_0\right)^2 \alpha_j''\left(\widehat{d}-d_0\right), \qquad (11)
$$

where α_j' and α_j'' denote the first and second derivative of α_j respectively for $\left(\widehat{d}-d_0\right) \to 0$. Using eq. (8) and eq. (10), we need to prove that,

$$
\lim_{n\to\infty}\sup \frac{1}{n}\sum_{t=1}^{n} E\left[E^*\left[\left(\left(\widehat{d}-d_0\right)\sum_{j=1}^{t-1}\alpha_j'\left(\widehat{d}-d_0\right) z_{t-j}^* + \left(\widehat{d}-d_0\right)^2\sum_{j=1}^{t-1}\alpha_j''\left(\widehat{d}-d_0\right) z_{t-j}^*\right)^2\right]\right] = 0. \qquad (12)
$$

This follows from the fact that the variance of $\widehat{d}-d_0$ goes to zero when $n \to \infty$,

$$
\lim_{n\to\infty}\sup E\left[E^*\left[\left(\widehat{d}-d_0\right)^2\right]\right] = 0. \qquad (13)
$$

Indeed, the first limit of the identity of eq. (12) can be written as,

$$\begin{aligned}&\lim_{n\to\infty}\sup\frac{1}{n}\sum_{t=1}^{n}E\left[E^{*}\left[\left(\left(\widehat{d}-d_0\right)\sum_{j=1}^{t-1}\alpha'_j\left(\widehat{d}-d_0\right)z^{*}_{t-j}\right)^{2}\right]\right]\\ &\le\lim_{n\to\infty}\sup E\left[E^{*}\left[\left(\widehat{d}-d_0\right)^{2}\right]\right]\lim_{n\to\infty}\sup\frac{1}{n}\sum_{t=1}^{n}E\left[E^{*}\left[\left(\sum_{j=1}^{t-1}\alpha'_j\left(\widehat{d}-d_0\right)z^{*}_{t-j}\right)^{2}\right]\right],\end{aligned}\tag{14}$$

where the second limit is finite. Following the limit theorem by Robinson (1995) in eq. (5) and eq. (14, 13), eq. (12) is proved.

Similarly,

$$\begin{aligned}\left\{\eta\left(F_{S_n^{*(\widehat{d},d_0)}},F_{S_n^{*(d_0,d_0)}}\right)\right\}^{2}&\le E\left[E^{*}\left[\left\|S_n^{*\left(\widehat{d},d_0\right)}-S_n^{*(d_0,d_0)}\right\|^{2}\right]\right]\\ &\le\frac{1}{n}\sum_{t=1}^{n}E\left[E^{*}\left[\left(B_n\left(Y_n^{*},H_n^{*}\right)\right)^{2}\right]\right]\cdot\frac{1}{n}\sum_{t=1}^{n}E\left[E^{*}\left[\left(y_t^{*}-h_t^{*}\right)^{2}\right]\right]\\ &\le\lim_{n\to\infty}\sup\frac{1}{n}\sum_{t=1}^{n}E\left[E^{*}\left[\left(B_n\left(Y_n^{*},H_n^{*}\right)\right)^{2}\right]\right]\cdot\frac{1}{n}\sum_{t=1}^{n}E\left[E^{*}\left[\left(y_t^{*}-h_t^{*}\right)^{2}\right]\right]\\ &\le\lim_{n\to\infty}\sup\frac{1}{n}\sum_{t=1}^{n}E\left[E^{*}\left[\left(B_n\left(Y_n^{*},H_n^{*}\right)\right)^{2}\right]\right]\cdot\lim_{n\to\infty}\sup\frac{1}{n}\sum_{t=1}^{n}E\left[E^{*}\left[\left(y_t^{*}-h_t^{*}\right)^{2}\right]\right],\end{aligned}$$

and hence we need to prove that,

$$\lim_{n\to\infty}\sup\frac{1}{n}\sum_{t=1}^{n}E\left[E^{*}\left[\left(y_t^{*}-h_t^{*}\right)^{2}\right]\right]=0.\tag{15}$$

As before we have that,

$$h_t^{*}=(1-L)^{\widehat{d}-d_0}\,y_t^{*}.\tag{16}$$

Now,

$$\begin{aligned}y_t^{*}-h_t^{*}&=-\left(h_t^{*}-y_t^{*}\right)=-\left(\sum_{j=1}^{t-1}\alpha_j\left(\widehat{d}-d_0\right)y^{*}_{t-j}\right)\\ &=-\left(\left(\widehat{d}-d_0\right)\sum_{j=1}^{t-1}\alpha'_j\left(\widehat{d}-d_0\right)y^{*}_{t-j}+\left(\widehat{d}-d_0\right)^{2}\sum_{j=1}^{t-1}\alpha''_j\left(\widehat{d}-d_0\right)y^{*}_{t-j}\right).\end{aligned}\tag{17}$$

We need to prove that,

$$\lim_{n\to\infty}\sup\frac{1}{n}\sum_{t=1}^{n}E\left[E^{*}\left[\left(-\left(\left(\widehat{d}-d_0\right)\sum_{j=1}^{t-1}\alpha'_j\left(\widehat{d}-d_0\right)y^{*}_{t-j}+\left(\widehat{d}-d_0\right)^{2}\sum_{j=1}^{t-1}\alpha''_j\left(\widehat{d}-d_0\right)y^{*}_{t-j}\right)\right)^{2}\right]\right]=0\tag{18}$$

and as before, the result follows. ■

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Table 1. ARFIMA (0, d, 0)

	Distribution of the Sample Mean												Distribution of the Gaussian Semi Parametric Estimator											
	d=0.2				d=0.4				d=0.8				d=0.2				d=0.4				d=0.8			
	Mean	Conf. Bounds		St. D.	Mean	Conf. Bounds		St. D.	Mean	Conf. Bounds		St. D.	Mean	Conf. Bounds		St. D.	Mean	Conf. Bounds		St. D.	Mean	Conf. Bounds		St. D.
n=100																								
"Exact" Moment	-0.005	-1.532	1.466	0.962	-0.007	-1.395	1.449	0.861	0.011	-1.012	0.981	0.627	0.201	0.130	0.271	0.046	0.398	0.328	0.470	0.045	0.803	0.728	0.873	0.046
NBB ($\lambda = 1/3$)	-0.004	-0.306	0.291	0.183	-0.011	-0.356	0.339	0.211	-0.001	-0.415	0.435	0.257	0.230	-0.218	0.650	0.269	0.325	-0.117	0.702	0.251	0.778	0.324	1.145	0.248
MBB ($\lambda = 1/3$)	-0.003	-0.314	0.312	0.190	-0.014	-0.424	0.384	0.250	0.000	-0.484	0.463	0.289	0.197	-0.254	0.606	0.264	0.397	-0.033	0.805	0.259	0.826	0.415	1.189	0.240
NBB ($\lambda = 1/4$)	-0.002	-0.279	0.289	0.171	-0.013	-0.363	0.334	0.215	-0.003	-0.430	0.441	0.262	0.224	-0.217	0.647	0.265	0.406	-0.030	0.817	0.264	0.850	0.439	1.188	0.228
MBB ($\lambda = 1/4$)	-0.004	-0.308	0.303	0.182	-0.017	-0.414	0.387	0.247	-0.001	-0.465	0.469	0.285	0.200	-0.275	0.619	0.268	0.407	-0.014	0.792	0.253	0.837	0.426	1.198	0.245
NBB ($\lambda = 1/5$)	-0.003	-0.292	0.264	0.171	-0.012	-0.358	0.335	0.219	-0.005	-0.422	0.432	0.264	0.216	-0.230	0.621	0.259	0.408	-0.047	0.828	0.259	0.828	0.386	1.184	0.256
MBB ($\lambda = 1/5$)	-0.003	-0.298	0.300	0.179	-0.015	-0.405	0.392	0.242	-0.002	-0.465	0.461	0.285	0.191	-0.267	0.618	0.274	0.405	-0.017	0.790	0.249	0.844	0.402	1.231	0.252
CBB	-0.002	-0.281	0.277	0.166	-0.020	-0.413	0.415	0.247	-0.006	-0.487	0.473	0.293	0.212	-0.230	0.598	0.256	0.399	-0.036	0.797	0.255	0.853	0.424	1.232	0.250
SB	0.003	-0.301	0.300	0.185	0.001	-0.201	0.208	0.122	-0.002	-0.141	0.133	0.083	0.108	-0.158	0.368	0.161	0.334	-0.023	0.604	0.189	1.084	0.758	1.458	0.214
Sieve AR	-0.013	-1.513	1.586	0.959	-0.042	-1.379	1.358	0.856	0.017	-0.963	0.985	0.599	0.035	-0.204	0.231	0.134	0.213	-0.146	0.432	0.179	0.694	0.506	0.832	0.101
n=400																								
"Exact" Moment	0.008	-1.622	1.663	0.978	0.064	-1.326	1.393	0.831	0.003	-0.927	0.927	0.589	0.201	0.130	0.272	0.046	0.401	0.329	0.472	0.046	0.803	0.728	0.872	0.046
NBB ($\lambda = 1/3$)	0.002	-0.254	0.270	0.155	-0.003	-0.388	0.384	0.227	0.005	-0.468	0.433	0.269	0.230	-0.023	0.458	0.142	0.451	0.180	0.710	0.163	0.858	0.599	1.083	0.147
MBB ($\lambda = 1/3$)	0.005	-0.289	0.300	0.175	0.004	-0.401	0.409	0.250	0.003	-0.461	0.459	0.283	0.232	0.004	0.457	0.137	0.447	0.169	0.690	0.162	0.872	0.619	1.095	0.146
NBB ($\lambda = 1/4$)	0.004	-0.241	0.260	0.154	0.003	-0.386	0.385	0.237	0.004	-0.438	0.424	0.269	0.219	-0.024	0.460	0.149	0.457	0.174	0.707	0.162	0.861	0.603	1.095	0.150
MBB ($\lambda = 1/4$)	-0.001	-0.277	0.275	0.169	0.005	-0.387	0.400	0.242	0.002	-0.444	0.448	0.279	0.227	-0.001	0.445	0.141	0.456	0.178	0.709	0.163	0.871	0.598	1.086	0.153
NBB ($\lambda = 1/5$)	0.004	-0.249	0.261	0.155	0.004	-0.386	0.381	0.233	-0.001	-0.443	0.431	0.270	0.212	-0.034	0.442	0.147	0.460	0.201	0.694	0.156	0.875	0.619	1.092	0.149
MBB ($\lambda = 1/5$)	-0.007	-0.281	0.267	0.165	0.004	-0.394	0.395	0.244	-0.001	-0.456	0.433	0.280	0.221	-0.016	0.452	0.143	0.452	0.162	0.700	0.161	0.877	0.609	1.098	0.147
CBB	0.000	-0.255	0.263	0.158	0.003	-0.410	0.411	0.250	0.005	-0.448	0.468	0.279	0.220	-0.019	0.450	0.145	0.445	0.165	0.700	0.162	0.861	0.585	1.087	0.155
SB	0.004	-0.290	0.323	0.188	-0.001	-0.197	0.191	0.122	0.003	-0.130	0.114	0.075	0.272	0.128	0.436	0.095	0.486	0.341	0.638	0.088	0.897	0.757	1.024	0.080
Sieve AR	0.022	-1.526	1.716	0.999	0.000	-1.558	1.413	0.870	0.021	-0.932	0.920	0.570	0.136	-0.027	0.239	0.086	0.228	-0.048	0.426	0.148	0.765	0.645	0.843	0.063
n=1000																								
"Exact" Moment	0.015	-1.500	1.475	0.886	-0.002	-1.372	1.310	0.802	-0.015	-1.107	1.010	0.638	0.199	0.130	0.271	0.044	0.399	0.330	0.470	0.045	0.804	0.733	0.871	0.044
NBB ($\lambda = 1/3$)	-0.004	-0.256	0.254	0.158	0.005	-0.377	0.380	0.234	-0.002	-0.477	0.478	0.296	0.213	0.020	0.382	0.108	0.357	0.158	0.530	0.112	0.795	0.603	0.966	0.108
MBB ($\lambda = 1/3$)	-0.010	-0.278	0.264	0.166	0.008	-0.385	0.415	0.248	-0.005	-0.502	0.506	0.303	0.201	0.002	0.373	0.115	0.366	0.181	0.547	0.113	0.793	0.601	0.963	0.107
NBB ($\lambda = 1/4$)	-0.009	-0.268	0.253	0.156	0.006	-0.370	0.409	0.238	-0.002	-0.485	0.470	0.295	0.197	0.016	0.358	0.104	0.372	0.180	0.560	0.115	0.790	0.592	0.967	0.115
MBB ($\lambda = 1/4$)	-0.005	-0.266	0.272	0.162	0.009	-0.394	0.399	0.245	-0.002	-0.504	0.506	0.304	0.204	0.013	0.376	0.108	0.364	0.176	0.545	0.113	0.791	0.587	0.967	0.114
NBB ($\lambda = 1/5$)	-0.006	-0.257	0.251	0.160	0.006	-0.376	0.400	0.240	0.000	-0.483	0.493	0.297	0.190	0.010	0.368	0.108	0.358	0.159	0.523	0.113	0.792	0.600	0.966	0.112
MBB ($\lambda = 1/5$)	-0.005	-0.284	0.276	0.166	0.007	-0.393	0.415	0.247	-0.003	-0.492	0.486	0.301	0.198	0.011	0.369	0.109	0.356	0.188	0.535	0.107	0.791	0.603	0.963	0.109
CBB	-0.005	-0.282	0.278	0.166	0.005	-0.404	0.413	0.245	-0.002	-0.493	0.500	0.303	0.197	-0.001	0.377	0.112	0.365	0.185	0.541	0.110	0.793	0.604	0.959	0.111
SB	0.002	-0.286	0.305	0.184	-0.006	-0.198	0.180	0.115	-0.001	-0.131	0.133	0.077	0.194	0.105	0.275	0.051	0.317	0.264	0.365	0.031	0.794	0.642	0.947	0.093
Sieve AR	-0.065	-1.728	1.471	1.030	0.008	-1.497	1.422	0.894	0.023	-1.082	1.078	0.656	0.144	-0.006	0.240	0.080	0.291	0.096	0.434	0.102	0.728	0.576	0.840	0.083

Table 2. ARFIMA (1, d, 0), $\phi = 0.8$

	Distribution of the Sample Mean													Distribution of the Gaussian Semi Parametric Estimator				
	d=0.2				d=0.4				d=0.8				d=0.2				d=0.4				d=0.8			
	Mean	Conf. Bounds		St. D.	Mean	Conf. Bounds		St. D.	Mean	Conf. Bounds		St. D.	Mean	Conf. Bounds		St. D.	Mean	Conf. Bounds		St. D.	Mean	Conf. Bounds		St. D.
n=100																								
"Exact" Moment	0.079	-5.340	5.630	3.396	0.223	-4.130	4.442	2.685	0.119	-3.227	3.492	2.067	0.211	0.132	0.273	0.045	0.411	0.331	0.474	0.045	0.812	0.733	0.873	0.044
NBB ($\lambda = 1/3$)	-0.018	-0.849	0.774	0.492	-0.016	-0.933	0.864	0.554	-0.030	-1.321	1.283	0.797	0.273	-0.174	0.668	0.254	0.520	0.100	0.913	0.258	0.840	0.408	1.207	0.243
MBB ($\lambda = 1/3$)	-0.002	-0.855	0.784	0.502	-0.010	-1.141	1.102	0.682	-0.034	-1.523	1.476	0.923	0.265	-0.157	0.662	0.247	0.485	0.067	0.899	0.258	0.842	0.422	1.212	0.244
NBB ($\lambda = 1/4$)	-0.006	-0.794	0.673	0.437	-0.012	-0.914	0.927	0.571	-0.031	-1.523	1.314	0.835	0.241	-0.187	0.618	0.250	0.484	-0.041	0.882	0.272	0.832	0.392	1.215	0.248
MBB ($\lambda = 1/4$)	-0.006	-0.823	0.741	0.470	-0.011	-1.125	1.078	0.673	-0.039	-1.556	1.439	0.933	0.250	-0.172	0.651	0.255	0.463	0.034	0.866	0.255	0.815	0.409	1.201	0.241
NBB ($\lambda = 1/5$)	-0.011	-0.759	0.707	0.443	-0.009	-0.983	0.944	0.574	-0.033	-1.394	1.301	0.839	0.254	-0.157	0.666	0.253	0.464	0.018	0.864	0.262	0.827	0.383	1.209	0.256
MBB ($\lambda = 1/5$)	-0.004	-0.831	0.730	0.469	-0.011	-1.115	1.079	0.669	-0.040	-1.603	1.437	0.920	0.269	-0.162	0.661	0.259	0.484	0.049	0.881	0.255	0.820	0.382	1.202	0.243
CBB	0.003	-0.841	0.890	0.513	-0.022	-1.286	1.237	0.763	-0.050	-1.790	1.568	1.000	0.164	-0.103	0.433	0.163	0.342	-0.326	0.886	0.384	0.711	0.327	1.032	0.217
SB	-0.002	-0.564	0.507	0.325	-0.015	-0.379	0.309	0.203	-0.004	-0.158	0.161	0.102	0.187	0.115	0.254	0.042	0.385	0.274	0.452	0.060	0.727	0.663	0.785	0.038
Sieve AR	-0.100	-5.498	5.105	3.265	-0.018	-4.431	4.356	2.708	-0.020	-3.231	2.998	1.947	0.060	-0.186	0.231	0.125	0.245	-0.091	0.436	0.166	0.678	0.443	0.836	0.123
n=400																								
"Exact" Moment	0.028	-7.278	7.089	4.247	-0.013	-5.782	5.974	3.588	0.132	-3.797	3.841	2.351	0.209	0.132	0.273	0.045	0.411	0.334	0.473	0.044	0.813	0.739	0.873	0.042
NBB ($\lambda = 1/3$)	-0.009	-1.011	0.974	0.608	0.014	-1.384	1.417	0.862	0.002	-1.887	1.796	1.123	0.278	0.039	0.507	0.142	0.448	0.150	0.691	0.167	0.867	0.600	1.109	0.156
MBB ($\lambda = 1/3$)	-0.005	-1.053	1.150	0.649	0.026	-1.544	1.608	0.934	0.014	-2.039	1.947	1.205	0.285	0.059	0.522	0.143	0.448	0.180	0.694	0.160	0.865	0.581	1.118	0.162
NBB ($\lambda = 1/4$)	-0.009	-0.952	0.952	0.575	0.033	-1.434	1.492	0.885	0.001	-1.951	1.854	1.163	0.286	0.043	0.500	0.142	0.442	0.165	0.684	0.161	0.850	0.551	1.085	0.159
MBB ($\lambda = 1/4$)	-0.004	-0.989	1.011	0.602	0.019	-1.518	1.578	0.922	0.007	-1.959	1.930	1.207	0.304	0.062	0.514	0.139	0.448	0.169	0.701	0.163	0.865	0.578	1.101	0.157
NBB ($\lambda = 1/5$)	-0.001	-0.933	0.982	0.575	0.033	-1.441	1.513	0.900	0.008	-1.971	1.925	1.172	0.292	0.054	0.525	0.142	0.447	0.171	0.696	0.161	0.852	0.558	1.078	0.154
MBB ($\lambda = 1/5$)	-0.003	-0.952	0.975	0.592	0.024	-1.509	1.603	0.929	0.020	-1.924	1.970	1.202	0.302	0.042	0.530	0.150	0.434	0.149	0.691	0.164	0.857	0.585	1.096	0.155
CBB	-0.023	-1.137	1.072	0.645	0.026	-1.608	1.625	0.963	0.005	-2.090	1.964	1.217	0.236	0.041	0.430	0.118	0.426	0.138	0.678	0.164	0.823	0.577	1.050	0.147
SB	-0.004	-0.290	0.254	0.185	0.000	-0.160	0.152	0.100	0.000	-0.075	0.074	0.058	0.219	0.197	0.248	0.016	0.412	0.377	0.448	0.021	0.809	0.786	0.834	0.015
Sieve AR	-0.123	-6.772	6.601	4.194	-0.114	-5.634	5.258	3.284	0.067	-3.488	3.426	2.110	0.156	0.007	0.243	0.076	0.221	-0.048	0.419	0.146	0.617	0.370	0.822	0.139
n=1000																								
"Exact" Moment	0.249	-7.617	8.488	4.788	0.078	-6.366	6.701	4.067	0.096	-4.843	4.635	2.808	0.215	0.134	0.276	0.045	0.413	0.340	0.475	0.042	0.817	0.737	0.874	0.042
NBB ($\lambda = 1/3$)	0.011	-1.050	1.064	0.637	0.025	-1.398	1.593	0.895	0.038	-2.065	1.888	1.183	0.174	-0.030	0.338	0.111	0.458	0.279	0.629	0.105	0.924	0.762	1.049	0.087
MBB ($\lambda = 1/3$)	0.007	-1.157	1.134	0.696	0.028	-1.487	1.576	0.955	0.029	-2.032	1.997	1.222	0.171	-0.024	0.354	0.114	0.460	0.272	0.642	0.113	0.930	0.783	1.053	0.082
NBB ($\lambda = 1/4$)	0.013	-1.007	1.036	0.632	0.023	-1.418	1.609	0.913	0.030	-2.136	1.901	1.198	0.165	-0.017	0.336	0.111	0.454	0.242	0.637	0.117	0.930	0.777	1.056	0.088
MBB ($\lambda = 1/4$)	0.017	-1.071	1.096	0.668	0.022	-1.452	1.663	0.957	0.032	-2.148	1.993	1.235	0.168	-0.019	0.339	0.110	0.451	0.262	0.627	0.110	0.930	0.782	1.062	0.083
NBB ($\lambda = 1/5$)	0.013	-1.099	1.063	0.657	0.036	-1.409	1.606	0.924	0.038	-2.015	1.907	1.206	0.155	-0.026	0.330	0.110	0.439	0.254	0.611	0.111	0.944	0.802	1.053	0.075
MBB ($\lambda = 1/5$)	0.013	-1.068	1.102	0.666	0.020	-1.426	1.644	0.951	0.036	-2.126	1.947	1.227	0.154	-0.024	0.327	0.111	0.443	0.253	0.622	0.111	0.947	0.815	1.053	0.072
CBB	0.010	-1.141	1.121	0.698	0.024	-1.551	1.647	0.970	0.029	-2.073	1.909	1.220	0.132	-0.063	0.304	0.111	0.387	0.265	0.503	0.072	0.892	0.725	1.022	0.092
SB	-0.004	-0.216	0.191	0.138	0.002	-0.094	0.099	0.066	0.000	-0.048	0.045	0.035	0.152	0.136	0.166	0.009	0.422	0.411	0.432	0.006	0.925	0.864	0.981	0.037
Sieve AR	0.188	-6.809	7.841	4.612	0.119	-6.615	6.579	4.220	0.082	-4.545	3.993	2.598	0.096	-0.072	0.229	0.095	0.329	0.177	0.439	0.086	0.808	0.737	0.847	0.036

Table 3. ARFIMA (0, d, 1), $\theta = 0.8$																								
	Distribution of the Sample Mean												Distribution of the Gaussian Semi Parametric Estimator											
	d=0.2				d=0.4				d=0.8				d=0.2				d=0.4				d=0.8			
	Mean	Conf. Bounds		St. D.	Mean	Conf. Bounds		St. D.	Mean	Conf. Bounds		St. D.	Mean	Conf. Bounds		St. D.	Mean	Conf. Bounds		St. D.	Mean	Conf. Bounds		St. D.
n=100																								
"Exact" Moment	0.142	-14.425	14.391	8.650	0.093	-12.913	11.984	7.530	0.070	-8.597	8.959	5.465	0.201	0.128	0.273	0.047	0.402	0.330	0.471	0.046	0.806	0.731	0.873	0.045
NBB ($\lambda = 1/3$)	0.050	-2.479	2.525	1.558	-0.025	-3.207	3.037	1.904	0.108	-3.636	3.853	2.338	0.185	-0.232	0.575	0.259	0.367	-0.087	0.787	0.264	0.964	0.632	1.232	0.174
MBB ($\lambda = 1/3$)	0.084	-2.643	2.780	1.704	0.072	-3.391	3.815	2.203	0.072	-3.881	4.269	2.554	0.226	-0.239	0.641	0.268	0.360	-0.083	0.764	0.255	0.927	0.583	1.223	0.195
NBB ($\lambda = 1/4$)	-0.021	-2.401	2.489	1.504	-0.053	-3.170	2.990	1.894	0.076	-3.742	3.871	2.371	0.259	-0.223	0.701	0.277	0.381	-0.048	0.784	0.254	0.939	0.602	1.233	0.195
MBB ($\lambda = 1/4$)	0.049	-2.531	2.546	1.594	0.037	-3.364	3.613	2.129	0.099	-3.984	4.355	2.563	0.233	-0.190	0.639	0.259	0.378	-0.069	0.780	0.261	0.931	0.592	1.215	0.192
NBB ($\lambda = 1/5$)	0.016	-2.410	2.444	1.479	-0.041	-3.129	2.943	1.881	0.082	-3.789	3.886	2.366	0.256	-0.199	0.656	0.265	0.392	-0.050	0.820	0.260	0.934	0.575	1.210	0.193
MBB ($\lambda = 1/5$)	0.047	-2.554	2.607	1.602	0.054	-3.313	3.807	2.165	0.074	-3.972	4.217	2.556	0.235	-0.233	0.663	0.271	0.383	-0.072	0.788	0.259	0.926	0.565	1.197	0.187
CBB	0.002	-2.389	2.389	1.504	0.062	-3.550	3.817	2.171	0.056	-4.177	4.167	2.637	0.268	-0.180	0.686	0.262	0.371	-0.101	0.794	0.263	0.920	0.583	1.226	0.192
SB	-0.009	-2.707	2.711	1.691	-0.017	-1.879	1.857	1.128	0.015	-1.159	1.242	0.733	0.297	-0.061	0.655	0.217	0.422	0.104	0.707	0.188	0.832	0.666	0.992	0.100
Sieve AR	-0.022	-13.694	14.442	8.886	-0.119	-12.771	12.238	7.484	0.218	-8.846	8.654	5.397	0.116	-0.110	0.240	0.109	0.208	-0.143	0.426	0.171	0.768	0.637	0.845	0.065
n=400																								
"Exact" Moment	-0.287	-15.017	13.567	8.683	-0.184	-12.263	12.104	7.383	-0.350	-8.364	8.234	5.185	0.203	0.128	0.271	0.046	0.404	0.330	0.472	0.046	0.805	0.731	0.870	0.044
NBB ($\lambda = 1/3$)	-0.001	-2.381	2.219	1.425	0.064	-3.152	3.300	1.972	-0.067	-4.321	3.699	2.501	0.157	-0.097	0.410	0.157	0.388	0.126	0.633	0.162	0.795	0.521	1.033	0.154
MBB ($\lambda = 1/3$)	0.001	-2.535	2.421	1.527	0.063	-3.400	3.499	2.126	-0.102	-4.448	4.266	2.643	0.146	-0.131	0.409	0.161	0.392	0.108	0.663	0.163	0.818	0.546	1.048	0.154
NBB ($\lambda = 1/4$)	0.041	-2.364	2.439	1.457	0.011	-3.236	3.223	1.996	-0.064	-4.417	4.026	2.563	0.151	-0.132	0.424	0.168	0.403	0.138	0.650	0.159	0.819	0.564	1.052	0.151
MBB ($\lambda = 1/4$)	0.037	-2.373	2.370	1.444	0.060	-3.403	3.603	2.100	-0.122	-4.510	4.037	2.599	0.153	-0.136	0.418	0.170	0.391	0.114	0.632	0.160	0.827	0.531	1.059	0.161
NBB ($\lambda = 1/5$)	0.037	-2.295	2.357	1.394	0.030	-3.334	3.292	2.042	-0.067	-4.287	4.087	2.576	0.160	-0.106	0.412	0.156	0.393	0.125	0.656	0.161	0.818	0.547	1.061	0.156
MBB ($\lambda = 1/5$)	0.024	-2.367	2.340	1.432	0.046	-3.363	3.447	2.116	-0.083	-4.444	4.134	2.628	0.155	-0.127	0.406	0.160	0.392	0.115	0.660	0.162	0.822	0.552	1.047	0.154
CBB	-0.009	-2.397	2.291	1.413	0.074	-3.397	3.445	2.103	-0.067	-4.524	4.183	2.655	0.146	-0.146	0.412	0.168	0.401	0.135	0.652	0.158	0.829	0.555	1.048	0.150
SB	-0.013	-2.532	2.462	1.538	-0.052	-1.664	1.537	0.976	-0.003	-1.040	1.031	0.628	0.084	-0.067	0.223	0.090	0.297	0.156	0.427	0.083	0.749	0.602	0.892	0.090
Sieve AR	-0.337	-15.299	13.762	9.073	0.426	-12.082	13.366	7.811	-0.116	-8.530	7.995	5.160	0.063	-0.162	0.229	0.121	0.291	0.081	0.434	0.111	0.763	0.631	0.845	0.066
n=1000																								
"Exact" Moment	0.200	-13.524	13.726	8.385	0.192	-13.379	13.305	7.921	0.052	-8.873	9.101	5.469	0.201	0.130	0.269	0.044	0.401	0.329	0.472	0.045	0.805	0.731	0.871	0.045
NBB ($\lambda = 1/3$)	0.004	-2.249	2.319	1.414	-0.063	-3.525	3.333	2.082	-0.022	-4.289	4.258	2.628	0.154	-0.030	0.329	0.109	0.463	0.267	0.640	0.114	0.888	0.716	1.037	0.099
MBB ($\lambda = 1/3$)	0.021	-2.352	2.324	1.453	-0.100	-3.596	3.274	2.117	0.001	-4.329	4.485	2.712	0.158	-0.023	0.333	0.107	0.450	0.266	0.618	0.110	0.898	0.727	1.041	0.097
NBB ($\lambda = 1/4$)	0.006	-2.302	2.223	1.391	-0.099	-3.491	3.372	2.088	0.013	-4.201	4.182	2.609	0.157	-0.036	0.338	0.111	0.451	0.263	0.617	0.114	0.898	0.723	1.040	0.100
MBB ($\lambda = 1/4$)	0.003	-2.386	2.280	1.442	-0.075	-3.696	3.395	2.129	0.008	-4.387	4.352	2.693	0.147	-0.049	0.334	0.115	0.450	0.261	0.622	0.112	0.900	0.726	1.047	0.097
NBB ($\lambda = 1/5$)	0.012	-2.369	2.218	1.388	-0.050	-3.473	3.279	2.084	0.018	-4.417	4.390	2.652	0.149	-0.033	0.320	0.110	0.456	0.275	0.634	0.115	0.905	0.727	1.052	0.100
MBB ($\lambda = 1/5$)	-0.019	-2.381	2.402	1.439	-0.087	-3.553	3.415	2.156	0.018	-4.263	4.509	2.672	0.151	-0.032	0.314	0.106	0.452	0.262	0.628	0.110	0.899	0.709	1.035	0.098
CBB	-0.009	-2.423	2.409	1.463	-0.070	-3.592	3.355	2.152	-0.033	-4.466	4.435	2.708	0.153	-0.036	0.327	0.112	0.455	0.263	0.623	0.110	0.895	0.726	1.033	0.095
SB	-0.100	-2.254	2.186	1.351	-0.011	-1.430	1.391	0.866	0.000	-0.893	0.904	0.554	0.182	0.139	0.223	0.026	0.385	0.258	0.498	0.074	0.909	0.823	0.983	0.048
Sieve AR	-0.016	-14.630	14.208	9.067	-0.551	-14.237	12.105	8.307	-0.153	-9.957	9.389	5.902	0.113	-0.060	0.235	0.091	0.368	0.244	0.444	0.063	0.777	0.644	0.845	0.065

Table 4. ARFIMA (1, d, 1), $\phi = -0.3$, $\theta = 0.4$

	Distribution of the Sample Mean												Distribution of the Gaussian Semi Parametric Estimator											
	d=0.2				d=0.4				d=0.8				d=0.2				d=0.4				d=0.8			
	Mean	Conf. Bounds		St. D.	Mean	Conf. Bounds		St. D.	Mean	Conf. Bounds		St. D.	Mean	Conf. Bounds		St. D.	Mean	Conf. Bounds		St. D.	Mean	Conf. Bounds		St. D.
n=100																								
"Exact" Moment	-0.011	-1.626	1.639	0.998	0.022	-1.514	1.580	0.942	0.002	-1.054	1.069	0.655	0.198	0.127	0.271	0.047	0.403	0.329	0.471	0.045	0.804	0.730	0.871	0.045
NBB ($\lambda = 1/3$)	0.003	-0.338	0.331	0.204	0.002	-0.352	0.345	0.216	0.009	-0.437	0.486	0.283	0.300	-0.116	0.676	0.245	0.420	-0.045	0.822	0.260	0.870	0.446	1.225	0.238
MBB ($\lambda = 1/3$)	0.008	-0.353	0.361	0.220	0.009	-0.400	0.433	0.255	0.006	-0.498	0.531	0.315	0.296	-0.088	0.673	0.240	0.408	-0.066	0.833	0.270	0.844	0.381	1.222	0.251
NBB ($\lambda = 1/4$)	0.007	-0.311	0.294	0.189	0.008	-0.348	0.383	0.223	0.011	-0.447	0.501	0.284	0.289	-0.102	0.667	0.236	0.412	-0.034	0.822	0.260	0.843	0.384	1.212	0.249
MBB ($\lambda = 1/4$)	0.008	-0.327	0.344	0.205	0.014	-0.391	0.433	0.254	0.003	-0.497	0.524	0.310	0.276	-0.145	0.663	0.244	0.418	-0.007	0.810	0.254	0.828	0.379	1.204	0.254
NBB ($\lambda = 1/5$)	0.003	-0.303	0.316	0.191	0.011	-0.345	0.386	0.224	0.011	-0.456	0.487	0.285	0.278	-0.132	0.666	0.242	0.430	-0.063	0.873	0.273	0.867	0.445	1.237	0.242
MBB ($\lambda = 1/5$)	0.004	-0.345	0.339	0.209	0.011	-0.393	0.425	0.250	0.002	-0.492	0.519	0.309	0.270	-0.149	0.664	0.245	0.413	-0.010	0.804	0.263	0.835	0.416	1.196	0.243
CBB	-0.003	-0.313	0.291	0.187	0.014	-0.403	0.418	0.250	-0.002	-0.493	0.544	0.316	0.279	-0.129	0.644	0.234	0.418	-0.053	0.839	0.277	0.846	0.442	1.189	0.231
SB	-0.001	-0.343	0.329	0.206	0.001	-0.212	0.222	0.132	-0.001	-0.140	0.150	0.088	-0.063	-0.277	0.120	0.119	0.287	-0.097	0.637	0.227	0.667	0.293	0.997	0.206
Sieve AR	0.072	-1.729	1.832	1.074	0.044	-1.400	1.631	0.943	-0.013	-1.050	1.070	0.663	0.121	-0.091	0.239	0.104	0.264	0.004	0.434	0.137	0.686	0.489	0.835	0.108
n=400																								
"Exact" Moment	0.001	-1.782	1.602	1.029	0.002	-1.533	1.494	0.906	-0.012	-1.046	0.984	0.639	0.198	0.127	0.271	0.046	0.401	0.329	0.472	0.045	0.804	0.731	0.873	0.045
NBB ($\lambda = 1/3$)	0.008	-0.278	0.288	0.172	0.006	-0.389	0.391	0.235	0.000	-0.496	0.491	0.296	0.163	-0.091	0.416	0.158	0.370	0.099	0.623	0.161	0.852	0.573	1.094	0.155
MBB ($\lambda = 1/3$)	0.005	-0.307	0.301	0.186	0.008	-0.413	0.436	0.265	-0.002	-0.519	0.489	0.310	0.168	-0.120	0.399	0.156	0.341	0.062	0.600	0.165	0.850	0.587	1.079	0.150
NBB ($\lambda = 1/4$)	0.003	-0.265	0.268	0.168	0.008	-0.414	0.380	0.244	-0.003	-0.495	0.474	0.297	0.157	-0.117	0.414	0.164	0.368	0.087	0.604	0.158	0.838	0.575	1.071	0.156
MBB ($\lambda = 1/4$)	0.008	-0.293	0.294	0.175	0.012	-0.424	0.432	0.255	-0.003	-0.522	0.470	0.307	0.148	-0.132	0.400	0.163	0.352	0.085	0.609	0.158	0.847	0.583	1.069	0.149
NBB ($\lambda = 1/5$)	0.006	-0.272	0.296	0.173	0.011	-0.396	0.413	0.244	-0.002	-0.502	0.473	0.297	0.165	-0.095	0.414	0.155	0.357	0.087	0.610	0.159	0.849	0.588	1.070	0.148
MBB ($\lambda = 1/5$)	0.011	-0.278	0.290	0.175	0.009	-0.417	0.428	0.258	0.000	-0.517	0.480	0.304	0.152	-0.108	0.417	0.159	0.365	0.095	0.602	0.154	0.858	0.591	1.098	0.153
CBB	0.010	-0.290	0.301	0.177	0.008	-0.441	0.426	0.260	-0.005	-0.522	0.473	0.305	0.150	-0.134	0.395	0.160	0.355	0.078	0.616	0.160	0.851	0.584	1.071	0.154
SB	0.000	-0.311	0.317	0.194	0.001	-0.211	0.200	0.122	-0.001	-0.130	0.130	0.079	0.172	0.046	0.296	0.076	0.201	0.026	0.350	0.101	0.842	0.619	1.053	0.130
Sieve AR	0.038	-1.620	1.663	1.020	0.058	-1.407	1.534	0.916	-0.044	-1.040	0.996	0.630	0.094	-0.117	0.236	0.111	0.201	-0.053	0.409	0.145	0.700	0.498	0.839	0.111
n=1000																								
"Exact" Moment	0.033	-1.532	1.642	0.978	0.015	-1.412	1.463	0.885	-0.016	-1.139	1.129	0.681	0.201	0.130	0.272	0.044	0.403	0.328	0.469	0.045	0.806	0.731	0.871	0.044
NBB ($\lambda = 1/3$)	0.002	-0.284	0.282	0.173	-0.010	-0.412	0.393	0.248	0.008	-0.498	0.512	0.307	0.182	-0.006	0.355	0.110	0.393	0.209	0.568	0.110	0.825	0.629	0.994	0.110
MBB ($\lambda = 1/3$)	0.006	-0.313	0.299	0.184	-0.017	-0.428	0.401	0.259	0.011	-0.524	0.533	0.319	0.166	-0.022	0.342	0.112	0.375	0.183	0.559	0.110	0.838	0.648	1.007	0.110
NBB ($\lambda = 1/4$)	0.005	-0.295	0.280	0.174	-0.013	-0.408	0.402	0.254	0.012	-0.505	0.523	0.310	0.159	-0.020	0.335	0.112	0.385	0.203	0.560	0.106	0.826	0.628	0.995	0.111
MBB ($\lambda = 1/4$)	0.008	-0.297	0.309	0.186	-0.013	-0.429	0.409	0.258	0.011	-0.517	0.532	0.319	0.165	-0.038	0.340	0.112	0.380	0.200	0.548	0.107	0.833	0.629	0.998	0.112
NBB ($\lambda = 1/5$)	0.003	-0.292	0.287	0.177	-0.013	-0.443	0.418	0.260	0.009	-0.512	0.532	0.313	0.172	-0.023	0.345	0.110	0.382	0.193	0.551	0.111	0.829	0.640	0.991	0.108
MBB ($\lambda = 1/5$)	0.004	-0.294	0.284	0.178	-0.017	-0.433	0.414	0.257	0.008	-0.526	0.523	0.319	0.165	-0.014	0.339	0.108	0.382	0.199	0.560	0.110	0.834	0.655	0.999	0.108
CBB	0.003	-0.306	0.306	0.179	-0.019	-0.435	0.407	0.260	0.010	-0.504	0.514	0.315	0.133	-0.049	0.308	0.109	0.381	0.194	0.560	0.112	0.837	0.640	1.004	0.110
SB	-0.004	-0.286	0.279	0.169	-0.007	-0.194	0.179	0.115	0.000	-0.119	0.114	0.071	0.188	0.144	0.228	0.025	0.383	0.342	0.426	0.026	0.787	0.685	0.879	0.059
Sieve AR	-0.012	-1.843	1.647	1.085	-0.023	-1.647	1.577	0.986	0.012	-1.095	1.107	0.667	0.113	-0.055	0.236	0.090	0.326	0.170	0.438	0.085	0.753	0.613	0.842	0.074

Table 5. ARFIMA (1, d, 1), $\phi = 0.3$, $\theta = -0.4$																								
	Distribution of the Sample Mean												Distribution of the Gaussian Semi Parametric Estimator											
	d=0.2				d=0.4				d=0.8				d=0.2				d=0.4				d=0.8			
	Mean	Conf. Bounds		St. D.	Mean	Conf. Bounds		St. D.	Mean	Conf. Bounds		St. D.	Mean	Conf. Bounds		St. D.	Mean	Conf. Bounds		St. D.	Mean	Conf. Bounds		St. D.
n=100																								
"Exact" Moment	0.017	-1.337	1.313	0.820	0.017	-1.145	1.239	0.745	0.011	-0.844	0.853	0.544	0.201	0.129	0.271	0.045	0.398	0.329	0.472	0.046	0.803	0.728	0.872	0.046
NBB ($\lambda = 1/3$)	0.001	-0.274	0.274	0.163	-0.006	-0.328	0.323	0.196	0.003	-0.373	0.379	0.235	0.183	-0.269	0.614	0.268	0.407	-0.022	0.804	0.255	0.909	0.537	1.242	0.219
MBB ($\lambda = 1/3$)	-0.006	-0.299	0.275	0.176	-0.006	-0.375	0.376	0.227	-0.001	-0.446	0.422	0.266	0.124	-0.326	0.563	0.269	0.377	-0.063	0.822	0.267	0.895	0.468	1.226	0.234
NBB ($\lambda = 1/4$)	0.000	-0.266	0.245	0.157	-0.007	-0.325	0.320	0.194	-0.002	-0.397	0.395	0.239	0.133	-0.331	0.559	0.266	0.427	-0.037	0.822	0.260	0.872	0.438	1.230	0.241
MBB ($\lambda = 1/4$)	-0.005	-0.292	0.275	0.173	-0.004	-0.358	0.363	0.220	0.001	-0.436	0.432	0.264	0.125	-0.338	0.534	0.270	0.418	-0.032	0.806	0.265	0.858	0.446	1.214	0.239
NBB ($\lambda = 1/5$)	-0.001	-0.251	0.251	0.156	-0.007	-0.325	0.329	0.197	0.000	-0.392	0.398	0.237	0.135	-0.331	0.567	0.269	0.434	-0.017	0.837	0.255	0.892	0.462	1.272	0.241
MBB ($\lambda = 1/5$)	-0.005	-0.302	0.270	0.171	-0.007	-0.346	0.360	0.219	0.002	-0.440	0.434	0.260	0.137	-0.315	0.561	0.267	0.402	-0.023	0.808	0.260	0.876	0.470	1.223	0.246
CBB	-0.007	-0.254	0.233	0.152	-0.005	-0.368	0.365	0.222	0.005	-0.448	0.446	0.265	0.133	-0.336	0.560	0.271	0.346	-0.086	0.749	0.256	0.869	0.467	1.204	0.225
SB	0.007	-0.292	0.280	0.171	-0.004	-0.187	0.172	0.106	0.000	-0.119	0.105	0.069	-0.049	-0.458	0.338	0.251	0.430	0.159	0.699	0.164	0.701	0.478	0.864	0.121
Sieve AR	-0.001	-1.350	1.396	0.844	-0.003	-1.191	1.188	0.744	-0.018	-0.888	0.857	0.544	0.040	-0.207	0.231	0.134	0.144	-0.250	0.418	0.206	0.733	0.549	0.841	0.090
n=400																								
"Exact" Moment	0.017	-1.373	1.314	0.802	-0.011	-1.233	1.197	0.754	0.018	-0.763	0.816	0.503	0.199	0.128	0.270	0.046	0.402	0.328	0.471	0.045	0.804	0.730	0.873	0.046
NBB ($\lambda = 1/3$)	0.005	-0.216	0.237	0.135	0.006	-0.312	0.325	0.199	-0.005	-0.384	0.388	0.238	0.363	0.131	0.606	0.143	0.449	0.173	0.694	0.158	0.794	0.515	1.033	0.155
MBB ($\lambda = 1/3$)	0.006	-0.233	0.255	0.149	0.008	-0.359	0.362	0.215	-0.004	-0.426	0.418	0.254	0.328	0.092	0.553	0.144	0.446	0.188	0.692	0.160	0.792	0.525	1.031	0.157
NBB ($\lambda = 1/4$)	0.004	-0.217	0.231	0.137	0.004	-0.321	0.324	0.198	-0.008	-0.393	0.390	0.243	0.337	0.086	0.564	0.147	0.446	0.175	0.705	0.164	0.794	0.514	1.021	0.154
MBB ($\lambda = 1/4$)	0.009	-0.224	0.254	0.142	0.007	-0.362	0.343	0.214	-0.013	-0.426	0.393	0.250	0.335	0.087	0.554	0.144	0.441	0.151	0.707	0.168	0.807	0.541	1.040	0.157
NBB ($\lambda = 1/5$)	0.003	-0.219	0.233	0.137	0.004	-0.336	0.341	0.205	-0.009	-0.407	0.403	0.246	0.321	0.076	0.549	0.144	0.448	0.163	0.695	0.163	0.807	0.513	1.043	0.159
MBB ($\lambda = 1/5$)	0.006	-0.228	0.226	0.138	0.007	-0.332	0.346	0.211	-0.008	-0.428	0.407	0.252	0.322	0.073	0.559	0.145	0.446	0.184	0.698	0.158	0.805	0.532	1.057	0.159
CBB	0.008	-0.224	0.225	0.139	0.007	-0.346	0.341	0.211	-0.009	-0.436	0.401	0.254	0.326	0.101	0.540	0.138	0.446	0.159	0.692	0.161	0.783	0.534	1.006	0.145
SB	-0.012	-0.262	0.250	0.158	0.004	-0.161	0.171	0.098	-0.003	-0.097	0.088	0.060	0.224	0.095	0.346	0.079	0.468	0.295	0.649	0.111	0.743	0.590	0.889	0.089
Sieve AR	-0.015	-1.432	1.313	0.854	-0.012	-1.219	1.190	0.728	-0.019	-0.864	0.785	0.502	0.187	0.097	0.245	0.050	0.309	0.119	0.438	0.105	0.699	0.537	0.833	0.092
n=1000																								
"Exact" Moment	0.028	-1.331	1.275	0.783	0.006	-1.209	1.219	0.718	0.019	-0.942	0.926	0.548	0.199	0.129	0.269	0.045	0.398	0.329	0.470	0.045	0.803	0.728	0.870	0.045
NBB ($\lambda = 1/3$)	-0.010	-0.223	0.225	0.141	-0.015	-0.346	0.295	0.200	0.009	-0.371	0.394	0.237	0.216	0.029	0.389	0.106	0.436	0.244	0.607	0.111	0.849	0.670	1.015	0.104
MBB ($\lambda = 1/3$)	-0.009	-0.243	0.233	0.146	-0.011	-0.361	0.328	0.209	0.007	-0.410	0.404	0.247	0.228	0.049	0.404	0.108	0.422	0.245	0.597	0.108	0.856	0.663	1.029	0.112
NBB ($\lambda = 1/4$)	-0.006	-0.224	0.219	0.134	-0.016	-0.350	0.309	0.203	0.011	-0.389	0.400	0.240	0.231	0.046	0.404	0.111	0.421	0.235	0.599	0.112	0.845	0.660	1.012	0.108
MBB ($\lambda = 1/4$)	-0.002	-0.223	0.235	0.143	-0.015	-0.359	0.324	0.208	0.008	-0.406	0.409	0.250	0.228	0.032	0.407	0.115	0.420	0.218	0.609	0.117	0.848	0.652	1.020	0.111
NBB ($\lambda = 1/5$)	-0.005	-0.234	0.214	0.139	-0.014	-0.353	0.318	0.202	0.008	-0.406	0.403	0.248	0.233	0.047	0.410	0.109	0.437	0.243	0.616	0.114	0.856	0.673	1.023	0.105
MBB ($\lambda = 1/5$)	-0.002	-0.221	0.235	0.140	-0.013	-0.358	0.333	0.209	0.008	-0.402	0.399	0.246	0.229	0.036	0.410	0.116	0.427	0.244	0.610	0.109	0.848	0.666	1.022	0.107
CBB	-0.004	-0.242	0.228	0.143	-0.015	-0.357	0.319	0.210	0.009	-0.400	0.408	0.249	0.198	0.003	0.370	0.113	0.414	0.224	0.591	0.113	0.851	0.679	1.021	0.103
SB	-0.005	-0.261	0.251	0.150	0.003	-0.153	0.177	0.103	-0.001	-0.091	0.079	0.054	0.216	0.123	0.297	0.053	0.459	0.374	0.544	0.052	0.768	0.648	0.883	0.072
Sieve AR	-0.011	-1.387	1.245	0.842	-0.016	-1.311	1.194	0.779	0.007	-0.823	0.888	0.524	0.133	-0.026	0.240	0.084	0.330	0.156	0.439	0.087	0.704	0.533	0.830	0.095

Table 6. ARFIMA (2, d, 0), $\phi_1 = 0.2$, $\phi_2 = 0.7$

	Distribution of the Sample Mean												Distribution of the Gaussian Semi Parametric Estimator											
	d=0.2				d=0.4				d=0.8				d=0.2				d=0.4				d=0.8			
	Mean	Conf. Bounds		St. D.	Mean	Conf. Bounds		St. D.	Mean	Conf. Bounds		St. D.	Mean	Conf. Bounds		St. D.	Mean	Conf. Bounds		St. D.	Mean	Conf. Bounds		St. D.
n=100																								
"Exact" Moment	0.087	-4.289	4.120	2.597	0.039	-3.083	3.104	1.938	0.055	-2.753	2.899	1.710	0.212	0.131	0.274	0.045	0.413	0.333	0.476	0.046	0.814	0.733	0.873	0.045
NBB ($\lambda = 1/3$)	-0.014	-0.581	0.574	0.350	0.021	-0.606	0.642	0.384	-0.012	-1.078	1.047	0.655	0.343	-0.056	0.719	0.242	0.529	0.067	0.901	0.246	1.001	0.702	1.237	0.165
MBB ($\lambda = 1/3$)	-0.004	-0.582	0.559	0.349	0.027	-0.757	0.797	0.479	-0.020	-1.373	1.299	0.804	0.364	-0.050	0.754	0.251	0.527	0.104	0.910	0.250	1.003	0.733	1.225	0.151
NBB ($\lambda = 1/4$)	-0.002	-0.491	0.481	0.297	0.020	-0.620	0.628	0.385	-0.018	-1.214	1.161	0.706	0.334	-0.063	0.707	0.231	0.519	0.069	0.932	0.256	0.999	0.709	1.220	0.161
MBB ($\lambda = 1/4$)	-0.004	-0.526	0.501	0.316	0.016	-0.734	0.805	0.468	-0.020	-1.392	1.315	0.823	0.347	-0.045	0.724	0.234	0.523	0.105	0.909	0.244	1.007	0.733	1.218	0.150
NBB ($\lambda = 1/5$)	-0.005	-0.509	0.502	0.300	0.024	-0.610	0.644	0.388	-0.008	-1.167	1.176	0.709	0.340	-0.029	0.717	0.229	0.512	0.062	0.927	0.264	1.003	0.748	1.235	0.150
MBB ($\lambda = 1/5$)	0.000	-0.518	0.501	0.316	0.021	-0.746	0.811	0.468	-0.019	-1.376	1.320	0.816	0.346	-0.034	0.723	0.236	0.512	0.071	0.897	0.250	1.012	0.777	1.223	0.142
CBB	0.005	-0.579	0.577	0.348	0.002	-0.868	0.929	0.547	-0.021	-1.623	1.494	0.933	0.374	-0.033	0.744	0.237	0.489	0.094	0.855	0.229	1.023	0.768	1.240	0.146
SB	-0.002	-0.400	0.421	0.250	-0.009	-0.257	0.253	0.154	0.000	-0.133	0.147	0.089	0.182	0.083	0.275	0.057	0.386	0.300	0.466	0.051	0.897	0.819	1.004	0.058
Sieve AR	-0.141	-4.229	4.017	2.520	0.007	-3.200	3.122	1.903	0.035	-2.511	2.657	1.594	0.112	-0.123	0.242	0.114	0.289	0.002	0.442	0.139	0.775	0.639	0.845	0.071
n=400																								
"Exact" Moment	-0.201	-5.692	5.168	3.361	0.078	-4.474	4.672	2.749	-0.011	-3.160	3.076	1.955	0.218	0.138	0.275	0.044	0.418	0.332	0.475	0.044	0.821	0.741	0.875	0.042
NBB ($\lambda = 1/3$)	0.001	-0.707	0.778	0.437	-0.014	-1.095	1.031	0.647	0.005	-1.521	1.409	0.880	0.288	0.010	0.537	0.156	0.426	0.148	0.686	0.164	0.875	0.598	1.114	0.157
MBB ($\lambda = 1/3$)	-0.017	-0.801	0.747	0.472	-0.022	-1.212	1.124	0.711	0.003	-1.661	1.472	0.946	0.276	0.009	0.521	0.158	0.427	0.161	0.682	0.159	0.875	0.631	1.090	0.144
NBB ($\lambda = 1/4$)	-0.020	-0.697	0.669	0.425	-0.008	-1.127	1.060	0.673	-0.006	-1.556	1.444	0.908	0.268	-0.006	0.511	0.157	0.393	0.125	0.648	0.164	0.864	0.589	1.092	0.151
MBB ($\lambda = 1/4$)	-0.016	-0.775	0.743	0.461	-0.020	-1.196	1.138	0.711	-0.012	-1.645	1.502	0.952	0.258	-0.016	0.522	0.160	0.393	0.117	0.647	0.161	0.872	0.611	1.077	0.147
NBB ($\lambda = 1/5$)	-0.018	-0.719	0.702	0.432	-0.007	-1.144	1.131	0.682	-0.010	-1.620	1.449	0.924	0.262	-0.021	0.535	0.162	0.387	0.087	0.648	0.164	0.862	0.615	1.095	0.152
MBB ($\lambda = 1/5$)	-0.022	-0.796	0.721	0.455	-0.022	-1.186	1.155	0.702	-0.015	-1.630	1.529	0.950	0.269	-0.006	0.525	0.159	0.393	0.137	0.634	0.152	0.854	0.579	1.082	0.154
CBB	-0.022	-0.866	0.832	0.512	-0.019	-1.225	1.187	0.739	-0.015	-1.744	1.578	0.982	0.306	0.151	0.478	0.101	0.430	0.194	0.679	0.148	0.875	0.696	1.043	0.112
SB	-0.005	-0.187	0.160	0.126	0.001	-0.080	0.090	0.068	0.000	-0.046	0.040	0.034	0.245	0.229	0.259	0.009	0.383	0.371	0.396	0.007	0.795	0.773	0.812	0.012
Sieve AR	0.004	-5.381	5.618	3.324	-0.029	-4.325	4.100	2.599	0.120	-2.485	2.902	1.644	0.098	-0.117	0.238	0.110	0.247	-0.018	0.430	0.138	0.738	0.582	0.842	0.081
n=1000																								
"Exact" Moment	0.040	-7.420	7.579	4.529	-0.288	-6.551	6.087	3.875	0.207	-4.398	4.708	2.749	0.229	0.149	0.277	0.039	0.428	0.348	0.477	0.039	0.831	0.748	0.877	0.039
NBB ($\lambda = 1/3$)	-0.016	-0.790	0.788	0.479	0.001	-1.212	1.192	0.736	-0.010	-1.460	1.450	0.896	0.207	0.012	0.384	0.116	0.470	0.271	0.643	0.110	0.812	0.631	0.994	0.111
MBB ($\lambda = 1/3$)	-0.005	-0.806	0.883	0.527	0.001	-1.276	1.285	0.787	0.000	-1.546	1.539	0.925	0.208	0.016	0.381	0.113	0.474	0.289	0.650	0.110	0.826	0.637	0.992	0.108
NBB ($\lambda = 1/4$)	-0.006	-0.788	0.817	0.487	0.016	-1.182	1.199	0.746	-0.007	-1.509	1.479	0.906	0.206	0.014	0.376	0.109	0.464	0.281	0.636	0.108	0.817	0.625	0.986	0.110
MBB ($\lambda = 1/4$)	-0.017	-0.877	0.877	0.519	0.002	-1.218	1.313	0.780	0.000	-1.530	1.549	0.922	0.200	0.014	0.372	0.113	0.458	0.272	0.626	0.112	0.814	0.627	0.995	0.112
NBB ($\lambda = 1/5$)	-0.014	-0.807	0.797	0.485	0.005	-1.236	1.288	0.765	-0.005	-1.499	1.515	0.905	0.180	-0.021	0.363	0.114	0.443	0.271	0.622	0.108	0.801	0.604	0.970	0.111
MBB ($\lambda = 1/5$)	-0.002	-0.804	0.829	0.498	0.006	-1.247	1.303	0.784	0.002	-1.496	1.497	0.918	0.192	0.006	0.368	0.111	0.441	0.256	0.614	0.109	0.805	0.604	0.978	0.111
CBB	-0.015	-0.860	0.855	0.517	0.008	-1.349	1.372	0.817	0.009	-1.523	1.522	0.926	0.185	0.004	0.355	0.106	0.453	0.269	0.624	0.110	0.867	0.738	0.988	0.076
SB	-0.001	-0.124	0.120	0.094	0.001	-0.067	0.070	0.055	0.001	-0.024	0.026	0.022	0.199	0.190	0.209	0.006	0.435	0.421	0.449	0.009	0.769	0.761	0.777	0.005
Sieve AR	0.059	-6.334	6.485	4.028	0.002	-5.544	5.335	3.321	-0.009	-3.350	3.587	2.150	0.123	-0.043	0.238	0.085	0.343	0.200	0.442	0.082	0.749	0.608	0.842	0.075

Table 7. ARFIMA (2, d, 0), $\phi_1 = 0.7$, $\phi_2 = 0.2$

	Distribution of the Sample Mean												Distribution of the Gaussian Semi Parametric Estimator											
	d=0.2				d=0.4				d=0.8				d=0.2				d=0.4				d=0.8			
	Mean	Conf. Bounds		St. D.	Mean	Conf. Bounds		St. D.	Mean	Conf. Bounds		St. D.	Mean	Conf. Bounds		St. D.	Mean	Conf. Bounds		St. D.	Mean	Conf. Bounds		St. D.
n=100																								
"Exact" Moment	0.124	-5.069	5.121	3.096	-0.044	-3.978	3.873	2.424	-0.082	-3.127	3.019	1.905	0.210	0.134	0.274	0.045	0.408	0.333	0.474	0.045	0.815	0.734	0.875	0.044
NBB ($\lambda = 1/3$)	0.000	-0.691	0.705	0.427	-0.012	-0.789	0.766	0.482	-0.041	-1.234	1.145	0.728	0.338	-0.064	0.705	0.243	0.424	-0.033	0.852	0.262	0.933	0.565	1.238	0.206
MBB ($\lambda = 1/3$)	0.017	-0.742	0.756	0.454	-0.003	-0.924	0.916	0.570	-0.039	-1.440	1.412	0.878	0.351	-0.063	0.718	0.231	0.407	-0.102	0.832	0.276	0.936	0.551	1.247	0.213
NBB ($\lambda = 1/4$)	0.005	-0.617	0.601	0.382	-0.007	-0.771	0.757	0.478	-0.039	-1.294	1.260	0.777	0.332	-0.061	0.707	0.241	0.409	-0.055	0.818	0.266	0.942	0.599	1.229	0.200
MBB ($\lambda = 1/4$)	0.018	-0.638	0.684	0.412	0.006	-0.925	0.940	0.566	-0.049	-1.463	1.403	0.876	0.338	-0.047	0.663	0.217	0.395	-0.046	0.786	0.256	0.930	0.575	1.244	0.207
NBB ($\lambda = 1/5$)	0.004	-0.598	0.610	0.372	-0.011	-0.797	0.764	0.476	-0.037	-1.276	1.248	0.768	0.345	-0.052	0.722	0.236	0.389	-0.074	0.830	0.269	0.937	0.569	1.227	0.200
MBB ($\lambda = 1/5$)	0.017	-0.630	0.683	0.408	0.013	-0.908	0.902	0.562	-0.049	-1.499	1.353	0.873	0.356	0.003	0.715	0.224	0.400	-0.054	0.842	0.274	0.937	0.584	1.215	0.198
CBB	0.030	-0.677	0.743	0.436	0.002	-1.032	1.091	0.645	-0.035	-1.519	1.513	0.947	0.339	-0.059	0.704	0.240	0.372	0.023	0.691	0.206	1.033	0.706	1.333	0.193
SB	-0.001	-0.432	0.477	0.281	0.000	-0.281	0.289	0.176	-0.001	-0.140	0.131	0.089	0.489	0.244	0.771	0.167	0.381	0.307	0.455	0.044	0.906	0.788	1.015	0.071
Sieve AR	-0.005	-4.858	4.932	3.039	-0.051	-3.959	3.578	2.298	0.009	-2.802	2.844	1.713	0.102	-0.100	0.239	0.107	0.141	-0.254	0.424	0.210	0.719	0.499	0.841	0.110
n=400																								
"Exact" Moment	-0.098	-6.509	6.292	3.878	-0.040	-5.477	5.390	3.297	0.065	-3.432	3.606	2.172	0.213	0.135	0.274	0.044	0.416	0.335	0.474	0.043	0.816	0.736	0.875	0.044
NBB ($\lambda = 1/3$)	-0.006	-0.860	0.878	0.525	-0.011	-1.380	1.306	0.813	0.046	-1.676	1.692	1.004	0.232	-0.034	0.494	0.161	0.506	0.249	0.738	0.152	0.879	0.628	1.089	0.141
MBB ($\lambda = 1/3$)	-0.001	-0.915	0.973	0.592	-0.004	-1.503	1.427	0.878	0.045	-1.680	1.738	1.076	0.239	-0.045	0.498	0.164	0.505	0.226	0.740	0.157	0.876	0.618	1.080	0.136
NBB ($\lambda = 1/4$)	-0.002	-0.856	0.872	0.529	-0.010	-1.400	1.376	0.834	0.036	-1.682	1.673	1.036	0.218	-0.044	0.478	0.156	0.511	0.268	0.753	0.152	0.885	0.651	1.072	0.132
MBB ($\lambda = 1/4$)	-0.010	-0.897	0.919	0.560	0.005	-1.533	1.488	0.881	0.047	-1.738	1.753	1.070	0.218	-0.045	0.466	0.161	0.505	0.243	0.737	0.153	0.878	0.627	1.084	0.138
NBB ($\lambda = 1/5$)	-0.021	-0.863	0.899	0.530	0.006	-1.414	1.402	0.843	0.040	-1.746	1.716	1.047	0.221	-0.045	0.482	0.163	0.511	0.233	0.744	0.157	0.896	0.668	1.071	0.126
MBB ($\lambda = 1/5$)	-0.015	-0.919	0.928	0.562	-0.002	-1.548	1.429	0.878	0.047	-1.684	1.773	1.072	0.210	-0.055	0.461	0.158	0.516	0.266	0.740	0.147	0.895	0.657	1.091	0.132
CBB	-0.006	-1.035	0.990	0.615	-0.004	-1.532	1.489	0.918	0.037	-1.787	1.780	1.098	0.198	0.009	0.389	0.117	0.479	0.234	0.738	0.155	0.884	0.653	1.060	0.130
SB	-0.005	-0.251	0.252	0.190	0.004	-0.127	0.145	0.099	0.001	-0.058	0.062	0.044	0.213	0.193	0.229	0.011	0.431	0.409	0.459	0.015	0.769	0.756	0.784	0.009
Sieve AR	-0.123	-6.312	6.207	3.844	-0.043	-4.850	4.791	3.020	0.094	-3.084	3.243	1.955	0.087	-0.131	0.234	0.113	0.296	0.085	0.438	0.110	0.773	0.661	0.844	0.057
n=1000																								
"Exact" Moment	0.218	-7.517	7.405	4.492	0.035	-6.478	6.350	3.973	-0.001	-4.487	4.545	2.763	0.219	0.141	0.274	0.043	0.420	0.343	0.476	0.041	0.826	0.742	0.876	0.041
NBB ($\lambda = 1/3$)	-0.003	-0.995	0.928	0.596	-0.007	-1.492	1.400	0.871	-0.033	-1.816	1.736	1.069	0.220	0.044	0.392	0.107	0.456	0.260	0.628	0.112	0.881	0.704	1.036	0.101
MBB ($\lambda = 1/3$)	0.000	-1.056	1.018	0.636	-0.003	-1.525	1.409	0.905	-0.044	-1.941	1.752	1.104	0.214	0.042	0.387	0.107	0.450	0.244	0.629	0.118	0.882	0.704	1.048	0.105
NBB ($\lambda = 1/4$)	0.006	-0.989	0.921	0.598	-0.004	-1.499	1.410	0.884	-0.039	-1.854	1.773	1.071	0.210	0.025	0.380	0.110	0.447	0.261	0.623	0.110	0.882	0.696	1.039	0.103
MBB ($\lambda = 1/4$)	0.009	-1.052	1.019	0.628	-0.008	-1.509	1.417	0.904	-0.042	-1.921	1.719	1.096	0.206	0.020	0.375	0.104	0.446	0.249	0.622	0.113	0.876	0.687	1.037	0.105
NBB ($\lambda = 1/5$)	0.010	-0.998	0.936	0.587	-0.001	-1.463	1.378	0.870	-0.042	-1.866	1.779	1.087	0.193	0.000	0.368	0.111	0.430	0.240	0.608	0.112	0.865	0.670	1.032	0.111
MBB ($\lambda = 1/5$)	0.007	-0.961	0.978	0.591	0.001	-1.481	1.465	0.889	-0.044	-1.881	1.760	1.095	0.193	0.001	0.365	0.110	0.432	0.233	0.606	0.111	0.871	0.687	1.036	0.108
CBB	0.009	-1.017	1.002	0.618	0.001	-1.470	1.446	0.910	-0.039	-1.870	1.762	1.106	0.220	0.060	0.366	0.095	0.425	0.257	0.599	0.107	0.871	0.722	1.001	0.083
SB	-0.003	-0.190	0.158	0.124	0.001	-0.085	0.097	0.068	-0.001	-0.039	0.033	0.029	0.191	0.183	0.201	0.006	0.426	0.414	0.439	0.008	0.842	0.832	0.852	0.006
Sieve AR	-0.058	-7.575	7.070	4.493	0.037	-6.004	5.848	3.657	0.003	-4.081	3.946	2.465	0.116	-0.053	0.232	0.089	0.372	0.247	0.446	0.063	0.773	0.650	0.844	0.061

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