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Abstract: A monetary policy rule is a function mapping any given output level of the economy to a corresponding rate of inflation. Such a rule is time-consistent if the central bank has no incentive to deviate from it. Within a simple dynamic model combining an output-inflation trade-off with rational private-sector expectations we study existence and properties of time-consistent monetary policy rules. It is shown that such rules exist only if (i) the central bank gives relatively high weight to price stability and relatively low weight to output stabilization and if (ii) the random shocks to the economy are not too strong. If time-consistent monetary policy rules exist, they are generically non-unique.

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Key words: Monetary policy, time-consistency, policy rules, inflation.

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1 Introduction

This paper studies existence and properties of time-consistent monetary policy rules in a simple dynamic framework. Before we start, we have to clarify what we mean by a policy rule and how policy rules are related to the problem of dynamic inconsistency.

Economists use the term *policy rule* with at least two different interpretations. In the literature on the dynamic inconsistency of low-inflation monetary policy the most important feature of a rule is that it provides a form of *commitment*. Barro and Gordon, for example, write in one of the most influential papers in this area that “rules are enforceable, so that the policymaker can commit the course of future policy (and thus of expectations)” [1, page 599]. Economists studying the practical implementation of monetary policy, on the other hand, interpret policy rules in a different way. In an important contribution to this literature Taylor writes that “rules are responsive, calling for changes in the money supply, the monetary base, or the short-term interest rate in response to changes of the price level or real income” [12, page 196]. Thus, this stream of literature emphasizes the *responsiveness* of rules or, in other words, the fact that they are *feedback rules*.

The concept of time-consistency (or dynamic consistency), in contrast, seems to be unambiguously defined. It describes a situation in which policymakers have no incentives to deviate from previously announced settings of the instrument variables. A policy rule in the sense of Barro and Gordon [1] does not make economic policy time-consistent but, rather, it eliminates the need for time-consistency. If policymakers are able to commit their future actions, then no one has to worry about any incentives to deviate from these actions. Economic policy conducted by a rule in the sense of Taylor [12], on the other hand, can be time-consistent or not depending on the incentives created by the rule. This has been pointed out clearly by McCallum, who identified the crucial property as “whether, in choosing the specific form of systematic behavior to be followed, the policymaker does or does not take account of how private-sector expectations are formed” [7, page 217]. Thus, only if we follow Taylor’s interpretation of economic policy rules, does it make sense to distinguish between policy rules that are time-consistent and those that are not. In the present paper we analyse under which conditions time-consistent Taylor-type monetary policy rules exist and what their properties are.

The framework of our analysis is the standard model from the dynamic inconsistency literature. More specifically, we use a model of monetary policy in the presence of an output-inflation trade-off that was proposed by Kydland and Precott [5] and elaborated on by Barro and Gordon [1]. The output-inflation trade-off is described by an expectations-augmented Phillips curve. The central bank can directly control the rate of inflation and it seeks to minimize deviations of output and inflation from given target levels. The private sector forms rational expectations about inflation. Existence and properties of time-consistent monetary policy rules in a static version of this model have been analysed in Sorger [11]. The present paper studies whether the results derived in [11] hold also in a fully dynamic model. As it turns out, this is indeed the case.

We start by studying a deterministic version of the model and extend it subsequently by including exogenous stochastic shocks to the economy. In the deterministic model we are

able to characterize the set of time-consistent monetary policy rules completely, whereas in the stochastic model we restrict the analysis to linear policy rules. Our main findings are as follows.

- Time-consistent monetary policy rules exist only if the central bank’s loss function gives relatively high weight to price stability and relatively low weight to output stability.
- Time-consistent monetary policy rules exist only if the stochastic shocks to the economy are not too strong.
- If time-consistent monetary policy rules exist at all, then they are in general non-unique. More specifically, there exist generically two linear rules and – at least in the deterministic model – infinitely many non-linear ones.

The first two results are closely related to each other, their common link being a destabilizing effect of policy rules on the expectations-augmented Phillips curve. If the exogenous shocks are very strong or if the central bank cares too much about output stability, then it will use a policy rule that prescribes high inflation even if output is only slightly below target. Since the private sector rationally anticipates this behavior, households and firms will expect high inflation if output falls below target. This effect on expectations makes the Phillips curve actually much steeper than it would be for any constant expected rate of inflation. In the case of strong exogenous shocks or in the case of a central bank that is very concerned about output stabilization, the additional instability generated by private-sector expectations is so profound that it rules out the existence of an equilibrium.¹ The first two results together provide therefore a new rationale for the appointment of conservative (i.e., inflation averse) central bankers. Whereas Rogoff [9] proves that delegating monetary policy to conservative central bankers leads to a socially preferred outcome, our results show that an economy in which monetary policy is conducted by a rule can become unstable and, hence, lack any equilibrium at all, if the central bankers are not sufficiently averse to deviations of the rate of inflation from its target.

The third result mentioned above has nothing to do with stability. It is merely a consequence of the high dimensionality of the strategy spaces. Formally, our models are differential games with Markovian strategy spaces. It is quite well-known that such games can have multiple equilibria, and the above mentioned property of our models is just another instance of this more general result. In this regard, it is worth mentioning that our models have a linear-quadratic structure. Therefore, the existence of equilibria consisting of linear policy rules is not surprising, whereas the existence of infinitely many equilibria consisting of non-linear rules is perhaps somewhat unexpected.

The paper is organised as follows. In section 2 the deterministic model is presented and analysed. Proofs for all results in section 2 are delegated to appendix A. Section 3 studies an extension of the model that includes exogenous stochastic shocks. All proofs for the results in section 3 can be found in appendix B. Finally, section 4 concludes the paper.

¹The destabilizing effect of policy rules has already been noted by Kydland and Prescott [5]; see the discussion of theorem 1 below.

2 The deterministic model: a complete characterization

The model considered in the present paper shares its basic features with the standard model from the dynamic inconsistency literature originating from Kydland and Precott [5] and Barro and Gordon [1]. In particular, we adopt the linear-quadratic structure that has been used predominantly in this literature; see Persson and Tabellini [8]. In static models of this kind, the output-inflation trade-off is typically described by a linear Phillips curve relation of the form $y = y_n + \alpha(\pi - \pi^e)$, where $y \in \mathbb{R}$ is logarithmic output, y_n is the natural level of output (also in logarithmic terms), $\pi \in \mathbb{R}$ is the rate of inflation, and $\pi^e \in \mathbb{R}$ is the expected rate of inflation. A dynamic version of this equation is

$$\dot{y}(t) = \gamma \{y_n + \alpha[\pi(t) - \pi^e(t)] - y(t)\} \quad \text{for all } t \geq 0, \quad (1)$$

where the three endogenous variables (output, inflation, and expected inflation) are functions of time t and where $\gamma > 0$ is a deterministic and exogenous adjustment speed parameter.² The natural level of output, y_n , and the parameter $\alpha > 0$ are exogenously given deterministic constants.³ Without loss of generality we assume that $y_n = 0$.

It is assumed that the central bank can directly control the inflation rate $\pi(t)$ and that it seeks to minimize a weighted average of the quadratic deviations of output and inflation from given target values $\bar{y}$ and $\bar{\pi}$, respectively. Following Barro and Gordon [1] we assume that the target level of output exceeds the natural level, that is $\bar{y} > 0$. It is no loss of generality to set $\bar{\pi} = 0$. Finally, we denote by $\beta > 0$ the relative importance given to price stability versus output stability and by $\rho > 0$ the time-preference rate of the central bank. It follows from these assumptions that the loss function of the central bank is given by

$$\int_0^{+\infty} e^{-\rho t} \{[y(t) - \bar{y}]^2 + \beta \pi(t)^2\} dt.$$

The central bank use a (Taylor-type) feedback rule to determine the inflation rate $\pi(t)$ at each point in time. Formally, a monetary policy rule is a function $\Pi : \mathbb{R} \mapsto \mathbb{R}$ which assigns to each possible state of the economy at time t , $y(t)$, a corresponding rate of inflation $\pi(t) = \Pi(y(t))$. Private agents have rational expectations and, hence, they anticipate the central bank's policy perfectly. In other words, private-sector expectations are given by $\pi^e(t) = \Pi(y(t))$, too. The central bank takes into account that private-sector expectations are given by $\pi^e(t) = \Pi(y(t))$, but it treats the function $\Pi(\cdot)$ in this relation as exogenously fixed. Thus, the dynamic Phillips curve relation (1), as seen by the central bank, has the form⁴

$$\dot{y}(t) = \gamma [\alpha \pi(t) - \alpha \Pi(y(t)) - y(t)] \quad \text{for all } t \geq 0.$$

The central bank does not have access to a commitment technology. It must therefore use a time-consistent monetary policy rule, that is, a rule that does not create any incentives for the central bank to deviate from that rule. To define the concept of a time-consistent monetary

²The static Phillips curve can be considered as the limit of (1) as γ approaches $+\infty$.

³Section 3 below will extend the model by including stochastic shocks to the economy.

⁴Recall that we have assumed $y_n = 0$.

policy rule formally, suppose that the economy is in state y_s at time s , and that private-sector expectations are given by $\pi^e(t) = \Pi(y(t))$ for all $t \geq s$. Under these assumptions the central bank faces the deterministic optimal control problem $P[y_s, s; \Pi(\cdot)]$ defined as follows:

$$\begin{aligned} & \text{minimize} && \int_s^{+\infty} e^{-\rho(t-s)} \{[y(t) - \bar{y}]^2 + \beta \pi(t)^2\} dt \\ & \text{subject to} && \dot{y}(t) = \gamma [\alpha \pi(t) - \alpha \Pi(y(t)) - y(t)] \quad \text{for all } t \geq s \\ & && y(s) = y_s. \end{aligned} \tag{2}$$

A function $\pi : [s, +\infty) \mapsto \mathbb{R}$ is a feasible control path for $P[y_s, s; \Pi(\cdot)]$ if there exists a unique function $y : [s, +\infty) \mapsto \mathbb{R}$ such that $y(s) = y_s$ and such that equation (2) holds. In this case, we call the function $y(\cdot)$ the state trajectory corresponding to the feasible control path $\pi(\cdot)$. A feasible control path is called optimal, if

$$\limsup_{T \rightarrow +\infty} \left\{ \int_s^T e^{-\rho(t-s)} \{[y(t) - \bar{y}]^2 + \beta \pi(t)^2\} dt - \int_s^T e^{-\rho(t-s)} \{[\tilde{y}(t) - \bar{y}]^2 + \beta \tilde{\pi}(t)^2\} dt \right\} \leq 0$$

holds for all pairs $(\tilde{\pi}(\cdot), \tilde{y}(\cdot))$, where $\tilde{\pi}(\cdot)$ is a feasible control path for problem $P[y_s, s; \Pi(\cdot)]$ and $\tilde{y}(\cdot)$ is the corresponding state trajectory. This optimality criterion requires that an optimal path catches up (at least asymptotically) with any other feasible path; see Gale [3].

Definition 1 A continuously differentiable function $\Pi : \mathbb{R} \mapsto \mathbb{R}$ is a *time-consistent monetary policy rule* or an *equilibrium* if, for all $s \in [0, +\infty)$ and all $y_s \in \mathbb{R}$, the problem $P[y_s, s; \Pi(\cdot)]$ has an optimal control path $\pi(\cdot)$ with corresponding state trajectory $y(\cdot)$ such that $\pi(t) = \Pi(y(t))$ holds for all $t \geq s$.

The assumption of continuous differentiability does not seem to be very restrictive. We believe that there do not exist any non-smooth time-consistent monetary policy rules. It should be noted that, from a formal point of view, our model is a differential game in which the two players, i.e. the central bank and the private sector, use stationary Markovian strategies. Any time-consistent monetary policy rule corresponds to a Markov-perfect Nash equilibrium of that game.⁵ To be able to state the main results of this section we define

$$\beta_1 = \frac{4\alpha^2\gamma^2}{(\rho + 2\gamma)^2} \quad \text{and} \quad \beta_2 = \frac{\alpha^2\gamma}{\rho + \gamma}.$$

Note that $0 < \beta_1 < \beta_2$ holds.

Theorem 1 (a) If $\beta < \beta_1$, then there does not exist a time-consistent monetary policy rule.
(b) If $\beta_1 < \beta < \beta_2$ then there exist infinitely many time-consistent monetary policy rules $\Pi(\cdot)$, and they all satisfies $\Pi'(y) < -1/\alpha$ for all $y \in \mathbb{R}$.
(c) If $\beta > \beta_2$ then there exist infinitely many time-consistent monetary policy rules $\Pi(\cdot)$. Exactly one of them satisfies $-1/\alpha < \Pi'(y) < 0$ for all $y \in \mathbb{R}$, whereas all others satisfy $\Pi'(y) < -1/\alpha$ for all y in an unbounded interval of the real line.

⁵See Dockner et al. [2] for an explanation of the terminology and the basic concepts used in differential games.

Throughout this section we focus on the case where $\beta \neq \beta_1$ and $\beta \neq \beta_2$. We shall not discuss the borderline cases in which $\beta = \beta_1$ or $\beta = \beta_2$. To understand the intuition of the results presented in theorem 1 first note that the right-hand side of equation (2) is increasing with respect to $y(t)$ if $\Pi'(y) < -1/\alpha$. If this is the case, then the economy described by (2) is unstable for any constant inflation rate $\pi(t) = \pi^*$. Only active monetary policy, that is, a non-constant inflation rate can help stabilizing the economy. In other words, if private agents expect a steeply downward sloping monetary policy rule, then the economic system itself tends to become unstable. On the other hand, if the economy is unstable, then the central bank must react strongly to output deviations and it must therefore choose a steeply downward sloping monetary policy rule. In the situation considered in theorem 1(a), the central bank attaches only very little weight to price stability but relatively much weight to output stability. This implies that it chooses high inflation even if output is just slightly below target. In other words, it chooses a very steep and downward sloping monetary policy rule. As we have seen above, such a policy destabilizes the economy and this, in turn, forces the central bank to choose an even steeper policy rule. If $\beta < \beta_1$, then theorem 1(a) shows that this process does not terminate and, as a consequence, there does not exist an equilibrium. Kydland and Prescott have already mentioned the destabilizing effect of economic policy under rational expectations. In a somewhat different context they write that “changes in policy induce changes in structure, which in turn necessitate reestimation and future changes in policy, and so on. We found for some not implausible structures that this iterative procedure does not converge, and, instead, stabilization efforts have the perverse effect of contributing to economic instability” [5, page 474]. If $\beta_1 < \beta < \beta_2$, the central bank’s incentive to choose a steeply downward sloping monetary policy rule is somewhat weaker because it cares more about price stability than in the previous case. Theorem 1(b) shows that there do exist time-consistent monetary policy rules (in fact, infinitely many of them), but that all of them have the property that $\Pi'(y) < -1/\alpha$. In other words, the weight given to output deviations is still so high that it destabilizes the economy, but in contrast to the case $\beta < \beta_1$, it is now possible and indeed optimal for the central bank to stabilize the economy by a time-consistent monetary policy rule. Theorem 1(c) considers the case $\beta > \beta_2$, in which the central bank’s loss function gives sufficiently high weight to price stability. In this situation the central bank reacts only moderately to output deviations. Because the private sector anticipates this, the economy itself remains relatively stable. As in the previous case there exist infinitely many time-consistent monetary policy rules. Exactly one of them has a slope that does not exceed $1/\alpha$ (in absolute value) and, therefore, makes the right-hand side of (2) globally decreasing. All other equilibria existing in this case have the property that the state equation (2) is increasing on an unbounded subinterval of the state space.

Theorem 1 provides a rationale for appointing conservative central bankers that is quite different from the one suggested by Rogoff [9]. Whereas Rogoff proves that delegating monetary policy to conservative central bankers leads to a socially preferred outcome, theorem 1 shows that an economy in which monetary policy is conducted by a rule can become unstable if the central bankers are not sufficiently conservative. If the weight given to price stability falls below β_1 , the instability becomes so profound that it rules out the existence of an equilibrium.

Theorem 1 already shows that every time-consistent monetary policy rule must be downward-sloping. This is intuitively plausible, as such a rule prescribes high inflation for low output levels and vice versa. However, even in the case $\beta > \beta_2$, many time-consistent policy rules have unexpected properties. To illustrate this, let us denote by $\mathcal{P}$ the set of all time-consistent monetary policy rules. Our next theorem gives some additional information about the set $\mathcal{P}$, and it describes the possible inflation rates that can occur in equilibrium. We define

$$\begin{aligned} s_1 &= -\frac{1}{\alpha} \left[1 + \frac{\rho}{2\gamma} + \sqrt{\left(1 + \frac{\rho}{2\gamma}\right)^2 - \frac{\alpha^2}{\beta}} \right], \\ s_2 &= -\frac{1}{\alpha} \left[1 + \frac{\rho}{2\gamma} - \sqrt{\left(1 + \frac{\rho}{2\gamma}\right)^2 - \frac{\alpha^2}{\beta}} \right], \\ \Pi_F &= -\frac{\alpha\gamma\bar{y}}{\alpha^2\gamma - \beta(\rho + \gamma)}. \end{aligned}$$

Note that s_1 and s_2 are negative numbers if $\beta > \beta_1$ and that Π_F is well-defined if $\beta \neq \beta_2$.

Theorem 2 (a) *If $\beta > \beta_1$ and $\beta \neq \beta_2$, then $\mathcal{P}$ contains the linear policy rules $\Pi_1(\cdot)$ and $\Pi_2(\cdot)$ given by $\Pi_1(y) = (1 + \alpha s_1)\Pi_F + s_1 y$ and $\Pi_2(y) = (1 + \alpha s_2)\Pi_F + s_2 y$ and, in addition, infinitely many non-linear policy rules.*

(b) *If $\beta_1 < \beta < \beta_2$, then it holds that*

$$\{\Pi(y) \mid \Pi(\cdot) \in \mathcal{P}\} = [\min\{\Pi_1(y), \Pi_2(y)\}, \max\{\Pi_1(y), \Pi_2(y)\}] \quad \text{for all } y \in \mathbb{R}.$$

(c) *If $\beta > \beta_2$, then it holds that*

$$\{\Pi(y) \mid \Pi(\cdot) \in \mathcal{P}\} = (-\infty, \min\{\Pi_1(y), \Pi_2(y)\}] \cup [\max\{\Pi_1(y), \Pi_2(y)\}, +\infty) \quad \text{for all } y \in \mathbb{R}.$$

Furthermore, in this case we have either $\lim_{y \rightarrow +\infty} [\Pi(y) - \Pi_2(y)] = \lim_{y \rightarrow -\infty} [\Pi(y) - \Pi_1(y)] = 0$ or $\lim_{y \rightarrow +\infty} [\Pi(y) - \Pi_1(y)] = \lim_{y \rightarrow -\infty} [\Pi(y) - \Pi_2(y)] = 0$.

Since the model under consideration is linear-quadratic, it is not surprising that there exist equilibria consisting of linear rules. What may strike the reader as a somewhat unexpected result is that there exist *two* linear equilibria and, in addition, infinitely many non-linear ones (if $\beta > \beta_1$ and $\beta \neq \beta_2$). However, note that the present model is formally a linear-quadratic differential game with Markovian strategy spaces. The possible non-uniqueness of linear equilibria in this kind of game has already been discussed by several authors (see, e.g., Lockwood [6]). The possible existence of non-linear equilibria in linear-quadratic differential games is also known (see, for example, Rowat [10]). Part (b) of theorem 2 shows that, if $\beta_1 < \beta < \beta_2$, every non-linear time-consistent monetary policy rule is bounded above by the maximum of the two linear equilibria, and it is bounded below by the minimum of the two linear equilibria. In the case $\beta > \beta_2$, on the other hand, it follows from part (c) of theorem 2 that every time-consistent monetary policy rule approaches either the maximum or the minimum of the two linear equilibria as $|y|$ becomes sufficiently large. In other words, in this case the time-consistent monetary policy rules are asymptotically linear.

By definition, we have $\pi(t) = \Pi(y(t))$ whenever $\Pi(\cdot)$ is an equilibrium. Because of equation (2) this implies that $y(t) = y(0)e^{-\gamma t}$ and, hence, $\lim_{t \rightarrow +\infty} y(t) = 0$. In the long-run, the economy converges to its natural output level. The long-run inflation rate under the monetary policy rule $\Pi(\cdot) \in \mathcal{P}$ is therefore given by $\Pi(0)$. Let us define $\mathcal{P}_0 = \{\Pi(0) \mid \Pi(\cdot) \in \mathcal{P}\}$. We can use the results in theorem 2 to characterize the set $\mathcal{P}_0$, i.e., the set of long-run inflation rates that can occur in equilibrium. From theorem 2(a) we know that $\Pi_i(0) = (1 + \alpha s_i)\Pi_F$, $i = 1, 2$. Using the definitions of s_1 , s_2 , and Π_F it is easy to see that

$$\begin{aligned} \Pi_1(0) &= \Pi_2(0) = \frac{(\rho + 2\gamma)^2 \bar{y}}{2\alpha\gamma\rho} & \text{if } \beta = \beta_1, \\ \lim_{\beta \nearrow \beta_2} \Pi_1(0) &= +\infty, \\ \lim_{\beta \searrow \beta_2} \Pi_1(0) &= -\infty, \\ \Pi_2(0) &= \frac{(\rho + \gamma)\bar{y}}{\alpha\rho} & \text{if } \beta = \beta_2, \\ \lim_{\beta \rightarrow +\infty} \Pi_1(0) &= \lim_{\beta \rightarrow +\infty} \Pi_2(0) = 0. \end{aligned}$$

Moreover, the function $\beta \mapsto \Pi_1(0)$ is strictly increasing on the two intervals $[\beta_1, \beta_2)$ and $(\beta_2, +\infty)$, whereas the function $\beta \mapsto \Pi_2(0)$ is strictly decreasing on $[\beta_1, +\infty)$. The graph of the correspondence $\beta \mapsto \mathcal{P}_0$, describing the long-run inflation rates that can be supported by time-consistent monetary policy rules, looks therefore as the shaded area in figure 1. We see that $\mathcal{P}_0$ is the compact interval $[\Pi_1(0), \Pi_2(0)] \subseteq \mathbb{R}_+$ for all $\beta \in (\beta_1, \beta_2)$. If $\beta > \beta_2$, however, the set $\mathcal{P}_0$ is the union of the two unbounded intervals $(-\infty, \Pi_1(0)]$ and $[\Pi_2(0), +\infty)$, where $\Pi_1(0) < 0$ and $\Pi_2(0) > 0$. Thus, not only can arbitrary high inflation rates be supported by time-consistent monetary policy rules but also arbitrary high rates of *deflation*. The former property has already been demonstrated in the static model used by Sorger [11]. Equilibria displaying long-run deflation, on the other hand, did not occur in that model.

3 The stochastic model: linear policy rules

In this section we consider a stochastic version of the model. More specifically, we replace the deterministic Phillips curve relation (1) by a stochastic differential equation of the form

$$dy(t) = \gamma \{y_n + \alpha[\pi(t) - \pi^e(t)] - y(t)\} dt + \sigma y(t) dw(t), \quad \text{for all } t \geq 0$$

where $w(\cdot)$ is a standardized Wiener process and $\sigma \geq 0$ a deterministic parameter. As before, we set the natural level of output equal to 0, that is, $y_n = 0$. Note that the term describing the supply shocks to the economy, $\sigma y(t) dw(t)$, is linearly increasing with respect to logarithmic output. This implies that the size of shocks is proportional to logarithmic output, an assumption that we have made for analytical tractability. The parameter σ measures the size of the shocks relative to logarithmic output. A high value of σ corresponds to large shocks whereas the special case $\sigma = 0$ describes the deterministic economy from the previous section.

The definition of a time-consistent monetary policy rule can be easily adapted to the stochastic setting. Because the private sector anticipates the policy rule $\Pi(\cdot)$, the central bank faces the following stochastic optimal control problem if the economy at time s is in state y_s :

$$\begin{aligned} \text{minimize} \quad & \mathbb{E} \int_s^{+\infty} e^{-\rho(t-s)} \{[y(t) - \bar{y}]^2 + \beta \pi(t)^2\} dt \\ \text{subject to} \quad & dy(t) = \gamma [\alpha \pi(t) - \alpha \Pi(y(t)) - y(t)] dt + \sigma y(t) dw(t) \quad \text{for all } t \geq s \\ & y(s) = y_s. \end{aligned}$$

Because the economy is now subject to random shocks, the central bank minimizes the *expected* loss. If we denote this problem again by $P[y_s, s; \Pi(\cdot)]$, definition 1 for a time-consistent monetary policy rule remains unchanged. From the formal point of view, our model is now a differential game with white noise. Solution concepts and techniques for this type of game are explained in Dockner et al. [2, chapter 8].

The analysis of the stochastic model is considerably more difficult than that of the deterministic version. We shall therefore not attempt to characterize the set of all time-consistent monetary policy rules but restrict ourselves to the study of linear equilibria, that is, time-consistent monetary policy rules of the form $\Pi(y) = A + sy$. Although this restriction is made primarily for analytical tractability, there is also another justification for it. First of all, linear policy rules are easy to understand and therefore more likely to be used by policymakers than non-linear ones. For example the monetary policy rule suggested by Taylor [12] is a linear one. Moreover, the results from section 2 suggest that there is a link between the existence of time-consistent monetary policy rules and the existence of *linear* equilibria. As a matter of fact, in the model discussed in the previous section, an equilibrium exists if and only if there exists a linear one. This is a consequence of the linear-quadratic structure of the game and it would indeed be very surprising if a linear-quadratic game has no equilibrium consisting of linear strategies while it admits an equilibrium consisting of non-linear ones.

To present our results we define

$$\beta_1 = \frac{4\alpha^2\gamma^2}{(\rho + 2\gamma - \sigma^2)^2} \quad \text{and} \quad \beta_2 = \begin{cases} \frac{\alpha^2\gamma^2}{(\rho + \gamma)(\gamma - \sigma^2)} & \text{if } \sigma^2 < \gamma, \\ +\infty & \text{if } \sigma^2 \geq \gamma. \end{cases}$$

Note that $0 < \beta_1 < \beta_2$ holds unless $\sigma^2 = \rho + 2\gamma$. As in the previous section, we shall not consider the borderline cases $\beta = \beta_1$ or $\beta = \beta_2$. Let us define

$$\begin{aligned} s_1 &= -\frac{1}{\alpha} \left[1 + \frac{\rho - \sigma^2}{2\gamma} + \sqrt{\left(1 + \frac{\rho - \sigma^2}{2\gamma}\right)^2 - \frac{\alpha^2}{\beta}} \right], \\ s_2 &= -\frac{1}{\alpha} \left[1 + \frac{\rho - \sigma^2}{2\gamma} - \sqrt{\left(1 + \frac{\rho - \sigma^2}{2\gamma}\right)^2 - \frac{\alpha^2}{\beta}} \right]. \end{aligned}$$

Note that s_1 and s_2 are real numbers if $\sigma^2 \neq \rho + 2\gamma$ and $\beta > \beta_1$. The following theorem deals with the existence of linear time-consistent monetary policy rules.

Theorem 3 (a) If $\sigma^2 \geq \rho + 2\gamma$ or $\beta < \beta_1$, then there do not exist real numbers A and s such that the function $\Pi(y) = A + sy$ is a time-consistent monetary policy rule.
(b) If $\sigma^2 < \rho + 2\gamma$, $\beta > \beta_1$, and $\beta \neq \beta_2$, then the function $\Pi(y) = A + sy$ is a time-consistent monetary policy rule if and only if $s \in \{s_1, s_2\}$ and A satisfies

$$\left(1 + \frac{\rho}{\gamma} + \alpha s\right) A - \frac{\alpha \bar{y}}{\beta} = 0. \quad (3)$$

The interpretation of this theorem is as follows. If the economy is exposed to strong supply shocks, i.e., if $\sigma^2 \geq \rho + 2\gamma$, then there does not exist a linear equilibrium. Note that the bound $\rho + 2\gamma$ depends positively on the rate of time-preference of the central bank and the adjustment speed in the dynamic Phillips curve relation. In other words, for a given variance of shocks, existence of an equilibrium consisting of linear policy rules requires that the economy adjusts quickly enough to the output level predicted by the static Phillips curve and that the central bank is not too myopic. The other necessary condition for the existence of linear time-consistent monetary policy rules stated in theorem 3 is $\beta > \beta_1$, which means that the central bank attaches sufficiently high weight to price stability. We have interpreted this condition already in the deterministic model from the previous section. Note that β_1 increases monotonically to $+\infty$ as σ^2 approaches $\rho + 2\gamma$. This implies that the set of weights β that are consistent with the existence of a linear equilibrium shrinks as the average size of the stochastic shocks relative to logarithmic output becomes large. Figure 2 illustrates the (β, σ^2) -parameter plane. Linear equilibria exist in the union of areas A , B , C , and D but nowhere else in the plane.⁶

Since we know from the analysis of the deterministic model that the stability properties of the economy play an important role for the existence of time-consistent monetary policy rules, the results stated in theorem 3 are not really surprising. We shall now study the structure of equilibria in more detail. To this end let us define

$$\beta_3 = \begin{cases} \frac{\alpha^2 \gamma}{\rho + \gamma - \sigma^2} & \text{if } \sigma^2 < \rho + \gamma, \\ +\infty & \text{if } \sigma^2 \geq \rho + \gamma. \end{cases}$$

Note that

$$\begin{aligned} \beta_1 &< \beta_2 = \beta_3 && \text{if } \sigma = 0, \\ \beta_1 &= \beta_3 = \alpha^2 && \text{if } \sigma^2 = \rho, \\ \beta_1 &< \beta_3 < \beta_2 && \text{if } 0 < \sigma^2 < \rho + \gamma \text{ and } \sigma^2 \neq \rho. \end{aligned}$$

For an illustration of the relative positions of β_1 , β_2 , and β_3 we refer to figure 2.

Theorem 4 Assume that $\sigma^2 < \rho + 2\gamma$ and $\beta > \beta_1$.

(a) It holds that s_1 and s_2 are negative numbers, that s_1 is increasing with respect to σ , and that s_2 is decreasing with respect to σ .

⁶There are no linear equilibria on the boundary between B and C . This boundary is the curve $\beta = \beta_2$.

- (b) If $\beta > \beta_3$, then it holds that $s_1 < -1/\alpha < s_2$.
(c) If $\sigma^2 \leq \rho$ and $\beta_1 < \beta < \beta_3$, then it holds that $s_1 < -1/\alpha$ and $s_2 < -1/\alpha$.
(d) If $\sigma^2 \geq \rho$ and $\beta_1 < \beta < \beta_3$, then it holds that $s_1 > -1/\alpha$ and $s_2 > -1/\alpha$.

Part (a) of the theorem shows that all linear time-consistent monetary policy rules are negatively sloped. As already explained in section 2, this is intuitively plausible. To understand parts (b)-(d) of the theorem, recall from section 2 that a critical value for the slope of the policy rules is $-1/\alpha$. If the slope is higher than $-1/\alpha$, that is, if the policy rule is relatively flat, then the economy is stable for any constant inflation rate. On the other hand, if the policy rule is so steep that its slope is smaller than $-1/\alpha$, then the economy is unstable for any constant inflation rate. The results of theorem 4 are best explained by means of figure 2.⁷ The subset of the (β, σ^2) -plane for which linear time-consistent monetary policy rules exist is partitioned into four areas A , B , C , and D . In areas B and C we have $\beta > \beta_3$ and it follows from theorem 4(b) that $s_1 < -1/\alpha < s_2$. In area A , on the other hand, we have $\sigma^2 \leq \rho$ and $\beta_1 < \beta < \beta_3$. Thus, in this area both s_1 and s_2 are smaller than $-1/\alpha$. Taking these two observations together we see that, as long as the parameters are located in A , B , or C , the basic conclusions from the deterministic model continue to hold. If the weight given to price stability, β , is relatively low, both linear policy rules are steeply sloped and the economy would be unstable for any constant inflation rate. If β is larger than β_3 , on the other hand, there exists one time-consistent monetary policy rule with slope larger than $-1/\alpha$ and one with slope smaller than $-1/\alpha$. In contrast to the deterministic model, however, the stochastic model allows for a third possibility. In area D , where $\beta_1 < \beta < \beta_3$ and $\sigma^2 \geq \rho$, there exist two linear time-consistent monetary policy rules with slope greater than $-1/\alpha$. The intuition for this result is that, in a situation in which the exogenous shocks are very large, any time-consistent monetary policy must be such that it stabilizes the economy even for a constant inflation rate.

Finally, we want to study the set of long-run inflation rates that can be supported by time-consistent monetary policy rules. We are particularly interested in finding out under which parameter constellations the stochastic model has equilibria generating long-run deflation similar to those that we encountered in the deterministic model. Since in equilibrium, $\pi(t) = \Pi(y(t))$ must hold, it follows that output evolves according to $dy(t) = -\gamma y(t)dt + \sigma y(t)dw(t)$. This equation implies that $\mathbb{E}y(t) = y_0 e^{-\gamma t}$ and, hence, $\lim_{t \rightarrow +\infty} \mathbb{E}y(t) = 0$. Because of the linearity of the policy rules $\Pi(\cdot)$ mentioned in theorem 3, expected long-run inflation is given by $\mathbb{E}[\lim_{t \rightarrow +\infty} \Pi(y(t))] = \Pi(\lim_{t \rightarrow +\infty} \mathbb{E}y(t)) = \Pi(0)$. Therefore, as in the deterministic case, it suffices to determine the conditions under which $\Pi(0)$ is negative. From theorem 3 we know that

$$\Pi(0) = A = \frac{\alpha\gamma\bar{y}}{\beta(\rho + \gamma + \alpha\gamma s)}.$$

This shows that the sign of $\Pi(0)$ is equal to the sign of $1 + \alpha s + \rho/\gamma$. It is straightforward to show that, for $s = s_2$, this expression is always positive such that the policy rule $\Pi_2(y) = A + s_2 y$ does not create long-run deflation. For $s = s_1$, on the other hand, the above expression is negative if and only if $\beta > \beta_2$. Thus, there exists a linear time-consistent monetary policy rule

⁷For simplicity of exposition, the following discussion neglects the fact that linear policy rule equilibria do not exist for $\beta = \beta_2$, i.e., on the boundary between areas B and C in figure 2.

leading to long-run deflation whenever (β, σ^2) is located in area B of figure 2. This means that there is no long-run deflation under a linear time-consistent monetary policy rule if exogenous stochastic shocks are strong, i.e., $\sigma^2 \geq \gamma$.

4 Conclusions

In this paper we have attempted to characterize the set of time-consistent monetary policy rules in a linear-quadratic, dynamic model. We have found that such policy rules do not exist if either the stochastic shocks to the economy are very strong or the central bank cares too much about output stability. In all other cases, there exist multiple time-consistent monetary policy rules. The paper therefore shows that the basic results derived in the static framework of Sorger [11] carry over to a fully dynamic model.

Our model has a linear-quadratic structure. Although this is certainly restrictive, it has become the workhorse model in this literature; see Persson and Tabellini [8]. In the deterministic version of the model we do not see any difficulty in generalizing our results to more general functional forms. The situation is different in the stochastic version of the model, because in this case we had to restrict our analysis to linear policy rules. If we would replace the linear-quadratic structure by a more general one, the model would no longer possess any equilibria consisting of linear rules and one would have to devise another method of analysis. This is left for future research.

Appendix A

This appendix contains the proofs of the results stated in section 2. We start by deriving an ordinary differential equation which every time-consistent monetary policy rule has to satisfy. We then analyse the solutions of this equation and, finally, we prove that all solutions do indeed qualify as time-consistent monetary policy rules.

Suppose that $\Pi : \mathbb{R} \mapsto \mathbb{R}$ is a time-consistent monetary policy rule. A continuously differentiable function $V : \mathbb{R} \mapsto \mathbb{R}$ satisfies the Hamilton-Jacobi-Bellman equation (HJB equation) for problem $P[y_0, 0; \Pi(\cdot)]$, if

$$\rho V(y) = \max \left\{ -(y - \bar{y})^2 - \beta \pi^2 + \gamma V'(y) [\alpha \pi - \alpha \Pi(y) - y] \mid \pi \in \mathbb{R} \right\} \quad (4)$$

holds for all $y \in \mathbb{R}$. The unique maximizer on the right-hand side of (4) is $\pi = \alpha \gamma V'(y) / (2\beta)$. Since in equilibrium the central bank must choose an optimal rate of inflation for every possible state of the economy, it follows that the optimal value function of problem $P[y_0, 0; \Pi(\cdot)]$ must solve the HJB equation. Moreover, because the optimal choice of the central bank in state y must coincide with the expected rate of inflation $\Pi(y)$, we obtain

$$\Pi(y) = \frac{\alpha \gamma}{2\beta} V'(y) \quad \text{for all } y \in \mathbb{R}, \quad (5)$$

where $V(\cdot)$ denotes the optimal value function of $P[y_0, 0; \Pi(\cdot)]$. Substituting $\pi = \Pi(y)$ and (5) into equation (4) we obtain

$$V(y) = -\frac{1}{\rho} \left[(y - \bar{y})^2 + \beta \Pi(y)^2 + \frac{2\beta}{\alpha} \Pi(y)y \right] \quad \text{for all } y \in \mathbb{R}. \quad (6)$$

Finally, by differentiating equation (6) with respect to y and using (5) it follows that $\Pi(\cdot)$ must satisfy the differential equation

$$\left(1 + \frac{\rho}{\gamma}\right) \Pi(y) + \Pi'(y)[\alpha \Pi(y) + y] + \frac{\alpha}{\beta}(y - \bar{y}) = 0 \quad \text{for all } y \in \mathbb{R}. \quad (7)$$

Our next step is to study the solutions of this equation. The following lemma characterizes all global solutions of equation (7) under the assumption that $\beta \neq \beta_1$ and $\beta \neq \beta_2$.

Lemma 1 (a) If $\beta < \beta_1$, then equation (7) has no global solution, i.e., there does not exist a function $\Pi : \mathbb{R} \mapsto \mathbb{R}$ which satisfies equation (7) for all $y \in \mathbb{R}$.

(b) If $\beta > \beta_1$ and $\beta \neq \beta_2$, then there exists an infinite family of functions $\Pi : \mathbb{R} \mapsto \mathbb{R}$ that solve equation (7) for all $y \in \mathbb{R}$. This family contains two linear functions given by $\Pi_1(y) = (1 + \alpha s_1)\Pi_F + s_1 y$ and $\Pi_2(y) = (1 + \alpha s_2)\Pi_F + s_2 y$ and infinitely many non-linear functions.

(c) Assume $\beta > \beta_1$ and $\beta \neq \beta_2$ and consider any function $\Pi(\cdot)$ belonging to the family of functions mentioned in part (b). Then it follows that $\Pi(\cdot)$ is continuously differentiable and strictly decreasing and that the following property holds: there exist real numbers y_+ , y_- , a_+ , a_- , A_+ , A_- , b_+ , b_- , B_+ , and B_- such that $b_+ < 0$, $b_- < 0$, $B_+ < 0$, $B_- < 0$, and such that

$$a_+ + b_+ y \leq \Pi(y) \leq A_+ + B_+ y \quad \text{for all } y \geq y_+$$

and

$$a_- + b_- y \leq \Pi(y) \leq A_- + B_- y \quad \text{for all } y \leq y_-.$$

PROOF: Equation (7) can be written in parametric form as the following system of two linear differential equations:

$$\begin{aligned} y'(x) &= y(x) + \alpha \Pi(x) \\ \Pi'(x) &= -\frac{\alpha}{\beta}[y(x) - \bar{y}] - \left(1 + \frac{\rho}{\gamma}\right) \Pi(x) \end{aligned} \quad (8)$$

We study the solutions of this system by means of a phase diagram analysis. It will be convenient to distinguish between the three cases $0 < \beta < \beta_1$, $\beta_1 < \beta < \beta_2$, and $\beta > \beta_2$. The corresponding phase diagrams are depicted in figures 3-5.

The isocline $y'(x) = 0$ is the line given by $\Pi = -y/\alpha$ and the isocline $\Pi'(x) = 0$ is the line determined by $\Pi = -\alpha\gamma(y - \bar{y})/[\beta(\rho + \gamma)]$. Both isoclines are downward sloping. If $\beta < \beta_2$, then the line $\Pi'(x) = 0$ is steeper than the line $y'(x) = 0$, whereas the opposite is true if $\beta > \beta_2$. The two isoclines intersect at the unique fixed point of (8) which we denote by F . The coordinates of F are $y = y_F := -\alpha\Pi_F$ and $\Pi = \Pi_F$. If $\beta < \beta_2$, then the fixed point F is located in the

south-east quadrant of the phase diagram, i.e., $y_F > 0$ and $\Pi_F < 0$. On the other hand, if $\beta > \beta_2$, then F is located in the north-west quadrant, i.e., $y_F < 0$ and $\Pi_F > 0$. The eigenvalues of (8) are $\sigma_1 = 1 + \alpha s_1$ and $\sigma_2 = 1 + \alpha s_2$.

If $\beta < \beta_1$, then both eigenvalues are complex numbers with negative real parts. It follows that the trajectories of (8) oscillate around the fixed point F , which implies that equation (7) has no globally defined solution; see figure 3.

Now assume that $\beta > \beta_1$. The eigenvectors corresponding to the eigenvalues σ_1 and σ_2 are

$$e_1 = \begin{pmatrix} 1 \\ s_1 \end{pmatrix} \quad \text{and} \quad e_2 = \begin{pmatrix} 1 \\ s_2 \end{pmatrix},$$

respectively. The eigenspaces spanned by these vectors are shown in figures 4 and 5 as lines E_1 and E_2 . The slopes of E_1 and E_2 are s_1 and s_2 , respectively.

If $\beta_1 < \beta < \beta_2$, then both eigenvalues are negative real numbers. The eigenspaces E_1 and E_2 are negatively sloped and steeper than both isoclines. We denote by A the area strictly above both E_1 and E_2 , by C the area strictly below both E_1 and E_2 , by B the area on or above E_2 and strictly below E_1 , and by D the area on or below E_2 and strictly above E_1 . All of these areas are invariant under the flow described by (8). Any trajectory of (8) in A intersects the locus $y'(x) = 0$ and, therefore, does not define a unique value $\Pi(y)$ for every $y \in \mathbb{R}$. In other words, such a trajectory does not correspond to a global solution of equation (7). An analogous argument shows that trajectories in C do not correspond to global solutions of equation (7). Now consider a curve in (y, Π) -space that arises out of pasting any trajectory in B to any trajectory in D . Such a curve forms a continuously differentiable and globally defined function $\Pi : \mathbb{R} \mapsto \mathbb{R}$ which solves (7) for all $y \in \mathbb{R}$. Obviously, there are infinitely many such curves and they are all downward sloping. Finally, the lines E_1 and E_2 form two further globally defined and continuously differentiable (in fact, linear) solutions of (7). It is easy to see that the equations of E_1 and E_2 are as stated in part (b) of the lemma. To prove part (c) of the lemma, just note that every curve $y \mapsto \Pi(y)$ constructed as explained above is bounded below by E_1 and bounded above by E_2 for all $y \geq y_F$. Analogously, these curves are bounded below by E_2 and bounded above by E_1 for all $y \leq y_F$.

If $\beta > \beta_2$, then σ_1 is negative and σ_2 is positive. The eigenspaces E_1 and E_2 are again negatively sloped but this time E_1 is steeper than both isoclines and E_2 is flatter than both isoclines. We denote by A the area strictly above both E_1 and E_2 , by C the area strictly below both E_1 and E_2 , by B the area strictly above E_2 and strictly below E_1 , and by D the area strictly below E_2 and strictly above E_1 . All of these areas are invariant under the flow described by (8). Any trajectory of (8) in B intersects the locus $y'(x) = 0$ and, therefore, does not define a unique value $\Pi(y)$ for every $y \in \mathbb{R}$. Such a trajectory is therefore not a global solution of equation (7). An analogous argument rules out trajectories in D . All other trajectories, i.e., the two eigenspaces E_1 and E_2 as well as all trajectories in A and C correspond to global solutions of equation (7). Note that all trajectories in A approach asymptotically the line E_2 as y approaches $+\infty$ and they approach asymptotically the line E_1 as y approaches $-\infty$. It is easy to see that this implies that all trajectories are asymptotically bounded above and below

by decreasing linear functions. An analogous argument shows that this property holds also for all trajectories in C . This completes the proof. $\triangleleft$

Part (a) of the lemma shows that the necessary equilibrium condition (7) has no solution if the central bank is too much concerned about output stability as compared with price stability, i.e., if $\beta < \beta_1$. It follows that in this case there does not exist a time-consistent monetary policy rule. If $\beta > \beta_1$ and $\beta \neq \beta_2$, on the other hand, lemma 1 shows that there are infinitely many solutions of the necessary equilibrium condition (7). Lemma 1(c) tells us that every solution of (7) is smooth, strictly decreasing, and bounded above and below by linear functions for sufficiently large and sufficiently small values of y . We shall now use these properties to prove that all solutions of (7) qualify as time-consistent monetary policy rules. To this end, we have to verify sufficient equilibrium conditions. This is done in the rest of this appendix by means of a series of lemmas. We start by showing that for every global solution of (7) the function $V : \mathbb{R} \mapsto \mathbb{R}$ defined by (6) satisfies the HJB equation.

Lemma 2 *If the function $\Pi : \mathbb{R} \mapsto \mathbb{R}$ satisfies (7) and the function $V : \mathbb{R} \mapsto \mathbb{R}$ satisfies (6), then it follows that equations (4) and (5) hold for all $y \in \mathbb{R}$.*

PROOF: Because of (6) we have

$$\rho V'(y) = -2(y - \bar{y}) - \frac{2\beta}{\alpha} \Pi'(y) [\alpha \Pi(y) + y] - \frac{2\beta}{\alpha} \Pi(y).$$

Combining this with (7) we obtain $\rho V'(y) = 2\beta \rho \Pi(y) / (\alpha \gamma)$ which is obviously equivalent to (5). The fact that $V(\cdot)$ satisfies the HJB equation (4) follows immediately from (5) and (6). $\triangleleft$

Next we show that $-V(y_0)$ is the value of the central bank's loss function if the central bank uses the policy rule $\Pi(\cdot)$ and the private sector has rational expectations.

Lemma 3 *Suppose the function $\Pi : \mathbb{R} \mapsto \mathbb{R}$ satisfies (7) and the function $V : \mathbb{R} \mapsto \mathbb{R}$ satisfies (6). If the feedback rule $\pi(t) = \Pi(y(t))$ is applied in problem $P[y_0, 0; \Pi(\cdot)]$, then $y(t) = y_0 e^{-\gamma t}$, $\lim_{t \rightarrow +\infty} e^{-\rho t} V(y(t)) = 0$, and*

$$\int_0^{+\infty} e^{-\rho t} \{ [y(t) - \bar{y}]^2 + \beta \pi(t)^2 \} dt = -V(y_0). \quad (9)$$

PROOF: Substituting the feedback rule $\pi(t) = \Pi(y(t))$ into the state equation of problem $P[y_0, 0; \Pi(\cdot)]$ we obtain $\dot{y}(t) = -\gamma y(t)$. Together with $y(0) = y_0$ this obviously implies $y(t) = y_0 e^{-\gamma t}$. Therefore we have $\lim_{t \rightarrow +\infty} e^{-\rho t} V(y(t)) = \lim_{t \rightarrow +\infty} e^{-\rho t} V(0) = 0$. From lemma 2 we know that the function $V(\cdot)$ satisfies the HJB equation (4) and that the maximum on the right-hand side of that equation is attained at $\pi = \Pi(y)$. Together with the state equation (2) this implies $[y(t) - \bar{y}]^2 + \beta \pi(t)^2 = -\rho V(y(t)) + V'(y(t)) \dot{y}(t)$ for all $t \in [0, +\infty)$. Multiplying this equation by $e^{-\rho t}$ and integrating from 0 to T we obtain

$$\int_0^T e^{-\rho t} \{ [y(t) - \bar{y}]^2 + \beta \pi(t)^2 \} dt = \int_0^T \frac{d}{dt} [e^{-\rho t} V(y(t))] dt = e^{-\rho T} V(y(T)) - V(y(0)). \quad (10)$$

Together with $\lim_{t \rightarrow \infty} e^{-\rho t} V(y(t)) = 0$ and $y(0) = y_0$ this proves (9). $\triangleleft$

In the following lemma we show that the central bank incurs a loss greater than or equal to $-V(y_0)$ along every feasible control path.

Lemma 4 *Suppose the function $\Pi : \mathbb{R} \mapsto \mathbb{R}$ satisfies (7) and the function $V : \mathbb{R} \mapsto \mathbb{R}$ satisfies (6). If $\tilde{\pi} : [0, +\infty) \mapsto \mathbb{R}$ is a feasible control path for problem $P[y_0, 0; \Pi(\cdot)]$ and $\tilde{y} : [0, +\infty) \mapsto \mathbb{R}$ is the corresponding state trajectory, then it follows that*

$$\liminf_{T \rightarrow +\infty} \int_0^T e^{-\rho t} \{[\tilde{y}(t) - \bar{y}]^2 + \beta \tilde{\pi}(t)^2\} dt \geq -V(y_0).$$

PROOF: Using the same arguments that led to equation (10), one can show

$$\int_0^T e^{-\rho t} \{[\tilde{y}(t) - \bar{y}]^2 + \beta \tilde{\pi}(t)^2\} dt \geq e^{-\rho T} V(\tilde{y}(T)) - V(y_0).$$

The reason why this is an inequality rather than an equality as in (10) is that the control path $\tilde{\pi}(\cdot)$ does not necessarily maximize the right-hand side of the HJB equation (4). If $\liminf_{T \rightarrow +\infty} e^{-\rho T} V(\tilde{y}(T)) \geq 0$ holds, then the proof is complete.

Now suppose that $\liminf_{T \rightarrow +\infty} e^{-\rho T} V(\tilde{y}(T)) < 0$. Obviously, this can only be the case if the trajectory $\tilde{y}(\cdot)$ is unbounded. Since the state trajectory corresponding to an optimal path in a stationary and deterministic infinite-horizon optimal control problem with a one-dimensional state space is always monotonic (see, e.g., Hartl [4]), this can only be the case if $\tilde{y}(t)$ grows monotonically to $+\infty$ or if $\tilde{y}(t)$ declines monotonically to $-\infty$. Suppose $\limsup_{t \rightarrow +\infty} \tilde{y}(t) = +\infty$. Because of monotonicity of $\tilde{y}(t)$, there exists $t_+ > 0$ such that $\tilde{y}(t) \geq y_+$ for all $t \geq t_+$, where y_+ is defined in lemma 1(c). From condition (5) it follows that

$$V(y) = \frac{2\beta}{\alpha\gamma} \int_{y_+}^y \Pi(x) dx + V_+$$

for some constant V_+ and all $y \in \mathbb{R}$. Together with lemma 1(c) and $\tilde{y}(t) \geq y_+$ for $t \geq t_+$ this equation implies that the inequality

$$V(\tilde{y}(t)) \geq \frac{2\beta}{\alpha\gamma} \int_{y_+}^{\tilde{y}(t)} (a_+ + b_+ x) dx + V_+$$

holds for all $t \geq t_+$, where $b_+ < 0$. Evaluating the integral on the right-hand side of this inequality we obtain $V(\tilde{y}(t)) \geq C_0 + C_1 \tilde{y}(t) + C_2 [\tilde{y}(t) - \bar{y}]^2$, where C_0 , C_1 , and C_2 are real numbers independent of t and where C_2 is negative. The assumption $\liminf_{t \rightarrow +\infty} e^{-\rho t} V(\tilde{y}(t)) < 0$ therefore implies $\limsup_{t \rightarrow +\infty} e^{-\rho t} [\tilde{y}(t) - \bar{y}]^2 > 0$. Using the monotonicity of $\tilde{y}(t)$ again, this leads to

$$\liminf_{T \rightarrow +\infty} \int_0^T e^{-\rho t} \{[\tilde{y}(t) - \bar{y}]^2 + \beta \tilde{\pi}(t)^2\} dt = +\infty,$$

which proves the lemma in the case when $\limsup_{t \rightarrow +\infty} \tilde{y}(t) = +\infty$. The case $\liminf_{t \rightarrow +\infty} \tilde{y}(t) = -\infty$ can be dealt with in exactly the same way. This completes the proof of the lemma. $\triangleleft$

Lemmas 3 and 4 together imply that the feedback law $\pi(t) = \Pi(y(t))$ generates an optimal control path for problem $P[y_0, 0; \Pi(\cdot)]$. This completes the proof of the results stated in section 2.

Appendix B

This appendix presents the proofs of the results stated in section 3. We start again by deriving an ordinary differential equation which every time-consistent monetary policy rule has to satisfy. We will then analyse the linear solutions of this equation and study which of them qualify as time-consistent monetary policy rules. We shall not deal with the non-linear solutions of the first-order condition.

Suppose that $\Pi : \mathbb{R} \mapsto \mathbb{R}$ is a time-consistent monetary policy rule. A twice continuously differentiable function $V : \mathbb{R} \mapsto \mathbb{R}$ satisfies the HJB equation for the stochastic optimal control problem $P[y_0, 0; \Pi(\cdot)]$, if

$$\rho V(y) = \max \left\{ -(y - \bar{y})^2 - \beta \pi^2 + \gamma V'(y)[\alpha \pi - \alpha \Pi(y) - y] + \sigma^2 y^2 V''(y)/2 \mid \pi \in \mathbb{R} \right\} \quad (11)$$

holds for all $y \in \mathbb{R}$. Proceeding in exactly the same way as in Appendix A, one can show that the policy rule $\Pi(\cdot)$ and the optimal value function $V(\cdot)$ must satisfy the conditions

$$V'(y) = \frac{2\beta}{\alpha\gamma} \Pi(y), \quad (12)$$

$$\left(1 + \frac{\rho}{\gamma}\right) \Pi(y) + \Pi'(y) \left[\alpha \Pi(y) + y \left(1 - \frac{\sigma^2}{\gamma}\right) \right] - \frac{\sigma^2}{2\gamma} \Pi''(y) y^2 + \frac{\alpha}{\beta} (y - \bar{y}) = 0, \quad (13)$$

$$V(y) = -\frac{1}{\rho} \left[(y - \bar{y})^2 + \beta \Pi(y)^2 + \frac{2\beta}{\alpha} \Pi(y)y - \frac{\beta \sigma^2}{\alpha\gamma} \Pi'(y)y^2 \right] \quad (14)$$

for all $y \in \mathbb{R}$. We are interested in (affine) linear solutions of equation (13). Substituting the general form $\Pi(y) = A + sy$ into (13) and collecting terms with equal powers of y , we find that $\Pi(y) = A + sy$ is a solution of equation (13) if and only if s satisfies

$$\alpha s^2 + \frac{\rho + 2\gamma - \sigma^2}{\gamma} s + \frac{\alpha}{\beta} = 0 \quad (15)$$

and A satisfies equation (3). It is straightforward to see that equation (15) has a real solution if and only if $\sigma^2 \neq \rho + 2\gamma$ and $\beta \geq \beta_1$. If $\sigma^2 \neq \rho + 2\gamma$ and $\beta > \beta_1$ it has two distinct real solutions given by s_1 and s_2 . It is also easy to see that $1 + \rho/\gamma + \alpha s \neq 0$ holds whenever $s \in \{s_1, s_2\} \subseteq \mathbb{R}$ and $\beta \neq \beta_2$. Therefore, if $\beta \neq \beta_2$ and s is a real solution of (15), then there exists a unique corresponding solution of (3). Furthermore it is obvious that s_1 and s_2 are negative if $\sigma^2 < \rho + 2\gamma$ and $\beta > \beta_1$, whereas both of these numbers are positive if $\sigma^2 > \rho + 2\gamma$ and $\beta > \beta_1$. We summarize these findings in the following lemma.

Lemma 5 (a) If $\sigma^2 = \rho + 2\gamma$ or $\beta < \beta_1$, then there do not exist real numbers A and s such that the function $\Pi(y) = A + sy$ satisfies equation (13).

(b) If $\sigma^2 < \rho + 2\gamma$, $\beta > \beta_1$, and $\beta \neq \beta_2$, then there exist two distinct pairs of real numbers, (A_1, s_1) and (A_2, s_2) , such that the function $\Pi_i(y) = A_i + s_i y$, $i = 1, 2$, satisfies equation (13). It holds that $s_1 < 0$ and $s_2 < 0$.

(c) If $\sigma^2 > \rho + 2\gamma$ and $\beta > \beta_1$, then there exist two distinct pairs of real numbers, (A_1, s_1) and (A_2, s_2) , such that $\Pi_i(y) = A_i + s_i y$ satisfies equation (13). It holds that $s_1 > 0$ and $s_2 > 0$.

It is straightforward to verify the properties of s_1 and s_2 that are stated in theorem 4. In the remainder of this appendix we will show that the decreasing linear solutions of equation (13) (see lemma 5(b)) are time-consistent monetary policy rules, whereas the increasing ones (see lemma 5(c)) are not. We proceed in a series of lemmas, the first of which can be proved in exactly the same way as lemma 2 in appendix A.

Lemma 6 *If the function $\Pi : \mathbb{R} \mapsto \mathbb{R}$ satisfies (13) and the function $V : \mathbb{R} \mapsto \mathbb{R}$ satisfies (14), then it follows that the HJB equation (11) and the first-order condition (12) hold for all $y \in \mathbb{R}$.*

The next lemma characterizes the optimal policy of the central bank in a finite-horizon version of our model given an arbitrary linear function describing private-sector expectations.⁸

Lemma 7 *Let A and s be arbitrary real numbers.*

(a) *For every $T > 0$ there exist unique functions $K(\cdot, T)$, $L(\cdot, T)$, and $M(\cdot, T)$ mapping $[0, T]$ to the real line, such that the function $V(y, t; T) = K(t, T)y^2 + L(t, T)y + M(t, T)$ satisfies the partial differential equation*

$$\begin{aligned} \rho V(y, t; T) - V_t(y, t; T) = & \max \left\{ -(y - \bar{y})^2 - \beta\pi^2 + \gamma V_y(y, t; T)[\alpha\pi - \alpha A - \alpha s y - y] \right. \\ & \left. + \sigma^2 y^2 V_{yy}(y, t; T)/2 \mid \pi \in \mathbb{R} \right\} \end{aligned} \quad (16)$$

and the boundary condition $V(y, T; T) = 0$ for all $y \in \mathbb{R}$.

(b) *The feedback law $\pi(t) = \alpha\gamma[2K(t, T)y(t) + L(t, T)]/(2\beta)$ generates an optimal control path for the stochastic optimal control problem $P[y_0, 0; \Pi(\cdot)]$, where $\Pi(y) = A + sy$ and where the time horizon is T (instead of $+\infty$).*

(c) *The limits $K = \lim_{T \rightarrow +\infty} K(t, T)$, $L = \lim_{T \rightarrow +\infty} L(t, T)$, and $M = \lim_{T \rightarrow +\infty} M(t, T)$ exist and are independent of t . The limit function $V(y) = Ky^2 + Ly + M$ satisfies equation (11) with $\Pi(y) = A + sy$.*

(d) *It holds that*

$$K = \frac{\beta}{\alpha^2 \gamma} \left[1 + \alpha s + \frac{\rho - \sigma^2}{2\gamma} - \sqrt{\left(1 + \alpha s + \frac{\rho - \sigma^2}{2\gamma} \right)^2 + \frac{\alpha^2}{\beta}} \right]$$

and

$$L \left[\frac{\alpha^2 \gamma^2}{\beta} K - (1 + \alpha s)\gamma - \rho \right] = 2\alpha\gamma AK - 2\bar{y}.$$

PROOF: Substituting $V(y, t; T) = K(t, T)y^2 + L(t, T)y + M(t, T)$ into (16), carrying out the maximization on the right-hand side of that equation, and collecting terms with equal powers of y , one can show that $V(y, t; T)$ is a solution of (16) if and only if the functions $K(\cdot, T)$,

⁸See [2, theorem 3.1] for a general statement of the optimality conditions that are used in lemma 7.

$L(\cdot, T)$, and $M(\cdot, T)$ satisfy the differential equations

$$K_t(t, T) = -\frac{\alpha^2 \gamma^2}{\beta} K(t, T)^2 + [2\gamma(1 + \alpha s) + \rho - \sigma^2] K(t, T) + 1, \quad (17)$$

$$L_t(t, T) = \left[(1 + \alpha s)\gamma + \rho - \frac{\alpha^2 \gamma^2}{\beta} K(t, T) \right] L(t, T) + 2\alpha\gamma AK(t, T) - 2\bar{y}, \quad (18)$$

$$M_t(t, T) = \rho M(t, T) - \frac{\alpha^2 \gamma^2}{4\beta} L(t, T)^2 + \alpha\gamma AL(t, T) + \bar{y}^2. \quad (19)$$

The boundary condition $V(y, T; T) = 0$ for all $y \in \mathbb{R}$ holds if and only if $K(T, T) = L(T, T) = M(T, T) = 0$.

Equation (17) is a Riccati differential equation for $K(\cdot, T)$. There is a unique solution of this equation which satisfies the boundary condition $K(T, T) = 0$. It is straightforward to show that $\lim_{T \rightarrow +\infty} K(t, T) = K$, where K is given in part (d) of the lemma. Note that

$$(1 + \alpha s)\gamma + \rho - \frac{\alpha^2 \gamma^2}{\beta} K = \frac{\rho + \sigma^2}{2} + \gamma \sqrt{\left(1 + \alpha s + \frac{\rho - \sigma^2}{2\gamma}\right)^2 + \frac{\alpha^2}{\beta}} > 0. \quad (20)$$

For any given function $K(\cdot, T)$, equation (18) is a linear differential equation for $L(\cdot, T)$ and, hence, there exists a unique solution of this equation which satisfies $L(T, T) = 0$. Moreover, if T is sufficiently large, $K(t, T)$ is close to K for all $t > 0$ such that

$$(1 + \alpha s)\gamma + \rho - \frac{\alpha^2 \gamma^2}{\beta} K(t, T) > 0$$

because of (20). This property implies that $\lim_{T \rightarrow +\infty} L(t, T) = L$, where K and L satisfy the second condition in part (d) of the lemma.

The existence of a unique solution of equation (19) satisfying the boundary condition $M(T, T) = 0$ and the existence of $\lim_{T \rightarrow +\infty} M(t, T)$ can be proved analogously. The fact that the limiting function $V(y)$ satisfies equation (11) with $\Pi(y) = A + sy$ is obvious. Finally, from the maximization of the right-hand side of (16) it follows that $\pi = \alpha\gamma V_y(y, t; T)/(2\beta)$, which proves part (b) of the lemma. Thus, the proof of the lemma is complete. $\triangleleft$

Let us first consider the case $\sigma^2 < \rho + \gamma$. To prove that the linear solutions of (13) are indeed equilibria, we apply the method described in [2, theorem 3.4].⁹ To this end, we first show that, under the assumption that $\sigma^2 < \rho + 2\gamma$ and that the central bank applies the feedback rule $\pi(t) = \Pi(y(t))$, the expected loss is given by $-V(y_0)$.

Lemma 8 *Assume $\sigma^2 < \rho + 2\gamma$. Suppose the function $\Pi : \mathbb{R} \mapsto \mathbb{R}$ defined by $\Pi(y) = A + sy$ satisfies equation (13) and the function $V : \mathbb{R} \mapsto \mathbb{R}$ satisfies (14). Furthermore, suppose the policy rule $\pi(t) = \Pi(y(t))$ is applied to problem $P[y_0, 0; \Pi(\cdot)]$. Then it follows that*

$$\mathbb{E} \int_0^{+\infty} e^{-\rho t} \{ [y(t) - \bar{y}]^2 + \beta \pi(t)^2 \} dt = -V(y_0).$$

⁹Although [2, theorem 3.4] is formulated for a deterministic optimal control problem, the result can easily be adapted to a stochastic setting.

PROOF: From lemma 6 we know that $V(\cdot)$ satisfies the HJB equation (11). Using this property it can be shown that¹⁰

$$\mathbb{E} \int_0^T e^{-\rho t} \{[y(t) - \bar{y}]^2 + \beta \pi(t)^2\} dt = -V(y_0) + \mathbb{E} [e^{-\rho T} V(y(T))] \quad \text{for all } T \geq 0.$$

It remains to be shown that $\lim_{T \rightarrow +\infty} \mathbb{E} [e^{-\rho T} V(y(T))] = 0$. Since $\pi(t) = \Pi(y(t))$, it follows from the state equation that $dy(t) = -\gamma y(t)dt + \sigma y(t)dw(t)$. Since $\Pi(\cdot)$ is linear it follows from (14) that $V(\cdot)$ is a quadratic polynomial, say, $V(y) = Ky^2 + Ly + M$. Using stochastic calculus it is straightforward to show that these properties imply

$$\mathbb{E} [e^{-\rho T} V(y(T))] = Ky_0^2 e^{(\sigma^2 - 2\gamma - \rho)T} + Ly_0 e^{-(\rho + \gamma)T} + Me^{-\rho T} \quad \text{for all } T \geq 0.$$

Because of $\sigma^2 < \rho + 2\gamma$ all three terms on the right-hand side of this equation converge to 0 as T approaches $+\infty$. This completes the proof of the lemma. $\triangleleft$

Suppose that $\sigma^2 < \rho + 2\gamma$, that $s \in \{s_1, s_2\}$, and that A satisfies (3). Substituting these values of A and s into the conditions of lemma 7(d), it is straightforward to verify that $K = \beta s / (\alpha \gamma)$ and that $L = 2\beta A / (\alpha \gamma)$. Because of lemma 7(b)-(c) it follows therefore that the optimal Markovian strategy for the finite-horizon problem of the central bank converges to the linear function $A + sy$ as the horizon T approaches $+\infty$. Together with lemma 8 and the stochastic version of [2, theorem 3.4], this proves that the linear solutions of equation (13) are time-consistent monetary policy rules if $\sigma^2 < \rho + 2\gamma$.

Now let us consider the case $\sigma^2 > \rho + 2\gamma$. In order to show that there does not exist a linear time-consistent monetary policy rule, it suffices to rule out that the two candidate solutions identified in lemma 5 are equilibria. Suppose therefore that $\Pi(y) = A + sy$, where A and s are given in lemma 5(c). Substituting these values into the conditions of lemma 7(d) we obtain

$$\frac{\alpha \gamma}{\beta} K - s = \frac{1}{\alpha} \left(2 + \frac{\rho - \sigma^2}{\gamma} \right). \quad (21)$$

We shall now show that the feedback rule $\pi(t) = \alpha \gamma [2Ky(t) + L] / (2\beta)$ is optimal, where K and L are specified in lemma 7(d). Since this feedback rule differs from the law $\pi(t) = \Pi(y(t))$, it will then follow that $\Pi(\cdot)$ is not an equilibrium. Because of lemma 7 and the stochastic version of [2, theorem 3.4], all we have to do is to show that $\lim_{T \rightarrow +\infty} \mathbb{E} [e^{-\rho T} V(y(T))] = 0$, where $V(y) = Ky^2 + Ly + M$ and where $y(\cdot)$ is the state process generated in problem $P[y_0, 0; \Pi(\cdot)]$ by the feedback law $\pi(t) = \alpha \gamma [2Ky(t) + L] / (2\beta)$. Because of (21) we have

$$\alpha [\pi(t) - \Pi(y(t))] = \alpha \left[\frac{\alpha \gamma [2Ky(t) + L]}{2\beta} - A - sy(t) \right] = \left(2 + \frac{\rho - \sigma^2}{\gamma} \right) y(t) + C \quad \text{for all } t \geq 0,$$

where C is a constant. Substituting this into the state equation of problem $P[y_0, 0; \Pi(\cdot)]$ it follows that

$$dy(t) = \gamma \left[\left(1 + \frac{\rho - \sigma^2}{\gamma} \right) y(t) + C \right] dt + \sigma y(t) dw(t) \quad \text{for all } t \geq 0.$$

¹⁰See, e.g., the proof of [2, theorem 8.4].

Using this equation it is straightforward to show that

$$\mathbb{E} \left[e^{-\rho T} V(y(T)) \right] = e^{-\rho T} \mathbb{E} \left[K y(T)^2 + L y(T) + M \right] = K_0 e^{(\rho+2\gamma-\sigma^2)T} + L_0 e^{(\gamma-\sigma^2)T} + M_0 e^{-\rho T}$$

for some constants K_0 , L_0 , and M_0 . The condition $\sigma^2 > \rho + 2\gamma$ implies therefore that $\lim_{T \rightarrow +\infty} \mathbb{E} \left[e^{-\rho T} V(y(T)) \right] = 0$ and the proof of the results stated in section 3 is complete.

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Figure 1: Equilibrium long-run inflation rates

Figure 2: The (β, σ^2) -parameter plane

Figure 3: Phase diagram for (7) if $\beta < \beta_1$

Figure 4: Phase diagram for (7) if $\beta_1 < \beta < \beta_2$

Figure 5: Phase diagram for (7) if $\beta > \beta_2$

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