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Abstract

This paper introduces a new long memory volatility process, denoted by Adaptive $FIGARCH$, or *A-FIGARCH,* which is designed to account for both long memory and structural change in the conditional variance process. Structural change is modeled by allowing the intercept to follow a slowly varying function, specified by Gallant (1984)'s flexible functional form. A Monte Carlo study finds that the *A-FIGARCH* model outperforms the standard *FIGARCH* model when structural change is present, and performs at least as well in the absence of structural instability. An empirical application to stock market volatility is also included to illustrate the usefulness of the technique.

Key words: *FIGARCH*, long memory, structural change, stock market volatility.
JEL classification: C15, C22, F31

1 Introduction

The purpose of this paper is to introduce a new long memory volatility process, denoted by Adaptive $FIGARCH$, or *A-FIGARCH,* which is designed to account for both long memory and structural change in the volatility processes of economic and financial time series. It is well known that most daily and high frequency financial time series exhibit quite persistent autocorrelation in their squared returns, power transformations of absolute returns, conditional variances and other measures of volatility. The seminal papers by Ding, Granger and Engle (1993) and Dacorogna et al. (1993) led to the development of the long memory stochastic volatility models of Breidt, Crato and de Lima (1998) and Harvey (1998), and the long memory ARCH models of Baillie, Bollerslev and Mikkelsen (1996), Bollerslev and Mikkelsen (1996) and Davidson (2004). While these models appear useful in describing many empirical volatility processes, there is understandably great interest in discerning the reasons and underlying causes for the widespread empirical finding of long memory in volatility. In particular, Granger and Ding (1996) have shown that contemporaneous aggregation of stable $GARCH(1,1)$ processes can result in an aggregate process that exhibits hyperbolically decaying autocorrelations, consistent with a long memory process. A related argument of Andersen and Bollerslev (1997) shows how the contemporaneous aggregation of weakly dependent information flow processes can produce the property of long memory in volatility. A further justification is provided by Muller et al. (1997), who suggest that long memory in volatility can arise from the reaction of short-term dealers to the dynamics of a proxy for the expected volatility trend (coarse volatility), which causes persistence in the higher frequency volatility, or (fine volatility) process.

While the above papers were concerned with the underlying causes of long memory volatility, other studies have essentially been more skeptical about the validity of the finding of the long memory property in volatility. In particular, it has been suggested that various types of structural change can explain extreme persistence of volatility, and can also generate a series that *appears* to have long memory. In particular, Mikosch and Starica (1998) and Granger and Hyung (2004) have presented theoretical and simulation evidence that spurious long memory can be detected from a time series with breaks. Moreover, while Granger and Hyung (2004) have found that an occasional breaks model provides an inferior forecasting performance than a long memory model for S&P500 absolute returns, for the same series Starica and Granger (2004) have found that a non stationary model, allowing for breaks in the unconditional variance, can outperform a long memory model in forecasting, but not at short horizons.[1] Furthermore, Diebold and Inoue (2001) have shown how Markov switching processes could generate long memory in the conditional mean, while Granger and Terasvirta (1999) have shown that a process that switches in sign has the characteristics of long memory.

The possible occurrence of structural breaks in conditional variance processes, generating extreme persistence of the $IGARCH$ form, appears to have been originally suggested by Lamoreaux and Lastrapes (1990) and Diebold (1986). Subsequent studies by Lobato and Savin (1998), Beine and Laurent (2000), Morana and Beltratti (2004) and Martens, van Dijk and de Pooter (2004) have suggested that an appropriate model for the volatility of financial returns should include the joint occurrence of long memory and structural change. These latter studies are generally consistent with previous literature such as Hamilton and Susmel (1994), which considered alternating regimes of high and low volatility, each one being characterized by strong persistence in their fluctuations. Economic explanations of the phenomenon have been suggested by Schwert (1989), who relates alternating volatility regimes to fluctuations in fundamental uncertainty and leverage effects over the business cycle. Also, Beltratti and Morana (2006) relate breaks in stock market volatility to monetary policy reactions in response to business cycle conditions.

Given the above summary of previous research, this present paper starts from the proposition that both long memory and structural breaks are likely to be present in the volatility processes of many economic and financial time series. The main contribution of this paper is then to present a model which allows for *both long memory and structural change* in a volatility process. The proposed model is named Adaptive $FIGARCH$, or *A-FIGARCH*, and augments the standard $FIGARCH$ model of Baillie, Bollerslev and Mikkelsen (1996) with a deterministic component, following Gallant (1984)'s flexible functional form. Hence the *A-FIGARCH* model allows for a stochastic long memory component and a deterministic break process component. The approach does not require pre-testing for the number of break points; nor does it require any smooth transition between volatility regimes; and has the advantage of being computationally straightforward.[2]

The rest of this paper is organized as follows. Section two introduces the *A-FIGARCH* model and its theoretical properties. Section three presents some Monte Carlo evidence for inference in the model and section four presents an empirical application based on equity market returns. The paper ends with a short concluding section.

[1] The finding that accounting for structural change may not be relevant for short-term forecasting is a robust finding in the literature. See for instance the discussion in Diebold and Inoue (2001) and the empirical results in Morana and Beltratti (2004).

[2] Indeed the proposed model is easily estimable with available menu-driven packages as for instance the G@RCH Ox interface.

2 The Adaptive *FIGARCH* Process

The Adaptive $FIGARCH$, or *A-FIGARCH* process is formed from two basic components of a long memory volatility process and a deterministic time-varying intercept which allows for breaks, cycles and changes in drift. By definition $\{y_t\}$ is a discrete time, real-valued stochastic process that is serially uncorrelated in its conditional mean, and has long memory type in its conditional variance process. Hence,

$$y_t \equiv \sigma_t z_t, \tag{1}$$

where $E_{t-1}[z_t] = 0$ and $Var_{t-1}[z_t] = 1$; σ_t is a positive, time-varying measurable function with respect to the information set available at time $t-1$, which is denoted as Ω_{t-1}. Hence, σ_t^2 is the time dependent conditional variance defined as $\sigma_t^2 = Var_{t-1}(y_t^2) = Var(y_t^2|\Omega_{t-1})$ and, following Baillie, Bollerslev and Mikkelsen (1996), is expressed as the long memory $FIGARCH(p, d, q)$ process

$$[1 - \beta(L)]\sigma_t^2 = w + [1 - \beta(L) - \phi(L)(1 - L)^d]y_t^2. \tag{2}$$

The process can be most easily motivated from representing $\{y_t^2\}$ as the $ARFIMA(m, d, q)$ model

$$\phi(L)(1 - L)^d y_t^2 = w + (1 - \beta(L))v_t, \tag{3}$$

where $v_t \equiv y_t^2 - \sigma_t^2$ is the innovation in the conditional variance. The long memory, fractional differencing parameter is denoted as d, and is allowed to be in the interval $0 < d < 1$, while the lag polynomials are defined as $\phi(L) = (1 - \alpha(L) - \beta(L))(1 - L)^{-d}$, where $\alpha(L) \equiv \alpha_1 L + ... + \alpha_q L^q$ and $\beta(L) \equiv \beta_1 L + ... + \beta_p L^p$. The polynomials $\phi(L)$ and $(1 - \beta(L))$ are assumed to have all their roots lying outside the unit circle. Moreover, $m = \max(p, q)$.

After rearrangement, an alternative representation for the $FIGARCH(p, d, q)$ model is

$$\sigma_t^2 = w[1 - \beta(1)]^{-1} + \left[1 - \phi(L)(1 - L)^d[1 - \beta(L)]^{-1}\right] y_t^2, \tag{4}$$

or

$$\sigma_t^2 = w[1 - \beta(1)]^{-1} + \lambda(L)y_t^2, \tag{5}$$

where $\lambda(L) \equiv \lambda_1 L + \lambda_2 L^2 + ...$, with $\lambda_i \geq 0$, for $i = 1, 2, ...$ and $w > 0$, for the conditional variance to be well defined, so that it is positive almost surely for every t. A key feature of the *FIGARCH* model is that for high lags, k, the distributed lag coefficients are $\lambda_k \simeq$

ck^{d-1}, where c is a positive constant; hence, the conditional variance can be expressed as a distributed lag of past squared returns with coefficients that decay at a slow, hyperbolic rate, which is consistent with the long memory property.

Recently, Conrad and Haag (2006) have provided two sets of sufficient conditions for the conditional variance process to be non negative almost surely. While the first set immediately implies the above condition, the second set is less restrictive, and in practice requires checking the non-negativity of only a finite number of the impulse response weights λ_is.

It is well known that for $0 < d \leq 1$ the $FIGARCH(p,d,q)$ process has an undefined unconditional variance. However, the process does possess a finite sum to its cumulative impulse response weights. This makes the $FIGARCH$ model different from other possible forms of long memory ARCH models, such as the class suggested by Karanassos, Pasaradakis and Sola (2004). However, following the arguments in Baillie, Bollerslev and Mikkelsen (1996), the $FIGARCH$ process does appear to be strictly stationary and ergodic for $0 \leq d \leq 1$.

As argued in the introduction, there are abundant motivations from the financial markets literature to allow for the possibility of structural instability in the volatility process. A straightforward, but quite powerful approach is to allow the intercept to be time dependent. Hence, the *A-FIGARCH*(p,d,q,k) process can be derived from the $FIGARCH(p,d,q)$ process by directly allowing the intercept w in the conditional variance equation to be time varying according to the Gallant (1994) flexible functional form. Hence, the model becomes

$$[1-\beta(L)]\left(\sigma_t^2 - w_t\right) = [1-\beta(L)-\phi(L)(1-L)^d]y_t^2, \tag{6}$$

where

$$w_t = w_0 + \sum_{j=1}^{k}[\gamma_j \sin(2\pi jt/T) + \delta_j \cos(2\pi jt/T)]. \tag{7}$$

Similarly to the $FIGARCH$ model, after rearrangement an alternative representation for the *A-FIGARCH*(p,d,q,k) model is

$$\sigma_t^2 = w_t + \left[1-\phi(L)(1-L)^d[1-\beta(L)]^{-1}\right]y_t^2, \tag{8}$$

or

$$\sigma_t^2 = w_t + \lambda(L)y_t^2. \tag{9}$$

In order for the conditional variance to be positive almost surely at each point in time, restrictions similar to those holding for the $FIGARCH(p,d,q)$ process have to be imposed, i.e. $w_t > 0$, for all t, and $\lambda_j \geq 0$, for all j. However, unlike the $FIGARCH$ process, the

A-FIGARCH process will not be ergodic and nor will it be strictly stationary, due to the time varying intercept component, modelled by the Gallant (1984) flexible functional form.

2.1 The *A-FIGARCH(1,d,1,k)* Process

A simple version of the model, which appears to be particularly useful in practice, is the *A-FIGARCH*$(1,d,1,k)$ process

$$[1-\beta L]\left(\sigma_t^2 - w_t\right) = [1-\beta(L)-\phi(L)(1-L)^d]y_t^2, \tag{10}$$

with w_t as defined in (7). On rearranging, an alternative representation for the *A-FIGARCH*$(1,d,1,k)$ model is then

$$\begin{aligned}\sigma_t^2 &= w_t + \left[1-(1-\beta L)^{-1}(1-L)^d(1-\phi L)\right]y_t^2 \\ &= w_t + \lambda(L)y_t^2,\end{aligned} \tag{11}$$

with $\lambda_0 = 1$, $\lambda_1 = d+\phi-\beta$, and, following Conrad and Haag (2006), $\lambda_i = \beta\lambda_{i-1} + (f_i - \phi)(-g_{i-1})$ $i > 1$, where $f_j = (j-1-d)/j$, for $j = 1, 2, ...$ and $g_j = f_j \cdot g_{j-1}$. As noted by Baillie, Bollerslev and Mikkelsen (1996), the non negativity of the conditional variance for the *FIGARCH*$(1,d,1)$ model can be ensured by the restrictions $w > 0$, $0 \leq \beta \leq \phi + d$ and $0 \leq d \leq 1-2\phi$. The last two constraints can be stated as $\beta - d \leq \phi \leq \frac{2-d}{3}$ and $d\left[\phi - \frac{1-d}{2}\right] \leq \beta(\phi - \beta + d)$, and are written by Chung (1999) as $0 \leq \phi \leq \beta \leq \beta d < 1$. Finally, Conrad and Haag (2006) have recently proposed alternative and less restrictive forms, and they show for the case of $0 < \beta < 1$, $\lambda_1 \geq 0$ and $\phi \leq f_2$; while for the case $-1 < \beta < 0$, $\lambda_1 \geq 0$, $\lambda_2 \geq 0$ and $\phi \leq f_2(\beta + f_3)/(\beta + f_2)$. Similar restrictions hold for the *A-FIGARCH* model.

2.2 Estimation

Estimation and inference for the parameters of the *A-FIGARCH* process can be facilitated by the familiar method of Quasi Maximum Likelihood Estimation ($QMLE$), where the Gaussian log likelihood

$$ln\{L(\theta, y_1, ..., y_T)\} = -0.5T\ln(2\pi) - 0.5\sum_{t=1}^{T}\{\ln(\sigma_t^2) + y_t^2\sigma_t^{-2}\}$$

is numerically maximized with respect to the vector of the parameters $\boldsymbol{\theta} = (d, \boldsymbol{\beta}', \boldsymbol{\phi}', \mathbf{w}', \boldsymbol{\gamma}')'$. Hence, the procedure implements simultaneous estimation of all the model's parameters, including those in the flexible functional form, which specify the time varying intercept in the conditional variance process. Under fairly general conditions, the asymptotic distribution of the $QMLE$ is

$$T^{1/2}\left(\hat{\boldsymbol{\theta}} - \boldsymbol{\theta}_0\right) \rightarrow N\{\mathbf{0}, \mathbf{A}(\boldsymbol{\theta}_0)^{-1}\mathbf{B}(\boldsymbol{\theta}_0)\mathbf{A}(\boldsymbol{\theta}_0)^{-1}\},$$

where $\boldsymbol{\theta}_0$ denotes the true value of the vector of parameters, and where $\mathbf{A}(\boldsymbol{\theta}_0)$ is the Hessian and $\mathbf{B}(\boldsymbol{\theta}_0)$ is the outer product gradient; both of which are evaluated at the true parameter values. Some results for the asymptotic properties of $QMLE$ can be established on the basis of dominance type arguments, using available results from the estimation of $IGARCH$ processes. Jensen and Rahbek (2004) have recently demonstrated that $QMLE$ has the properties of consistency and asymptotic normality when applied to the $IGARCH(1,1)$ process, which exhibits non stationarity and non ergodicity, similarly to that of the $FIGARCH(1,d,1)$ process. Although a formal proof is beyond the scope of this paper, it is expected that similar results can be expected to hold for the *A-FIGARCH*$(1,d,1,k)$ process. However, it is worth noting that the conditions required by Jensen and Rahbek (2004) are less stringent than those imposed by Lee and Hansen (1994) and Lumsdaine (1996), where the consistency and asymptotic normality of the $QMLE$ was initially shown for the strictly stationary and ergodic case. Clearly the $IGARCH(1,1)$ case is in some sense an "extreme" situation. In particular, Jensen and Rahbek (2004) assume that $z_t \sim i.i.d.(0,1)$, with $Var(z_t^2) = k < \infty$, and that the true parameters satisfy the condition $E\ln(\alpha_0 z_t^2 + \beta_0) \geq 0$, where α_0 and β_0 denote the true values of the parameters α and β, i.e. the squared innovation and lagged conditional variance parameters, respectively, in the *GARCH*(1,1) model. Hence, the requirements do not depend on further higher moment conditions and cover the integrated and explosive cases.[3] Moreover, Jensen and Rahbek (2004) have shown that the asymptotic properties of the estimator still hold for any initial values σ_0^2 and y_0^2, and any value of ω. This allows conditioning on the sample mean value of y_t^2, which is $\frac{1}{T}\sum_{t=1}^{T} y_t^2$, for σ_0^2 and y_0^2, as is usually implemented in the estimation of $GARCH$ models. Finally, it is important to note that results concerning consistency and asymptotic normality of the $QMLE$ have been obtained for the general strictly stationary and ergodic $GARCH(p,q)$ process; see Berkes et al. (2003). However, results for the non-stationary and non ergodic case currently only exist for the $GARCH(1,1)$ process, which is fortunately the most widely used model in applied econometric work.

[3] Lee and Hansen (1994) assume that $E\ln(\alpha_0 z_t^2 + \beta_0) < 0$, which is a necessary and sufficient condition for the stationarity of the $GARCH(1,1)$ process. This latter condition is in fact implied by the condition that $\alpha_0 + \beta_0 \leq 1$.

The numerical maximization of the log likelihood function is implemented by using the asymptotically equivalent method of minimizing the conditional sum of squares function, which neglects starting values. Many previous studies have presented simulation evidence which shows that neglecting initial conditions has minimal effects on parameter estimation of long memory models in either of the first two conditional moments, given a sample size of at least 100 observations. See for instance the results in Baillie, Chung and Tieslau (1996) for the $ARFIMA$ model with stable $GARCH(1,1)$ innovations and also Baillie, Bollerslev and Mikkelsen (1996) for the $FIGARCH$ case.

3 Simulation Results

This section reports some quite detailed Monte Carlo evidence on the impact of estimating *A-FIGARCH* models under different data generating process scenarios. All the estimated *A-FIGARCH* models are contrasted and compared with the properties of estimated $FIGARCH$ models from the same data generating process across all the replications. All of the experiments specify an uncorrelated process y_t for the mean, but with various forms of long memory and structural breaks, or time dependent intercept for its conditional variance process. In particular, the martingale with *FIGARCH(p,d,0)* model, with $p = (0,1)$, has been employed, which data generating process is

$$y_t = \sigma_t \varepsilon_t$$

$$\varepsilon_t \sim NID(0,1)$$

$$\sigma_t^2 = w_t + (1-L)^d y_t^2 \text{ when } p = 0 \tag{12}$$

$$\text{and,} \tag{13}$$

$$(1-\beta L)(\sigma_t^2 - w_t) = [(1-\beta L)(1-L)^d] y_t^2 \text{ when } p = 1. \tag{14}$$

Three different designs were focused upon:

Design 1 has a constant intercept of $w_t = w = 0.5$, and corresponds to the standard case without structural breaks in the conditional variance.

Design 2 has a step change in the intercept at the midpoint of the sample, where the intercept is doubled at this point. Hence,

$$w_t = \begin{cases} 0.5 & t = 1, ..., T/2 \\ 1 & t = T/2+1, ..., T \end{cases}.$$

Design 3 has two step changes equally spaced throughout the sample where the intercept increases eight fold, one third of the way through the sample and then decreases four fold after two thirds of the sample. Hence,

$$w_t = \begin{cases} 0.5 & t = 1, ..., T/3 \\ 4 & t = T/3 + 1, ..., 2T/3 \\ 1 & t = 2T/3 + 1, ..., T \end{cases} .$$

These three designs were each simulated for three different values of the long memory parameter, given by $d = (0.15, 0.30, 0.45)$, and for three values for the short memory parameter $\beta = (0, 0.15, 0.30)$. Clearly, the estimation of the *A-FIGARCH* model should prove superfluous in design 1, while the interest in designs 2 and 3 centers on the performance of *QMLE* when the pure *martingale-FIGARCH* process and the new *martingale-A-FIGARCH* models are estimated in the presence of structural breaks in the intercept of the conditional variance. Hence, for designs 1, 2 and 3 the estimated models are the $FIGARCH(p, d, 0)$, with $p = (0, 1)$,

$$\begin{aligned} y_t &= \sigma_t z_t \\ z_t &\sim NID(0, 1) \\ (1 - \beta L)\sigma_t^2 &= w + [(1 - \beta L)(1 - L)^d] y_t^2, \end{aligned} \tag{15}$$

and the *A-FIGARCH*$(p, d, 0, k)$ model, with $p = (0, 1)$,

$$\begin{aligned} y &= \sigma_t z_t \\ z_t &\sim NID(0, 1) \\ (1 - \beta L)(\sigma_t^2 - w_t) &= [(1 - \beta L)(1 - L)^d] y_t^2, \\ w_t &= w_0 + \sum_{j=1}^{k} [\gamma_j \sin(2\pi jt/T) + \delta_j \cos(2\pi jt/T)]. \end{aligned}$$

The *A-FIGARCH* models were estimated for each design with one to four pairs of trigonometric terms included, i.e. $k = (1, 2, 3, 4)$. The number of simulated observations for each design is 10,000 observations, which includes the discarded first 7,000, leaving with simulated processes of sample size equal to 3,000 observations. Following Baillie, Bollerslev and Mikkelsen (1996), the order of the truncation in estimation has been set to 1,000 observations. Finally, 500 Monte Carlo replications were employed in all of the designs. In Tables 1 through 3 the Monte Carlo bias (bias), root mean square error ($RMSE$) and the standard error (s.e.) of the estimator, are reported for the $p = 0$ case. The results for the $p = 1$ case are reported in Tables 4 through 6. The ability of the models in fitting the conditional variance process has also been assessed by means of the root mean square forecast error statistic ($RMSFE_{\sigma}$), which is computed as

$$RMSFE_{\sigma} = \frac{1}{500}\sum_{j=1}^{500} RMSFE_j$$

$$RMSFEj = \sqrt{\frac{1}{T}\sum_{t=1}^{T}\left(\hat{\sigma}_t^2 - \sigma_t^2\right)^2},$$

where $\hat{\sigma}_t^2$ is the estimated conditional variance process and σ_t^2 is the actual conditional variance process.

The simulation experiments reveal several general points concerning the performance of the different estimators of the long memory parameter d. For case 1, where there is no structural change, the application of the *A-FIGARCH* model should clearly be unnecessary since the intercept is a constant. First, the estimate of the long memory parameter obtained from the *A-FIGARCH* estimation has approximately the same degree of small sample bias as the corresponding estimate from the estimation of the $FIGARCH$ model. This result appears consistent across all the designs. However, the most interesting result is the reduction in $RMSE$ of the estimate of the d parameter from using *A-FIGARCH* compared with estimation from the regular $FIGARCH$ model. The reduction in RMSE appears to noticeably increase as the level of persistence (value of d) increases. These results suggest that there is no additional cost from using the *A-FIGARCH* model as opposed to the $FIGARCH$ model, even when there is no structural break in the conditional mean. The interpretation of this is intriguing and suggests that the time dependent intercept is also somehow adjusting for parameter uncertainty in the estimation of d.

For cases 2 and 3, where the intercept is subject to structural breaks, apart from the low persistence case ($d = 0.15$), the degree of bias in the estimates of d is very small for both estimators. However, the bias is again always smaller for the *A-FIGARCH* model compared to the pure *FIGARCH* model. Moreover, the $RMSE$ of the estimate of d is generally lower from the *A-FIGARCH* estimation compared to the corresponding $FIGARCH$ estimation. Finally, the generally superior performance of the estimate of d from the estimation of the *A-FIGARCH* model relative to the standard $FIGARCH$ model, is robust across the three different values of d used in the designs, with the improvement increasing as the the degree of persistence increases. Hence, the Gallant flexible functional form seems to work quite well in the *A-FIGARCH* model estimation framework, doing a good job in terms of modeling the structural change in the intercept.

Interestingly, from Tables 4 through 6, it can also be noted that neglecting structural breaks does not only lead to an upward biased estimate of the fractional differencing parameter, as already found for the $p = 0$ case, but also in the estimate of the stationary autoregressive parameter, β. This latter finding is particularly evident when the degree of persistence is low, as in the $d = 0.15$ case. The upward bias in the estimate of d from the regular $FIGARCH$ estimation appears to be mitigated by the inclusion of the trigonometric components in the *A-FIGARCH* estimation. The improved performance of the estimation of d tends to increase with the degree of persistence of the series. Hence, estimation of the *A-FIGARCH* shows a superior performance relatively to the *FIGARCH* model in terms of bias and RMSE in all the designs. Interestingly, the greatest improvement is in the $d = 0.45$ case, which is the one mostly relevant for financial applications. In this case there is a 145% reduction in bias and a 60% reduction in RMSE obtained from using the *A-FIGARCH* model, relatively to the *FIGARCH* model.

Overall, the above results indicate potentially significant gains from using the *A-FIGARCH* specification, and certainly no perceptible losses, even in the absence of structural breaks. The possible loss of efficiency in using an unnecessary, over-parameterized *A-FIGARCH* model specification does not appear to be an issue. It may be that the estimation from smaller sample sizes would find losses in efficiency of the estimation of d. Since a sample size of $T = 3,000$ is quite common for finance applications, the situation from smaller sample sizes was not investigated.

The final point of interest concerns the overall goodness of fit of the models as indicated by the $RMSFE_\sigma$ statistic. Only when the degree of persistence is low ($d = 0.15$) and there is no structural change, does the inclusion of the adaptive component not yield any improvement in the goodness of fit statistic. In particular, in Table 1 and Table 4 for case 1 the estimation of high order ($k = 3$, or $k = 4$) adaptive components decreases the goodness of fit.

Therefore, in the light of the Monte Carlo evidence, it seems preferable to include the adaptive non linear trend component in the specification for the conditional variance equation at the out set, since no negative consequences for estimation may be expected, apart from the case of weak long memory, which however does not seem to be relevant for financial returns. Then, following a general to specific methodology, the best fitting parsimonious model may be obtained. Moreover, for the cases investigated in the Monte Carlo exercise, there is no evidence of an improvement in the performance of the model by the inclusion of polynomial terms beyond the second or third order.

4 Applications to Stock Market Volatility

This section of the paper reports estimation of *A-FIGARCH* and $FIGARCH$ models for the S&P500 returns. The time span is from January 3, 1928 through February 15, 2007, which realizes a total of T = 20,863 observations and is a long enough period for the likely occurrence of multiple structural breaks in volatility. For the practical implementation of the *A-FIGARCH* method, an important consideration is the determination of the order of the trigonometric terms in the Gallant flexible functional form, in addition to the order of the specification of the stationary components in the conditional mean and conditional variance equations. In the reported results the Schwartz BIC information criterion is used for model selection. Since the conditional mean exhibited some small degree of autocorrelation, an AR(2) term was eventually included in the mean equation. Hence, the following *AR(2)-FIGARCH*$(1, d, 1)$ model

$$\begin{aligned}(1-\theta_1 L-\theta_2 L^2)y_t &= \mu+\varepsilon_t \\ \varepsilon_t &= \sigma_t z_t \\ z_t &\sim NID(0,1) \\ [1-\beta L]\sigma_t^2 &= w+[1-\beta L-(\phi L)(1-L)^d]\varepsilon_t^2,\end{aligned}$$

and the *AR(2)-A-FIGARCH*$(1, d, 1, k)$ model

$$\begin{aligned}(1-\theta_1 L-\theta_2 L^2)y_t &= \mu+\varepsilon_t \\ \varepsilon_t &= \sigma_t z_t \\ z_t &\sim NID(0,1) \\ [1-\beta L](\sigma_t^2-w_t) &= [1-\beta L-(\phi L)(1-L)^d]\varepsilon_t^2 \\ w_t &= w_0+\sum_{j=1}^{k}[\gamma_j\sin(2\pi jt/T)+\delta_j\cos(2\pi jt/T)]\end{aligned}$$

were estimated for the S&P500 returns series, and the results are reported in Table 7. The SBC criterion indicates that the inclusion of trigonometric cosine components makes an important contribution to the general goodness of fit of the models. This finding is consistent with evidence on the presence of structural breaks previously detected for S&P500 returns, as reported by Lobato and Savin (1998), Granger and Hyung (2004), Starica and Granger (2004) and Beltratti and Morana (2006). On comparison of the estimated parameters for the

FIGARCH and *A-FIGARCH* models, it can be seen that neglecting the presence of structural breaks leads to an inferior fit of the conditional variance process, while the estimated persistence and autoregressive parameters are not statistically different across models. This is fully to be expected given the previously described simulation evidence. The SBC criterion suggests that the inclusion of trigonometric terms up to the third order ($k = 3$) is desirable from a specification perspective, with no additional improvements beyond this order. The estimated conditional standard deviation by the preferred *A-FIGARCH*$(1, d, 0, 3)$ model is plotted in Figure 1. Finally, the consequences of neglecting structural change can be clearly noted in Figure 2, where the conditional standard deviations from the *FIGARCH*$(1, d, 0)$ model and the *A-FIGARCH*$(1, d, 0, 3)$ model have been plotted for four sub-periods, randomly chosen, of 100 days each. As shown in the plots, due to neglecting the break process, the estimated conditional standard deviation process from the *FIGARCH*$(1, d, 0)$ model can show a noticeable bias, both upward or downward, relatively to the one obtained by the *A-FIGARCH*$(1, d, 0, 3)$ model.

5 Conclusions

This paper has introduced the new Adaptive *FIGARCH* or *A-FIGARCH* process to model volatility, which is designed to account for both long memory and structural change in the conditional variance process. Structural change is modeled by allowing the intercept to follow a slowly varying function, specified by Gallant (1984)'s flexible functional form. A detailed simulation experiment finds that the *A-FIGARCH* model outperforms the standard *FIGARCH* model when structural change is present, and performs at least as well in the absence of structural instability. Overall, there appear to be significant gains in terms of bias and efficiency from using the *A-FIGARCH* specification. An empirical application to stock market volatility is also included to illustrate the usefulness of the technique.

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Table 1: Monte Carlo results, $A\text{-}FIGARCH(0,d,0,k)$ and $FIGARCH(0,d,0)$ models

$T = 3000$

$A\text{-}FIGARCH(0,d,0,k)$; $d = 0.15$

		$bias_d$	$RMSE_d$	$s.e._d$	$RMSFE_\sigma$
	m_1	.036	.029	.028	.028
$k=1$	m_2	.081	.017	.010	.468
	m_3	.089	.020	.012	.587
	m_1	−.004	.020	.020	.028
$k=2$	m_2	.075	.016	.010	.467
	m_3	.091	.020	.012	.587
	m_1	−.010	.022	.022	.035
$k=3$	m_2	.081	.016	.010	.476
	m_3	.090	.020	.012	.587
	m_1	−.012	.021	.021	.038
$k=4$	m_2	.081	.017	.010	.477
	m_3	.090	.019	.011	.586

$FIGARCH(0,d,0)$; $d = 0.15$

		$bias_d$	$RMSE_d$	$s.e._d$	$RMSFE_\sigma$
	m_1	.001	.021	.021	.023
$k=0$	m_2	.102	.020	.010	.488
	m_3	.107	.023	.012	.595

Key: The table reports results for a simulation study. The table indicates the bias, root mean square error (RMSE) and standard error (s.e.) for estimation of the fractional differencing parameter d from a sample size of T =3,000 observations. The table also reports the goodness of fit criterion (RMSFE$_\sigma$) for the conditional variance process; and all results are based on 500 replications. Three different break configurations have been investigated, i.e. the case of no break (m_1), the case of a single break point (m_2) and the case of two break points (m_3), employing up to a fourth order trigonometric expansion ($k = 0, 1, 2, 3, 4$) for the adaptive component.

Table 2: Monte Carlo results: $A\text{-}FIGARCH(0,d,0,k)$ and $FIGARCH(0,d,0)$ models

$T = 3000$

$A\text{-}FIGARCH(0,d,0,k)$; $d = 0.30$

		$bias_d$	$RMSE_d$	$s.e._d$	$RMSFE_\sigma$
	m_1	−.004	.024	.024	.034
$k=1$	m_2	.017	.022	.021	.280
	m_3	.036	.026	.025	.382
	m_1	.002	.023	.023	.033
$k=2$	m_2	.003	.021	.021	.254
	m_3	.034	.027	.026	.380
	m_1	−.010	.023	.023	.033
$k=3$	m_2	.005	.021	.021	.266
	m_3	.027	.028	.027	.417
	m_1	−.011	.023	.023	.036
$k=4$	m_2	.004	.020	.020	.263
	m_3	.028	.029	.028	.373

$FIGARCH(0,d,0)$; $d = 0.30$

		$bias_d$	$RMSE_d$	$s.e._d$	$RMSFE_\sigma$
	m_1	.005	.035	.035	.047
$k=0$	m_2	.025	.022	.021	.278
	m_3	.044	.030	.028	.421

Key: As for Table 1.

Table 3: Monte Carlo results: $A\text{-}FIGARCH(0,d,0,k)$ and $FIGARCH(0,d,0)$ models

$T = 3000$

$A\text{-}FIGARCH(0,d,0,k)$; $d = 0.45$

		$bias_d$	$RMSE_d$	$s.e._d$	$RMSFE_\sigma$
	m_1	$-.017$	.024	.024	.039
$k=1$	m_2	$-.012$	.034	.034	.347
	m_3	.005	.035	.035	.433
	m_1	$-.007$	.056	.042	.166
$k=2$	m_2	.003	.021	.021	.254
	m_3	.001	.027	.025	.368
	m_1	$-.020$	.028	.028	.062
$k=3$	m_2	$-.030$	.036	.036	.335
	m_3	$-.003$	.040	.040	.400
	m_1	$-.018$	.029	.029	.063
$k=4$	m_2	$-.027$	.039	.038	.444
	m_3	$-.004$	.037	.037	.313

$FIGARCH(0,d,0)$; $d = 0.45$

		$bias_d$	$RMSE_d$	$s.e._d$	$RMSFE_\sigma$
	m_1	.013	.035	.035	.059
$k=0$	m_2	$-.012$	.049	.049	.470
	m_3	.036	.040	.037	.484

Key: As for Table 1.

Table 4: Monte Carlo results: $A\text{-}FIGARCH(1,d,0,k)$ and $FIGARCH(1,d,0)$ models

$T = 3000$

$A\text{-}FIGARCH(1,d,0,k)$; $d = 0.15$, $\beta = 0.15$

		$bias_d$	$RMSE_d$	$s.e._d$	$bias_\beta$	$RMSE_\beta$	$s.e._\beta$	$RMSFE_\sigma$
	m_1	−.008	.028	.028	−.016	.029	.029	.026
$k = 1$	m_2	.202	.062	.022	.186	.065	.030	.427
	m_3	.216	.082	.035	.191	.085	.048	.540
	m_1	−.009	.032	.032	−.016	.033	.033	.028
$k = 2$	m_2	.152	.039	.016	.120	.042	.025	.429
	m_3	.158	.054	.029	.124	.056	.040	.542
	m_1	−.020	.029	.029	−.014	.030	.030	.034
$k = 3$	m_2	.151	.037	.015	.126	.041	.025	.434
	m_3	.154	.041	.018	.112	.040	.026	.535
	m_1	−.027	.031	.030	−.033	.033	.032	.040
$k = 4$	m_2	.167	.044	.016	.147	.047	.026	.433
	m_3	.196	.072	.034	.174	.075	.045	.536

$FIGARCH(1,d,0)$; $d = 0.15$, $\beta = 0.15$

		$bias_d$	$RMSE_d$	$s.e._d$	$bias_\beta$	$RMSE_\beta$	$s.e._\beta$	$RMSFE_\sigma$
	m_1	−.004	.030	.030	−.011	.032	.032	.022
$k = 0$	m_2	.220	.071	.022	.205	.074	.032	.439
	m_3	.244	.098	.047	.225	.097	.038	.545

As for Table 1; and also including the bias, root mean square error (RMSE) and standard error (s.e.) for the first order autoregressive parameter β.

Table 5: Monte Carlo results: $A\text{-}FIGARCH(1,d,0,k)$ and $FIGARCH(1,d,0)$ models

$T = 3000$

$A\text{-}FIGARCH(1,d,0,k)$; $d = 0.30$, $\beta = 0.30$

		$bias_d$	$RMSE_d$	$s.e._d$	$bias_\beta$	$RMSE_\beta$	$s.e._\beta$	$RMSFE_\sigma$
	m_1	.002	.042	.042	−.008	.044	.044	.031
$k=1$	m_2	.130	.055	.038	.118	.057	.043	.287
	m_3	.168	.070	.041	.155	.071	.047	.373
	m_1	−.007	.039	.039	−.015	.041	.041	.029
$k=2$	m_2	.087	.037	.027	.076	.034	.031	.280
	m_3	.137	.067	.048	.123	.068	.053	.371
	m_1	−.021	.039	.038	−.028	.040	.039	.032
$k=3$	m_2	.083	.031	.024	.069	.033	.028	.280
	m_3	.126	.056	.040	.111	.057	.045	.307
	m_1	−.027	.038	.038	−.034	.039	.038	.034
$k=4$	m_2	.095	.040	.030	.085	.042	.035	.284
	m_3	.147	.067	.046	.134	.068	.050	.377

$FIGARCH(1,d,0)$; $d = 0.30$, $\beta = 0.30$

		$bias_d$	$RMSE_d$	$s.e._d$	$bias_\beta$	$RMSE_\beta$	$s.e._\beta$	$RMSFE_\sigma$
	m_1	−.002	.047	.047	−.010	.049	.049	.033
$k=0$	m_2	.149	.070	.048	.138	.071	.052	.302
	m_3	.178	.106	.078	.190	.106	.082	.366

Key: As for Table 4.

Table 6: Monte Carlo results: $A\text{-}FIGARCH(1,d,0,k)$ and $FIGARCH(1,d,0)$ models

$T = 3000$

$A\text{-}FIGARCH(1,d,0,k)$; $d = 0.45$, $\beta = 0.30$

		$bias_d$	$RMSE_d$	$s.e._d$	$bias_\beta$	$RMSE_\beta$	$s.e._\beta$	$RMSFE_\sigma$
	m_1	−.006	.042	.042	−.008	.050	.050	.040
$k = 1$	m_2	.044	.051	.049	.049	.060	.058	.209
	m_3	.085	.061	.053	.087	.070	.063	.285
	m_1	−.005	.044	.044	−.007	.051	.051	.045
$k = 2$	m_2	.022	.046	.049	.031	.050	.045	.200
	m_3	.066	.060	.055	.071	.067	.062	.301
	m_1	−.021	.051	.050	−.019	.056	.056	.049
$k = 3$	m_2	.023	.055	.055	.033	.060	.059	.232
	m_3	.055	.061	.058	.064	.070	.066	.270
	m_1	−.027	.050	.049	−.023	.054	.053	.049
$k = 4$	m_2	.021	.056	.055	.030	.060	.059	.228
	m_3	.062	.061	.058	.072	.069	.064	.286

$FIGARCH(1,d,0)$; $d = 0.45$, $\beta = 0.30$

		$bias_d$	$RMSE_d$	$s.e._d$	$bias_\beta$	$RMSE_\beta$	$s.e._\beta$	$RMSFE_\sigma$
	m_1	.032	.079	.078	.029	.079	.078	.063
$k = 0$	m_2	.073	.071	.066	.078	.077	.071	.260
	m_3	.108	.085	.073	.112	.093	.080	.331

Key: As for Table 4.

Table 7: Estimation of *A-FIGARCH*(1,d,1,k) and *FIGARCH*(1,d,1) models to S&P500 Returns

	FI	AFI(1)	AFI(2)	AFI(3)	AFI(4)
μ	0.050 (.006)	0.050 (.006)	0.051 (.006)	0.051 (.006)	0.051 (0.006)
θ_1	0.115 (.008)	0.115 (.008)	0.115 (.008)	0.115 (.008)	0.115 (0.008)
θ_2	-0.036 (.008)	-0.036 (.008)	-0.036 (.008)	-0.036 (.008)	-0.036 (0.008)
w	0.058 (.008)	0.072 (.009)	0.068 (.009)	0.071 (.009)	0.071 (0.009)
β	0.230 (.044)	0.223 (.030)	0.223 (.029)	0.219 (.029)	0.218 (0.030)
d	0.330 (.037)	0.329 (.024)	0.331 (.023)	0.328 (.024)	0.326 (0.024)
δ_1		0.035 (.010)	0.014 (.008)	0.013 (.009)	0.012 (0.009)
δ_2			-0.037 (.009)	-0.029 (.009)	-0.027 (0.009)
δ_3				0.019 (.010)	0.015 (0.009)
δ_4					-0.011 (0.009)
LB_{10}	23.950	27.448	27.463	27.107	27.288
LB_{50}	82.062	86.154	87.141	87.027	87.171
LB^2_{10}	16.913	16.532	16.306	16.154	16.271
LB^2_{50}	48.286	47.551	47.842	48.890	49.701
sk	-0.421	-0.419	-0.408	-0.407	-0.407
ku	6.688	6.731	6.537	6.486	6.456
sb	0.130	0.123	0.095	0.088	0.088
nsb	0.000	0.001	0.000	0.000	0.000
psb	0.073	0.049	0.050	0.048	0.047
SBC	2.5628	2.5617	2.560	2.5599	2.5602

The sample is from January 3 1928 through February 15 2007, for a total of T = 20,863 observations. The asymptotic standard errors are reported in parenthesis beside corresponding parameter estimates. The diagnostic statistics are LB which denotes the Ljung-Box test for serial correlation in the standardized residuals, LB^2 is the Ljung-Box test for serial correlation in the squared standardized residuals, *sk* is the index of skewness and *ku* is the index of kurtosis. The Ljung Box statistics are computed from the first 10 and 50 sample autocorrelations. Finally, *sb* denotes the p-value of the sign bias t-test, *nsb* the p-value of the negative size bias t-test, *psb* the p-value of the positive size bias t-test, while *SBC* is the Schwarz-Bayes information criterion. The estimated models are the $ARFIMA(1, d, 0)$ model (FI) and the $A\text{-}ARFIMA(1, d, 0, k)$ model (AFI(k)), with $k = 1, 2, 3, 4$.

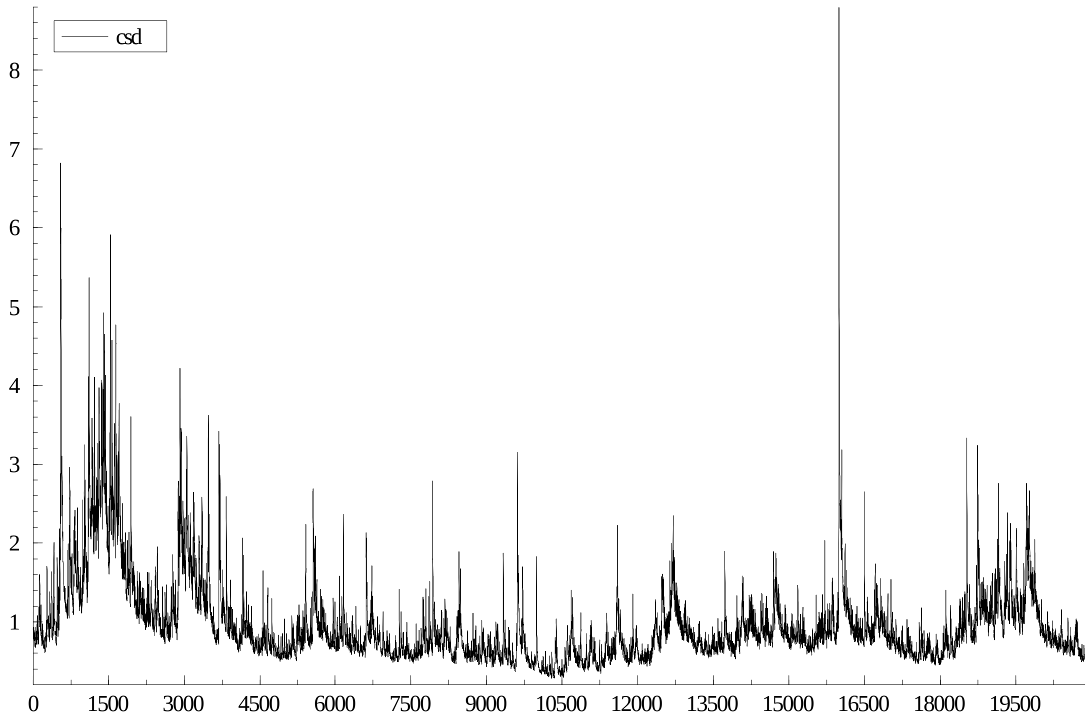

Figure 1. S&P 500 conditional standard deviation process (csd), A-FIGARCH(1,d,0,3) model.

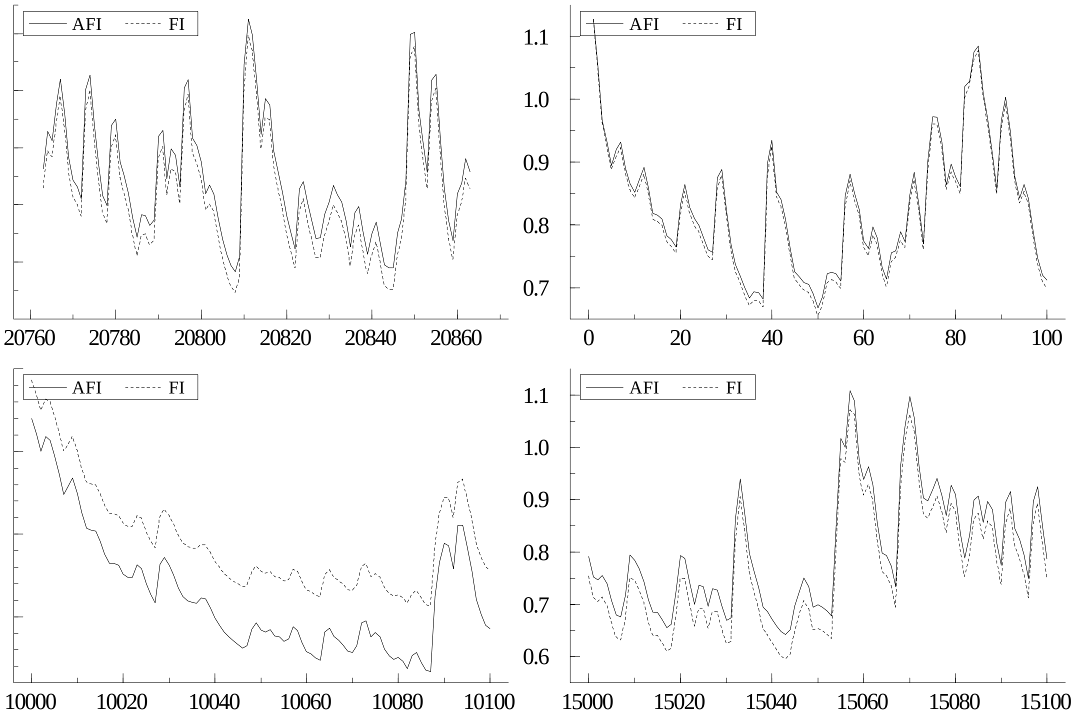

Figure 2: Estimated conditional standard deviations from the FIGARCH(1,d,0) model (FI) and the A-FIGARCH(1,d,0,3) model (AFI).

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