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Abstract

In this paper we introduce a pricing model for a European call option when the price of the underlying stock (asset) follows a random walk with Markov Chain type of shifts in the drift and volatility parameters according to the regime that the stock market lies in, at a given period of time. We show that the model can explain the main stylised facts of the option pricing literature and substantially reduce the BS option pricing biases when allows for time-varying transition probabilities between the regimes of the stock market. version preferences, based on traded option prices data.

JEL Classification: G10, G13, C22

Keywords: Markov regime switching, Option pricing, Volatility smile

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1 Introduction

Since Black and Scholes (1973) originally developed their option pricing model, many extensions have been suggested incorporating stochastic volatility or jumps into the model to improve its empirical performance, adequately.[1] These extensions were mainly motivated by evidence that the conditional mean or volatility of stock returns vary over time. In this paper, we suggest a discrete-time European call option pricing model which considers that the mean and volatility changes of stock returns are driven by a common state variable modelled by a Markov chain, according to the Markov Regime Switching (MRS) model of Hamilton (1989). The state variable is assumed to randomly change between two regimes of the market: the bear and bull according to some transition probabilities. The bull regime is identified by our data as having higher mean and lower volatility, while the bear the inverse: lower mean and higher volatility.

The MRS extension of the Black-Scholes (BS) model that we introduce in this paper was motivated by recent evidence of MRS type of changes in stock (asset) returns.[2] The model can incorporate many characteristics of the stock (asset) prices or returns into option pricing. First, it can account for negative skewness or leptokurtosis of stock returns [see Ghysels, Harvey and Renault (1996), for a survey], since it implies a conditional density of the future stock return which is a mixture of normal distributions.[3] Second, it can capture systematic shifts (or serial correlation in stock returns) [see Lo and Wang (1995), *inter alia*], since the state variable is considered to be time-dependent. Third, it can allow for discontinuous shifts in both the

[1]See Merton (1976), Hull and White (1987), Heston (1993), Lo and Wang (1995), and Bates (1996), *inter alia*.

[2]See Turner, Startz and Nelson (1989), Hamilton and Susmel (1994), and Bollen, Gray and Whaley (2000), *inter alia*.

[3]It can also justify growing efforts of using mixtures of normal distributions to calculate risk neutral densities of stocks prices [see Bahra (1996), and Soderlind and Svenson (1997), inter alia].

mean and volatility of stock returns [see Lamourex and Lastrapes (1992), *inter alia*], because of the discrete-time nature of the state variable. Finally, it can encounter for leverage effects in stock prices [see Turner, Startz and Nelson (1989), *inter alia*], since it allows the conditional mean and volatility of the underlying stock to be related through the same, state variable.

The analytic formula of the MRS option pricing model that the paper derives can be thought of as a weighted average of option prices which have a similar functional form with the BS model and are conditional on the number of periods that the market will stay in the bear regime until the expiration date of the option. This number of periods is known as the sojourn time of the Markov chain process and its distribution over which the conditional option prices are averaged out captures the effects of the stock market regime shifts on the European call option price, over its entire life. These type of effects can not be hedged out under risk neutral arbitrage arguments, since there is no asset that is perfectly correlated with them.[4]

The paper suggests two versions of the MRS: with constant and time varying transition probabilities between the regimes. The second version was motivated by recent evidence of time varying transition probabilities in asset returns [see, Gray (1996), *inter alia*]. To model the time variation in the transition probabilities we adopt a non-linear generalized autoregressive scheme which bounds the transition probabilities between zero and one, and updates the previous period transition probabilities according to the previous period innovation of the stock return. The incorporation of the innovation term into the above scheme can be thought of as introducing learning effects into the MRS option pricing model.

The paper evaluates the performance of the MRS model to adequately price option data in two ways: First, by investigating if it can explain the mispecification of the BS model, known as the BS volatility smile, and second by checking if it can reduce the BS option price biases, compared with a

[4]See also Merton (1973, 1976), Cox and Ross (1976), and Naik (1993).

benchmark extension of the BS model which *ex post* considers for any type of mispecification of the volatility smile, across moneyness and maturity. The results of our empirical analysis provide strong evidence that the version of the MRS model with time-varying transition probabilities can substantially improve the performance of the BS, especially for shorter term ITM or ATM categories of options where the extensions of the BS model with jumps or stochastic volatility can not significantly outperform it.[5] For these categories, we find that this version of the MRS model performs equally well to the benchmark extension of the BS model. We show that responsible for this successful performance of the MRS option pricing model is its ability to generate stock returns with adequate enough degree and persistency of negative skewness. This can be attributed to the tendency of the stock market to stay in the bull regime for most periods of time.

The paper is organized as follows. In Section 2 we set the assumptions of the MRS model and derive an analytic solution of the European call price. In Section 3 we evaluate the ability of the MRS model to explain some of the main stylized facts of the option pricing literature and to price traded option data adequately. Section 4 concludes the paper.

[5]See Baksi, Cao and Chen (1997), and Aït-Sahalia and Lo (1998), for recent evidence.

2 The MRS Option Pricing Model

2.1 Distribution properties of the MRS model of stock returns

2.1.1 Model setup

Let the logarithm of the stock price, defined $y_t = \log Y_t$, follow the MRS stochastic process:

$$y_{t+1} = y_t + \mu_{t+1} - \frac{1}{2}\sigma_{t+1}^2 + \sigma_{t+1}\epsilon_{t+1}, \tag{1}$$

with $\quad \mu_{t+1} = \mu_0 + \mu_1 S_{t+1},\ \sigma_{t+1}^2 = \sigma_0^2 + \sigma_1^2 S_{t+1},\ \text{and} \quad \epsilon_{t+1} \sim NIID\,(0,1)\,,$

where S_t denotes the state (regime) that the stock market lies in, at time t. Equation (1) implies that the one-period stock return, defined $Y_{t+\tau}/Y_t) - 1$, has conditional on the current stock market information set, $\mathcal{I}_t$, mean and volatility parameters which are equal to μ_{t+1} and σ_{t+1}, respectively.

Consider that S_t follows a Markov chain and that there are two regimes driving the stock market, i.e. $S_t \in \{0,\ 1\}$. The movements between the two regimes are dictated by the transition matrix of probabilities, $\mathbf{P}$, given by

$$\mathbf{P} = \begin{bmatrix} p_{00} & p_{10} = 1 - p_{11} \\ p_{01} = 1 - p_{00} & p_{11} \end{bmatrix}, \tag{2}$$

where p_{ij} declares the transition (switching) probability of moving from state i to state j. The first regime, denoted by "0", is characterized by the pair of mean and volatility parameters $(\mu_0,\ \sigma_0^2)$, while the second by the pair $(\mu_0 + \mu_1, \sigma_0^2 + \sigma_1^2)$. This definition of the regimes means that the first has the lowest volatility. Given that this regime will be identified by our stock data as having the highest mean, it will be referred to as the bull market regime, henceforth. The second regime will be referred to as the bear regime.

2.1.2 Probability density of the MRS model

An option pricing formula for the MRS model can be obtained if we characterize the conditional on $\mathcal{I}_t$ probability density function (pdf) of the stock log-price (or its implied log-return), implied by the model, over the option maturity interval, τ. To this end, consider that $\mathcal{I}_t$ includes, in addition to all historical stock prices, the present and past values of the state variable, S_t, i.e. $\mathcal{I}_t = \{y_t, S_t, y_{t-1}, S_{t-1}, \dots\}$. This assumption means that the stock market investors recognize the current regime of the stock market. The future realizations of the regime are considered to be unknown and independent of the innovation term ϵ_t, for all t. The last assumption means that the investors are surprised by regime changes [see Turner, Startz and Nelson (1989)].

For notation convenience, write (1) as

$$y_{t+1} = y_t + \tilde{\mu}_{t+1} + \sigma_{t+1}\epsilon_{t+1} \tag{3}$$

where $\tilde{\mu}_{t+1} = \tilde{\mu}_0 + \tilde{\mu}_1 S_{t+1}$, with $\tilde{\mu}_0 = \mu_0 - \frac{1}{2}\sigma_0^2$ and $\tilde{\mu}_1 = \mu_1 - \frac{1}{2}\sigma_1^2$. The conditional on $\mathcal{I}_t$ pdf of the stock log-price, $y_{t+\tau}$ (or its implied τ-period log-return, defined $\tilde{y}_{t,\tau} = y_{t+\tau} - y_t$), denoted as $f_y\left(y_{t+\tau}|\mathcal{I}_t\right)$, can be studied by writing (after forward substitution) equation (4) as

$$y_{t+\tau} = y_t + \tau\tilde{\mu}_0 + \tilde{\mu}_1 \sum_{h=1}^{\tau} S_{t+h} + \omega_{t,\tau}, \tag{4}$$

where $\omega_{t,\tau} = \sum_{h=1}^{\tau} \sigma_{t+h}\epsilon_{t+h}$.

Equation (4) indicates that it not feasible to derive a known pdf for $y_{t+\tau}$, which could facilitate the derivation of a closed form option formula for the MRS model. This happens since $y_{t+\tau}$ contains the sum of products of random variables, $\omega_{t,\tau}$, plus the sum of the future sequence of states $\{S_{t+h}\}_{h=1}^{\tau}$,

$\sum_{h=1}^{\tau} S_{t+h}$. This sum, defined as

$$Z_{t,\tau} = \sum_{h=1}^{\tau} S_{t+h}, \tag{5}$$

is a random variable which is known as the sojourn time of the Markov Chain [see Darroch and Morris (1968), and Norris (1997)]. Given that $S_{t+\tau}$ is a binary process taking values 0 and 1, $Z_{t,\tau}$ takes values, ζ, in the set $\{0, ..., \tau\}$. The values of ζ reflect the number of periods that the stock market spends in regime "1" between the periods $t+1$ and $t+\tau$.

Although it is difficult to derive a known function form of $f_y\left(y_{t+\tau}|\mathcal{I}_t\right)$, we can obtain an analytic formula of it which enables us to calculate its values. This can be done by applying Bayes' rule and noticing that ϵ_t and S_t are independent. In so doing, define the extension of the information set $\mathcal{I}_t$ with the sequence of the values of $Z_{t,\tau}$, $\{Z_{t,\tau} = \zeta\}$,

$$\mathcal{H}_{t,\tau}(\zeta) = \mathcal{I}_t \cup \left\{Z_{t,\tau} = \zeta\right\}. \tag{6}$$

The assumptions of independency between ϵ_t and S_t, and across ϵ_t, for all t, imply that the conditional on $\mathcal{H}_{t,\tau}(\zeta)$ distribution of $\omega_{t,\tau}$ is normal, given by

$$\omega_{t,\tau}|\mathcal{H}_{t,\tau}(\zeta) \sim N\left(0, \tau\sigma_0^2 + \sigma_1^2 \sum_{h=1}^{\tau} S_{t+i}\right). \tag{7}$$

By Bayes' rule and the result of equation (7), we can write $f_y\left(y_{t+\tau}|\mathcal{I}_t\right)$ as

$$f_y\left(y_{t+\tau}|\mathcal{I}_t\right) = \sum_{\zeta=0}^{\tau} f_y\left(y_{t+\tau}, \zeta|\mathcal{H}_{t,\tau}(\zeta)\right) \Pr\left[Z_{t,\tau} = \zeta|\mathcal{I}_t\right], \tag{8}$$

where

$$
\begin{aligned}
f_y\left(y_{t+\tau}, \zeta | \mathcal{H}_{t+\tau}(\zeta)\right) &= \left[2\pi\sigma\left(\zeta\right)^2\right]^{-\frac{1}{2}} \exp\left\{-\frac{\left[y_{t+\tau} - y_t - \mu\left(\zeta\right)\right]^2}{2\sigma\left(\zeta\right)^2}\right\}, & (9)\\
\tilde{\mu}\left(\zeta\right) &= \tau\tilde{\mu}_0 + \zeta\tilde{\mu}_1, & (10)\\
\sigma\left(\zeta\right)^2 &= \tau\sigma_0^2 + \zeta\sigma_1^2. & (11)
\end{aligned}
$$

Equation (8) indicates that the pdf $f_y\left(y_{t+\tau}|\mathcal{I}_t\right)$, implied by the MRS model, constitutes a mixture of the $(\tau+1)$ normal pdf $f_y\left(y_{t,\tau}, \zeta|\mathcal{H}_{t,\tau}(\zeta)\right)$. These densities are conditional on the sojourn time values, ζ, and are appropriately weighted with the corresponding conditional probabilities of these values, denoted $\Pr\left[Z_{t,\tau} = \zeta|\mathcal{I}_t\right]$. The pdf $f_y\left(y_{t+\tau}|\mathcal{I}_t\right)$ can be evaluated based on estimates of the vector of parameters of the MRS stochastic process (1), denoted by $\Theta' = (\mu_0, \mu_1, \sigma_0, \sigma_1, p_{00}, p_{11})$, and the probabilities of the sojourn time values, $\Pr\left[Z_{t,\tau} = \zeta|\mathcal{I}_t\right]$. Estimates of Θ can be obtained by estimating (1) by the Maximum Likelihood (ML) method, suggested by Hamilton (1989). The values $\Pr\left[Z_{t,\tau} = \zeta|\mathcal{I}_t\right]$ can be obtained using the functional form of the pdf of $Z_{t,\tau}$, derived by Pedler (1971) [see also Darroch and Morris (1968)].[6] An alternative way of calculating $\Pr\left[Z_{t,\tau} = \zeta|\mathcal{I}_t\right]$ is to use Kedem's

[6]Pedler shows that the conditional on a given state, $S_t = i$, probability that $Z_{t+\tau} = \zeta$, denoted $\Pr\left[Z_{t+\tau} = \zeta|S_t = i\right]$, is given by

$$
\begin{aligned}
\Pr\left[Z_{t+\tau} = \zeta|S_t = i\right] &= p_{00}^{\zeta} p_{11}^{\tau-\zeta} \left\{F\left(-\tau+\zeta, -\zeta; 1; l\right)\right.\\
&\qquad \left. - d p_{11}^{-1} F\left(-\tau+\zeta+i, -\zeta+1-i; 1; l\right)\right\}, \qquad (12)
\end{aligned}
$$

where $i \in \{0,1\}$, $d = p_{00}p_{11} - p_{01}p_{10}$, $l = \frac{p_{01}p_{10}}{p_{00}p_{11}}$, and $F\left(a;b;c;l\right)$ denotes the hypergeometric function

$$
F(a;b;c;l) = \sum_{\kappa=0}^{\infty} \frac{(a)_\kappa (b)_\kappa}{(c)_\kappa} \frac{l^\kappa}{\kappa!},
$$

where $(a)_\kappa$, $(b)_\kappa$, and $(c)_\kappa$ are the Pochhammer terms, which are defined as $(a)_k = a(a+1)...(a+k-1)$ [see Abadir (1999) for a survey of Hypergeometric functions].

(1981) algorithm.

There two special cases of the pdf $f_y(y_{t+\tau}|\mathcal{I}_t)$, given by equation (8), which are interesting to analyze. The first is for $\tau = 1$ -the case of the one-period log-return $\tilde{y}_{t,1}$. Then, $f_y(y_{t+\tau}|\mathcal{I}_t)$ reduces to

$$\begin{aligned} & f_y(\tilde{y}_{t,1}|\mathcal{I}_t) \qquad (13) \\ = & \sum_{\zeta=0}^{1} \left[2\pi\left(\sigma_0^2+\zeta\sigma_1^2\right)\right]^{-\frac{1}{2}} \exp\left\{-\frac{\left[\tilde{y}_{t,1}-(\tilde{\mu}_0+\zeta\tilde{\mu}_1)\right]^2}{2\left(\sigma_0^2+\zeta\sigma_1^2\right)}\right\} \Pr\left[S_{t+1}=\zeta|\mathcal{I}_t\right], \end{aligned}$$

since $Z_{t,1} = S_{t+1}$ when $\tau = 1$. Equation (13) shows that the conditional distribution of the period log-return $\tilde{y}_{t,1}$ is not normal, as it is often assumed in the stochastic volatility or jumps extensions of the BS model. But, it is a mixture of two normal distributions, each of which depends on the particular regime of the economy: "0" or "1", at time $t+1$. Evidence that the distribution given by (13) can fit into the stock market data better than the normal can be found in the studies of Turner, Startz and Nelson (1989), and Bahra (1996), *inter alia.*

The second special case of (8) is for τ going to infinity, $\tau \to \infty$. Then, $f_y(y_{t+\tau}|\mathcal{I}_t)$ becomes normal. This is proved in the Appendix, and is established in the following proposition.

Proposition 1 *Let stochastic process (1), with its underlying assumptions, hold. Then, as $\tau \to \infty$:*

$$y_{t+\tau}|\mathcal{I}_t \sim N\left(y_t+\tau\tilde{\mu}_\infty, \tau\sigma_\infty^2\right), \qquad (14)$$

where $\tilde{\mu}_\infty = (\tilde{\mu}_0+\pi_1\tilde{\mu}_1)$, $\sigma_\infty^2 = (\pi_0\pi_1\sigma_S^2+\sigma_0^4+\pi_1\sigma_1^4+2\sigma_0^2\sigma_1^2\pi_1)$, and σ_S^2 defined in the Appendix.

The above proposition claims that a random walk model with constant drift, given by $\tilde{\mu}_\infty$, and variance of the disturbance term, given by σ_∞^2, implies a pdf which can approximate the pdf of the MRS model, for sufficiently large τ. The result of the proposition is consistent with evidence that deviations of stock returns from the normal distribution become less apparent, as investment horizon, τ, increases [see Campbell, Lo and MacKinlay (1997) for a survey of the literature].

2.1.3 Moments over longer horizons

The density function $f_y(y_{t+\tau}|\mathcal{I}_t)$, given by (8), being a mixture of normal density functions can allow for different degrees of skewness and excess (over the normal distribution) kurtosis of the τ-period log-return, $\tilde{y}_{t,\tau}$ [or the stock return $(Y_{t+\tau}/Y_t) - 1$]. To verify this, we next derive the centered moments of $f_y(y_{t+\tau}|\mathcal{I}_t)$, for fixed τ.

Let $M_r(\tau) = E_t[y_{t+\tau} - E_t y_{t+\tau}]^r$ be the r-th order centered moment of $y_{t+\tau}$ conditional on I_t, where the conditional expectations operator $E_t = E[\cdot|\mathcal{I}_t]$ is used for notational convenience. It can be easily seen from equation (4) that the centered first ($r = 1$) moment $M_1(\tau)$ is zero, for all τ, since $E_t[Z_{t+\tau} - E_t Z_{t+\tau}] = 0$. The centered moments for $r \geq 2$ are given in the following Proposition.

Proposition 2 *Let stochastic process (1), with its underlying assumptions, hold. Then, the centered moments $M_r(\tau)$, for $r \geq 2$ are given by*

$$\begin{aligned} & M_r(\tau) \\ = & \sum_{\zeta=0}^{\tau}\sum_{\kappa=0}^{r}\binom{r}{\kappa}\tilde{\mu}_1^{r-\kappa}\left(\zeta - E_t[Z_{t,\tau}]\right)^{r-\kappa} c_\kappa \sigma(\zeta)^\kappa \Pr[Z_{t,\tau} = \zeta|\mathcal{I}_t], \end{aligned} \tag{15}$$

where c_κ is the κ-th centered moment of the standardized normal distribution, and $E_t[Z_{t,\tau}] = \sum_{\zeta=0}^{\tau}\zeta\Pr[Z_{t,\tau} = \zeta|\mathcal{I}_t]$.

The proof of the proposition is given in the Appendix.[7]

[7]Note that the moments of the stock return $(Y_{t+\tau}/Y_t - 1)$ can be derived by replacing $\tilde{\mu}_1$ with μ_1 into (15).

The moments $M_r(\tau)$ can be used to examine the behavior of skewness and kurtosis of the return $\tilde{y}_{t,\tau}$, over different investment horizons, τ. Since $\tilde{y}_{t,\tau}$ is compounded by one-period log-returns, it is instructive at this point to investigate the generating sources of skewness and kurtosis based on $\tilde{y}_{t,1}$. To this end, we derive the first four centered unconditional moments of $\tilde{y}_{t,1}$.[8] These can be obtained from (15) by setting $\tau = 1$ and evaluating the involved expectations at the ergodic levels of probabilities. This yields[9]

$$\begin{aligned}
M_1(1) &= \mu_0 + \pi_1\tilde{\mu}_1, \\
M_2(1) &= \pi_0\pi_1\tilde{\mu}_1^2 + \sigma_0^2 + \pi_1\sigma_1^2, \\
M_3(1) &= \left(\pi_0^3 - \pi_1^3\right)\tilde{\mu}_1^3 + 3\pi_0\pi_1\tilde{\mu}_1, \\
M_4(1) &= \pi_0\pi_1\left(\pi_0^3 + \pi_1^3\right)\tilde{\mu}_1^4 + 6\pi_0\pi_1\tilde{\mu}_1^2\left(\sigma_0^2 + \pi_1\sigma_1^2\right) \\
&\quad +3\pi_0\sigma_0^4 + 3\pi_1(\sigma_0^2 + \sigma_1^2)^2.
\end{aligned} \tag{16}$$

As was expected, the moments given by (16) indicate that the conditional on $\mathcal{I}_t$ pdf of $\tilde{y}_{t,1}$ can allow for different degrees of skewness, defined $Sk = M_3(1)/M_2(1)^{\frac{3}{2}}$, and excess kurtosis, defined $Ku = M_4(1)/M_2(1)^2 - 3$, depending on the values of the vector of parameters Θ.[10] The functional form of the third moment, $M_3(1)$, indicates that a necessary and sufficient condition for the existence of skewness is that $\tilde{\mu}_1 \neq 0$. This condition requires the existence of a regime shift in the conditional mean of the stock return $(Y_{t+1}/Y_t) - 1$. In contrast to the coefficient of skewness, the fourth moment $M_4(1)$ indicates that there is not a unique cause of excess kurtosis in the stock

[8]For the one period return, $\tilde{y}_{t,1}$, extensions of these moments for a finite (more than two) number of states and various alternative presentations of (1) are given by Timmermann (1999).

[9]Conditional on the current state of the economy, S_t, moments can be obtained by replacing the ergodic probabilities, π_i, where $i \in \{0,1\}$, with the filter probabilities $\Pr[S_t = i|\mathcal{I}_t]$ [see Hamilton (1989)]

[10]Note that the functional forms given by (16) are consistent with those of Timmermann (1999), who considers a different parameterization of (1).

return. This can be attributed either to shifts in the mean or variance.[11]

2.2 Option Pricing with the MRS model

2.2.1 Option pricing formula

Having obtained an analytic form of the conditional on the information set, $\mathcal{I}_t$, density function of the stock log-price, $y_{t+\tau}$, (or its implied return $\tilde{y}_{t,\tau}$), implied by the MRS model (1), in this subsection we introduce an option pricing formula of a European call, for the model. In so doing, we assume that interest rates are deterministic and we impose the following risk neutral local-arbitrage condition on the τ-period stock return, $(Y_{t+\tau}/Y_t) - 1$

$$\tau r_t(\tau) = \tau\mu_0 + \mu_1 E_t Z_{t,\tau}, \tag{17}$$

where $r_t(\tau)$ is the τ-period free-risk (deterministic) interest rate [see Heston and Nandi (1997), *inter alia*]. In words, this condition states that the current τ-period interest rate is equal to the conditional on $\mathcal{I}_t$ expected return of the τ-period stock return. Under this condition, the conditional mean of the log-return $\tilde{y}_{t,\tau}$ will satisfy the following restriction

$$\tau\mu_0 = \tau(r_t(\tau) - \frac{1}{2}\sigma_0^2) + (\tilde{\mu}_1 Z_{t,\tau} - \mu_1 E_t\left[Z_{t,\tau}\right]). \tag{18}$$

Let $C_t(\tau)$ denote the value of a European Call, at time t, that matures at date $t+\tau$ and has exercise price K. The European call option can be valued

[11] Note that the above coefficients of skewness and kurtosis can become time-varying if the probabilities in (16) are evaluated at their conditional levels. Evidence of time varying skewness has been recently provided by Harvey and Siddique (2000).

as

$$\begin{aligned} C_t(\tau) &= e^{-\tau r_t(\tau)} \hat{E}_t\left[(Y_{t+\tau} - K)^+\right] \\ &= e^{-\tau r_t(\tau)} \int_K^{+\infty} \frac{Y_{t+\tau} - K}{Y_{t+\tau}} \hat{f}\left(y_{t+\tau}|\mathcal{I}_t\right) dy_{t+\tau}, \end{aligned} \tag{19}$$

where $\hat{E}_t(\cdot)$ and $\hat{f}(\cdot)$ denote the risk neutral measures of the expectation and probability density function of the call option payoff function $(Y_{t+\tau} - K)^+ = \max\{Y_{t+\tau} - K, 0\}$.

Equation (19) indicates that an analytic formula of call price $C_t(\tau)$ can be derived when the risk neutral density function (RND), $\hat{f}\left(y_{t+\tau}|\mathcal{I}_t\right)$, is specified. To this end, we consider that the location parameter of $\hat{f}\left(y_{t+\tau}|\mathcal{I}_t\right)$ satisfies restriction (18), implied by the risk neutral local-arbitrage condition (17). The other distribution parameters of $\hat{f}_y\left(y_{t+\tau}|\mathcal{I}_t\right)$ are assumed to be identical to those of $f_y\left(y_{t+\tau}|\mathcal{I}_t\right)$ [see also Brennan (1979), *inter alia*]. This specification of $\hat{f}\left(y_{t+\tau}|\mathcal{I}_t\right)$ means that there is only one source of risk that is priced in stock market equilibrium. This comes from the innovation error ϵ_t. The risk of a regime change in the stock return is assumed to be diversifiable.[12] Based on this specification of $\hat{f}\left(y_{t+\tau}|\mathcal{I}_t\right)$, an analytic formula of the European call price, $C_t(\tau)$, is obtained in the following proposition.

Proposition 3 *Let the log-price, y_t, of the underlying stock follow the MRS process (1), with its assumptions, and $\hat{f}\left(y_{t+\tau}|\mathcal{I}_t\right)$ be the RND function. Then, the European call option price, $C_t(\tau)$, is given by*

$$C_t(\tau) = e^{-\tau r_t(\tau)} \sum_{\zeta=0}^{\tau} C_t(\tau, \zeta) \Pr\left[Z_{t,\tau} = \zeta|\mathcal{I}_t\right], \tag{20}$$

[12] Analogues assumptions are made by Merton (1976), Hull and White (1987), and Naik (1993) in derivating analytic formulas of option prices when the undelying stock undertakes Poisson driven jumps in the mean, and continuous or discontinuous changes in the volatility function, respectively.

where

$$C_t(\tau,\zeta) = e^{\tau r_t(\tau)+\mu_1(\zeta - E_t[Z_{t,\tau}])}\Phi\left[d_1(\zeta)\right] - K\Phi\left[d_2(\zeta)\right]\Pr\left[Z_{t,\tau}=\zeta|\mathcal{I}_t\right],$$

$$d_1(\zeta) = \frac{\log(Y_t/K)+\tau(r_t(\tau)-\frac{1}{2}\sigma_0^2)+(\tilde{\mu}_1\zeta - E_t[Z_{t,\tau}])}{\sigma(\zeta)} + \sigma(\zeta), \;\; d_2(\zeta) = d_1(\zeta) - \sigma(\zeta)$$

and $\Phi(\cdot)$ *denotes the cumulative normal distribution.*

The proof of the proposition is given in the appendix.

Proposition 3 demonstrates that the European call option price implied by the MRS model (1) can be thought of as a weighted average of the option prices $C_t(\tau,\zeta)$, for all ζ, which are conditional on the number of periods that the stock market can stay in regime "1" until the expiration date of the option. Each conditional option price is weighted with the corresponding probability of ζ to occur.[13] The distribution of ζ over which the conditional option prices are averaged out captures the effects of possible regime shifts on the option price, over the entire life of the option. These effects can not disappear under the risk neutral arbitrage condition (18), since there is no asset that is perfectly correlated with the state variable S_t.

There are two interesting special cases where (20) can be simplified. The first is when there is no regime change, i.e. $S_t = 0$ with probability one, for all t. Then, $\Pr\left[Z_{t+\tau}=0|\mathcal{I}_t\right]=1$, and hence (20) reduces to the familiar, one-state BS formula. The second special case of (20) is when τ is sufficiently large. Then, $C_t(\tau)$ can be sufficiently approximated by

$$C_t(\tau) = e^{-\tau r_t(\tau)}\left\{e^{\tau r_t(\tau)}\Phi\left[d_1(y_t)\right] - K\Phi\left[d_2(y_t)\right]\right\}, \tag{21}$$

where $d_1 = \frac{\log(Y_t/K)+(\tau r_t(\tau)-\frac{1}{2}\sigma_\infty^2)}{\sigma_\infty\sqrt{\tau}} + \sigma_\infty\sqrt{\tau}$ and $d_2 = d_1 - \sigma_\infty\sqrt{\tau}$. This result is the outcome of Proposition 1, stating that the MRS model (1) can be sufficiently approximated by a random walk model with drift $\tilde{\mu}_\infty$ and volatility

[13]Note that the functional form of (1) is a consequence of the fact the MRS model implies a probability density function of the future stock price which is a mixture of normal densities.

σ_∞, as $\tau \to \infty$. It implies that the BS model may be successfully employed in pricing a European call option with very long maturity.

The MRS option pricing model, given by (20), differs from other models considering discontinuous shifts in volatility subject to Markov chains [see Naik (1993)] or driven by a threshold model [see Duan *et al* (1999)], recently introduced in the literature, with respect to the following main points. First, it differs from both the above models since it considers that the conditional density of the one-period stock return is not normal, which is assumed by the other models, but a mixture of normal distributions [see equation (13)]. This is due to the assumption that both the conditional mean and volatility of the MRS model (1) depend on the future value of the state variable S_{t+1}, rather than the current. For this property, the MRS model can account for the effects of skewness and leptokurtosis into option pricing, even for one-period to maturity options.[14] Second, it differs from Duan's *et al* model, since it enables us to account for the leverage effects by allowing the mean of the stock return to be negatively correlated with the volatility, when $\mu_1 < 0$. In Duan's *et al* model, the shifts in the mean are assumed to be positively correlated to the volatility ones according to the predictions of the conditional CAPM. To account for leverage effects, Naik introduced jumps into the mean of his model, when the regime changes.

2.2.2 Option pricing with time-varying transition probabilities

The above MRS option pricing model considers that the transition probabilities between the regimes are constant. In this subsection we extend the model to allow for time varying transition probabilities following recent evidence that these type of MRS models can better describe assets prices [see Gray (1996)].

To model time variation in the transition probabilities we consider the

[14]Note that for one-period to maturity options the other models reduce to the BS model with deterministic volatility.

following model:

$$p_{ij,t} = g\left(\alpha_{ij} + k_{ij} p_{ij,t-1} + \phi_{ij} u_{t-1}\right), \qquad i \neq j \in \{0,1\}, \tag{22}$$

where $p_{ij,t}$ denotes the time-varying transition probabilities, the error term u_{t-1} is defined as $u_{t-1} = \sigma_{t-1}\epsilon_{t-1}$, and

$$g : \mathbb{R} \rightarrow [0,1] : g(x) = \begin{cases} 0, & \text{if } x \leqslant 0 \\ x^3\left(6x^2 - 15x + 10\right), & \text{if } 0 < x \leqslant 1 \\ 1, & \text{if } x > 1 \end{cases}, \tag{23}$$

for any real value of x.

Equation (22) is a composite function of two component functions: The first is the linear model $\alpha_{ij} + k_{ij} p_{ij,t-1} + \phi_{ij} u_{t-1}$ which revises the transition probabilities $p_{ij,t}$ from their previous period values, $p_{ij,t-1}$, according to the previous period error term u_{t-1}.[15] This term introduces a feedback learning mechanism into the MRS model. Note that if $\phi_{ij} = k_{ij} = 0$, the MRS model with time varying probabilities degenerates to that with constant transition probabilities. The second component function of (22), $g(x)$, ensures that the mapping from the set of real numbers, $\mathbb{R}$, into the closed interval $[0,1]$ of the values of $p_{ij,t}$ is achieved in a sufficiently smooth way. The fifth order spline of $g(x)$ is chosen in such a way that $g(x)$ to be a continuous twice differentiable function. The derivation of $g(x)$ and some of its main properties are discussed in the Appendix.[16]

The specification of the time varying transition probabilities, given by (22), does not alter the functional form and intuition of the RND function,

[15] Note that this autoregressive scheme of the transition propabilities is similar in spirit to the ARCH/GARCH models of volatility [see Engle (1982) and Bollerslev (1986)].

[16] The second order differentiability of $g(x)$ enable us to obtain linear forecasts of the transition probabilities and their conditional variance by applying Taylor's expansion [see an earlier version of the paper].

$\hat{f}(y_{t+\tau}|\mathcal{I}_t)$, and the option pricing formula (20), given that $\hat{f}(y_{t+\tau}|\mathcal{I}_t)$ is evaluated, as before, maintaining the assumption that the price of the regime change risk is zero. It only modifies the way of calculating the values sojourn time probabilities, $\Pr[Z_{t,\tau} = \zeta|\mathcal{I}_t]$. To calculate these probabilities we will employ a Monte Carlo simulation method.[17] This method is described in the Appendix. Its application requires estimates of the vector of parameters, Θ, and the current values of $p_{ij,t}$ and u_t. The latter can be obtained by an appropriate modification of Hamilton's (1989) filter to accommodate (22).

3 Evaluation of the MRS option pricing model

The aim of this section is to evaluate the ability of both versions of the MRS model: with constant (MRS-CN) and time varying (MRS-TV) transition probabilities to explain some of the main stylized facts of the option pricing literature, and to reduce the BS option pricing biases, considerably. Our empirical analysis has the following order. First, we investigate which version of the model can generate those distribution characteristics of the S&P500 index return which can better match the shape of the BS volatility smile observed in reality, across different areas of moneyness and maturity. To see this, we are based on ML estimates of the parameters of each specification of the model using a sample of S&P500 index price data from January 1, 1988 to December 31, 1993. Having seen this, we next compare the empirical performance of the alternative specifications of the model in reducing the BS option pricing biases with a benchmark extension of it which assumes that the BS volatility is a deterministic function of the maturity and strike price of the option. This is done based on Aï t-Sahalia and Lo's (1998) option price data set. This set consists of daily observations of pairs of European

[17] A closed for solution of $\Pr[Z_{t,\tau} = \zeta|\mathcal{I}_t]$ does not seem possible to be obtained due to the involved nonlinearity of $g(x)$ and the dependency of $p_{ij,t+h}$, across $h = 1, 2..., \tau$.

call and put option prices for S&P500 index options, traded on the Chicago Board Options Exchange between January 4, 1993 and December 31, 1993.[18]

3.1 Investigating the properties of the MRS model

3.1.1 Fitting the MRS model into the S&P500 returns

The estimation results of the parameters of the MRS-CN and MRS-TV specifications for the one-period log-return are reported in Table B.1. Panel A of the table gives the estimates of the MRS-CN specification, while Panel B of the MRS-TV. The estimates of Θ are obtained based on the EM algorithm [see Titterington, Smith and Makov (1985), and Hamilton (1990)]. The results of the table clearly indicate that the MRS-TV specification outperforms the MRS-CN, since it has higher log-likelihood function value. This can be also verified by the statistical significance of the estimates of the coefficients ϕ_{ij} and k_{ij}.

Both versions of the MRS model agree that there are two distinct stock market regimes. The first, denoted by "1", constitutes the bear market regime, and it characterized by a lower mean, $\tilde{\mu}_0 + \tilde{\mu}_1$, (which is not different than zero) and higher volatility $(\sigma_0^2 + \sigma_1^2)^{1/2}$ of the log-return $\tilde{y}_{t,1}$.[19] The second, denoted by "0", is the bull market regime, and it is characterized by higher mean, μ_0, and lower volatility σ_0. These results confirm that there is an inverse relationship between the mean and volatility parameters across the two regimes which is recognized in the literature as evidence of leverage effects in the stock price.[20]

[18] Option price data on the S&P500 index have been used by many other authors in empirical invetsigations of the option pricing puzzles [see Bates (1996), Jackwerth and Rubinstein (1996), Baksi, Cao and Chen (1997) , and Dumas, Fleming and Whaley (1998), *inter alia*].

[19] We also find that this characterization of the two regimes also holds for the stock return $(Y_{t+1}/Y_t) - 1$.

[20] See also Turner, Startz and Nelson (1989).

The estimates of the parameters of the time-varying transition probabilities equation (22) are consistent with the interpretation of (22) as a mechanism of introducing feedback learning effects of the stock market into the MRS model. The signs of α_{ij} and ϕ_{ij} reveal that the effect of the error term u_{t-1} on the probability $p_{ij,t}$ is inversely related to the state of the economy, for any t. In particular, the negative sign of ϕ_{01} means that a positive value of u_{t-1} will decrease the transition probability from the bull to bear market regime, p_{01}. This can be interpreted as reflecting a positive reaction of the stock market investors to news of higher returns, as part of their learning behavior. By similar reasoning, we can explain the positive sign of ϕ_{10}.

To see how the transition probabilities vary within our sample period, in Figure C.1 we graphically present the estimates of $p_{01,t}$ and $p_{10,t}$ (see the lower and upper parts of the figure, respectively). The graphs of the figure reveal significant fluctuations of the transition probabilities, especially of $p_{10,t}$ -the transition probability from the bear to bull market state. This probability oscillates around values closed to zero for most of the time periods before the end of 1990. Afterwards, it shifts and oscillates around values closed to unity indicating that there is an apparent structural break in the series of $p_{10,t}$, occurred after the end of 1990. The break point of $p_{10,t}$ may reflect the considerable improvement of the stock market uncertainty conditions after the 1987 crash situation.[21] The decrease of investors' uncertainty about the prevailing stock market regime conditions after the end of 1990 can be also confirmed by Figure C.1.B, which presents the in-sample estimate of the probability that the market stays in the bull regime, "0", i.e. $\Pr[S_t = 0|I_t]$, over the whole sample period. This graph indicates that the stock market seems to stay most of the time in the bull regime after the end of 1990. This is particularly true for the year 1993 where the estimates of $p_{ij,t}$ will be used for option pricing.

Summing up, the results of this subsection support the view that the ver-

[21] For similar evidence see Schwert (1998).

sion of the MRS model with time-varying transition probabilities constitutes a better specification of the S&P500 data generating process than the version with constant probabilities.

3.1.2 Distributional properties of the alternative MRS specifications

Given the estimates of the parameters of the alternative specifications of the MRS model, in this subsection we present graphs of the pdf $f_y(y_{t+\tau}|\mathcal{I}_t)$, for each one of the specifications, separately, and for $\tau = \{1, 63, 126, 252\}$ [see Figures C.2.A-D].[22] We also present graphs of their corresponding skewness and excess kurtosis coefficients with respect to τ [see Figures C.3]. The values of $f_y(y_{t+\tau}|\mathcal{I}_t)$ are calculated based on the standardized version of the density function formula (8).[23] The values of the skewness and leptokurtosis coefficients are calculated based on the centered moment formula (15). To estimate the sojourn time probabilities $\Pr[Z_{t+\tau} = \zeta|\mathcal{I}_t]$ involved in the above formulas, we used Kedem's (1981) algorithm, for the MRS-CN specification, and the Monte Carlo simulation method described in the Appendix, for the MRS-TV.

Inspection of the figures leads to the following conclusions. First, the pdf of both versions of the MRS model are leptokurtic and negatively skewed, as evidence based on time series or option prices data reveals [24] As the moment analysis of Section 2.2 indicates, the negative skewness of $f_y(y_{t+\tau}|\mathcal{I}_t)$ can be attributed to the negative value of $\tilde{\mu}_1$, implied by the estimate of $\tilde{\mu}_0 + \tilde{\mu}_1$,

[22] For interesting comparisons with the pdf implied by the BS model, the figures also contain the standardised normal density function.

[23] The densities are standardised based on their first and second moments given by (15).

[24] See [see Jarrow and Rudd (1982), Jackwerth and Rubinstein (1996), Bahra (1996), Aït-Sahalia and Lo (1998), and Abadir and Rockinger (1999), *inter alia*]. Note that the graphs of $f_y(y_{t+\tau}|\mathcal{I}_t)$ can be compared with those of the RND function $\hat{f}_y(y_{t+\tau}|\mathcal{I}_t)$, since we assume that risk neutrality does not change the shape of the pdf, but only its location parameter.

and the fact that the ergodic probability π_0 is bigger than π_1. This means that negative skewness can be interpreted as evidence that the stock market tends to stay in the bull regime for a higher number of periods than in the bear regime. As τ increases, the densities $f_y(y_{t+\tau}|\mathcal{I}_t)$, for both versions of the model, approach the normal density, and the skewness and leptokurtosis coefficients tend to zero, which are consistent with the predictions of Proposition 1.

The second conclusion which can be drawn from the above figures is that the version of the MRS model with time-varying transition probabilities produces higher and more persistent degrees of negative skewness and leptokurtosis than that with constant, especially of skewness. Although the deviations of the skewness coefficient from zero seem to follow a similar nonlinear, convex pattern with τ, for both versions of the model, they substantially differ in terms of their magnitude and their rate of convergence towards zero, as τ increases. The MRS-TV specification can generate higher and more persistent degrees of skewness than the MRS-CN. This is due to the time-varying nature of the transition probabilities and the learning effects, involved in the MRS-TV specification, which amplify the degree of negative skewness.

3.1.3 Explaining the volatility smile of the BS model

Evidence of leptokurtosis and negative skewness of the pdf (or its risk neutral version) of the τ-period log-return $q_{t,\tau}$ is often offered as an explanation of the volatility smile of the BS model [see Ghysels, Harvey and Renault (1996), and Aït-Sahalia and Lo (1998), *inter alia*]. This is a convex relationship between the implied volatility of the BS model when is fitted into traded option price data and the stock/strike price ratio, Y_t/K, known as moneyness.

The aim of this subsection is to examine which version of the MRS option pricing model can better explain two main features of the smile.[25] The first is its inverse relationship with the maturity interval, τ, i.e. the fact that

[25] See Bates (1996), and Bakshi, Cao and Chen (1997), *inter alia*.

it appears stronger for shorter term maturity options rather than longer. The second is that it becomes stronger as we move from deep in-the-money (ITM) or out-the-money (OTM) to at-the-money (ATM) options. To see this, in Figure C.4.A-D we present graphs of the volatility smile implied by BS when is fitted into option price data generated according to the MRS-CN and MRS-TV specifications. This is done for values of τ which correspond to those of Figures C.2.

Inspection of the graphs of the figures clearly indicate that the version of the MRS model with time-varying transition probabilities can better resemble the shape of the BS volatility smile based on actual data [see Bakshi, Cao and Chen (1997) *inter alia*], especially at the short or medium end of the maturity structure of the call option prices, i.e. $\tau = \{66, 126\}$. The graphs of the volatility smile functions show that the MRS-TV specification can generate smiles of greater amplitude than the MRS-CN meaning that it can account for a higher degree of mispecification of the BS model. Given the calibration analysis of the previous subsection, this property of the MRS-TV specification can be attributed to its ability to generate stock returns with higher degree of negative skewness and leptokurtosis than the MRS-CN, especially of skewness.

3.2 An empirical appraisal of the MRS option pricing model

Having established that both versions of the MRS model can explain some of the main stylized facts of the stock and option pricing literature, in this subsection we assess their empirical performance to adequately price traded options. To examine this, we compare the pricing performance of the alternative versions of the model with the standard BS model and an extension of it based on a deterministic function of volatility in the maturity interval and the strike price of the option [see Rubinstein (1994), and Dumas, Fleming and Whaley (1998), *inter alia*]. This extension of the BS model can be though of as the benchmark against to which the alternative versions of the MRS model should be compared with, since it considers for a variety of mispecifications of the volatility function across different levels of moneyness and maturity.

The relative performance of all the above, alternative option pricing models is judged by calculating the percentage and absolute pricing errors between the traded option prices, $C_t(\tau)$, and those predicted by the MRS-CN and MRS-TV specifications, denoted by $C_t^{MRS-CN}(\tau)$ and $C_t^{MRS-TV}(\tau)$, respectively, the standard BS, $C_t^{BS}(\tau)$, and the BS with the deterministic volatility function, $C_t^{BS-DVF}(\tau)$.[26] This is done across different levels of moneyness and maturity. To separate any possible effects of dividend payments or risk aversion on the parameter estimates of the option prices formulas, the risk-neutral interest rate, $r_t(\tau)$, adjusted for the dividend rate, $\delta_t(\tau)$, i.e. $r_t(\tau) - \delta_t(\tau)$, is calculated by the spot-forward arbitrage relationship

$$r_t(\tau) - \delta_t(\tau) = \frac{\ln\left(F_t(\tau)/Y_t\right)}{\tau}, \tag{24}$$

where $F_t(\tau)$ is the date-t forward price of the S&P500 of maturity τ [see Aït-Sahalia and Lo (1998)].

3.2.1 Pricing errors

The values of the percentage and absolute measures of the alternative option pricing models considered are reported in Table B.2. Following Bakshi, Cao and Chen (1997), the table presents results for six moneyness and three term-

[26] The percentage pricing errors are sample averages of the traded prices minus those predicted by the models, divided by the traded prices. This measure of the errors is suitable in indicating the direction (sign) of the misspecification of the option pricing models. The absolute pricing errors are sample averages of the absolute differences between the traded prices and the predicted by the models. One advanrage of this measure, compared with the percentage, us that is a better indicator of the actual magnitude of the pricing errors, since it is not sensitive to very small option prices which tend to amplify the percentage measure.

to-expiration option price categories. In particular, a call is categorized to be far OTM if $Y_t/K \leq 0.94$; OTM if $Y_t/K \in (0.94, 0.97)$; ATM if $Y_t/K \in [0.97, 1.03]$; ITM if $Y_t/K \in (1.03, 1.06)$; and far ITM if $Y_t/K \geq 1.06$. By the term-to-expiration, a call option contract is classified as short term if $\tau \leq 60$ calendar days; medium-term if $\tau \in (60, 180)$; and long-term if $\tau \geq 180$ and thus shape the pattern of option mispricing across moneyness or maturity.

The results of the table clearly indicate that the MRS-TV version of the MRS model, in contrast to the MRS-CN, constitutes an important improvement upon the BS model in reducing the option prices biases. This is supported by the values of both the percentage and absolute measures of option pricing errors, reported in the table. The MRS-TV specification leads to an overall reduction of the BS percentage pricing errors by about 60 percent and the absolute errors by more than 100%. The improvements of the MRS-TV model are very important especially for ATM or ITM short term call options where other extensions of the BS model, such as the stochastic volatility model with or without jumps do not seem to significantly outperform the model [see Bakshi, Cao and Chen (1997)]. Note that for these moneyness areas, the MRS-TV model performs even better than the BS-DVF model, which can be though of as the in-sample benchmark model. The only category of options where the BS seems to outperform the MRS-TV model is that of medium or longer term far ITM options.[27]

The superiority of the specification of the MRS model with time–varying transition probabilities to reduce the BS option pricing biases, substantially, for the ATM and ITM short or medium term options can be attributed to the ability of this specification to produce stock returns with adequate enough degree of skewness, especially for short-medium period returns, as

[27] One reason for this may be that the option prices of this moneyness category are almost equal to their intrinsic values, and thus the effects of the regime changes in the mean and volatility encountered by the MRS models do not have any profound effect on option prices.

the calibration analysis of Section 3.1.2 and 3.1.3 reveals. This happens since the model allows for regime shifts in its conditional or volatility which are quite persistent, over time, due to the time-varying nature of the transition probabilities.

4 Conclusions

This paper introduces a pricing model for a European call option when the price of the underlying stock (asset) follows a random walk model with Markov Chain type of shifts in its drift and volatility parameters according to the regime that the stock market lies in, at any period of time. The option price formula that the paper derives is a weighted average of option prices conditional on different paths of future regime changes in the stock market. The functional form of this formula is a consequence of the fact the Markov Regime Switching (MRS) model implies a probability density function of the future stock price which is a mixture of normal densities, each depending on a path of future regime changes of the stock market. Being a mixture of normal distributions, the probability density of the future stock price, underlying the option, can allow for different degrees of skewness and kurtosis. This is shown by deriving the moments of the probability density function of the MRS model, over different investment horizons.

In evaluating the ability of the MRS model to explain some of the main stylized facts of the option pricing literature, we fit two versions of the model into the S&P500 stock index return, for the period between 1988 and 1993. The first assumes constant transition probabilities between the bull and bear regimes of the market, while the second time-varying based on a model which incorporates learning effects into the MRS model. The paper shows that the version of the MRS with time varying transition probabilities fits better into the S&P500 return in terms of the log-likelihood maximum value. This version of the model can generate leptokurtic stock returns with higher degree and persistency of negative skewness than the version with constant transition probabilities, and can thus better explain the shape of the BS volatility smile, observed in reality.

When applying both versions of the MSR model to traded options data to assess their empirical performance relative to the BS model, we find evidence that the version of the model with time varying transition probabilities considerably reduces the option pricing biases of the BS. The reductions are substantial for categories of moneyness and maturities of option prices, such as short term at-the-money or in-the-money options. Note that for these moneyness categories, extensions of the BS model, such as the stochastic volatility model with or without jumps, require either more frequent and more significant number of jumps in the underlying stock price changes or more persistent and negatively correlated with the stock price changes volatility shocks in order to generate stock returns with a substantial degree of negative skewness to be consistent with the data [see Bates (1996), Bakshi, Cao and Chen (1997), and Aït-Sahalia and Lo (1998)].

A Appendix

In this Appendix, we prove the results of Propositions 2.1-2.3 and derive the functional form of equation (23) given in the main text. We also give a brief description of the simulation method that we use to calculate the probabilities of the sojourn time values, $\Pr[Z_{t,\tau} = \zeta | \mathcal{I}_t]$, employed in the evaluation of the MRS-TV option pricing model.

A.1 Proof of Proposition 2.1

The proposition follows immediately by (4) noticing the two asymptotic (over τ) results. The first result is:

$$\frac{Z_{t,\tau} - \tau\pi_1}{\sqrt{\tau\sigma_S^2}} \xrightarrow{d} N(0,1), \tag{25}$$

where " $\xrightarrow{d}$ " signifies convergence in distribution. This result follows by application of the Central Limit Theorem (CLT) for dependent Bernoulli trials to $Z_{t,\tau}$ [see Kedem (1980)]. If there are two regimes, "0" and "1", it can be shown that the unconditional (ergotic) mean of $Z_{t,\tau}$ is given by $\tau\pi_1$ and has asymptotic (long run variance) give by $\sigma_S^2 = \frac{\pi_0\pi_1(p_{00}+p_{11})}{2-(p_{00}+p_{11})}$.

The second result that is used to prove the proposition is:

$$\frac{\sum_{h=1}^{\tau}(\sigma_0 + \sigma_1 S_{t+h})\epsilon_{t+h}}{\sqrt{\tau\left(\sigma_0^4 + \pi_1\sigma_1^4 + 2\sigma_0^2\sigma_1^2\pi_1\right)}} \xrightarrow{d} N\left(0,1\right). \tag{26}$$

This result follows by the assumptions that ϵ_t is $NIID$ [see the definition of (1)], and S_t and ϵ_t are independent, for all t. These two assumptions imply that $(\sigma_0^2 + \sigma_1^2 S_{t+h})\epsilon_{t+h}$ is an independent process which has mean equal to zero and asymptotic variance $(\sigma_0^4 + \pi_1\sigma_1^4 + 2\sigma_0^2\sigma_1^2\pi_1)$. This enables us to apply the CLT to the standardized sum of $(\sigma_0 + \sigma_1 S_{t+h})\epsilon_{t+h}$.

A.2 Proof of Proposition 2.2

Using the independence assumption between S_t and ϵ_t, for all t, write the centered moment, $M_r(\tau)$, as

$$\begin{aligned}
& M_r(\tau) \\
= & \int_{-\infty}^{\infty} (y_{t+\tau} - E_t[y_{t+\tau}])^r f_y(y_{t+\tau}|\mathcal{I}_t)\, dy_{t+\tau} \\
= & \sum_{\zeta=0}^{\tau} \left\{ \int_{-\infty}^{\infty} (y_{t+\tau} - y_t - \tau\tilde{\mu}_1 - \tilde{\mu}_1 E_t[Z_{t+\tau}])^r f_y(y_{t+\tau}|\mathcal{H}_{t,\tau}(\zeta))\, dy_{t+\tau} \right\} \\
& \Pr[Z_{t,\tau} = \zeta|\mathcal{I}_t], \qquad (27)
\end{aligned}$$

where $E_t[y_{t+\tau}] = y_t + \tau\tilde{\mu}_0 + \tilde{\mu}_1 E_t[Z_{t,\tau}]$ by the underlying assumptions of (1).

Using Binomial expansion, write $(y_{t+\tau} - y_t - \tau\tilde{\mu}_0 - \tilde{\mu}_1 E_t[Z_{t,\tau}])^r$ as

$$\begin{aligned}
& (y_{t+\tau} - y_t - \tau\tilde{\mu}_0 - \tilde{\mu}_1 E_t[Z_{t,\tau}])^r \\
= & [(y_{t+\tau} - \tilde{\mu}(\zeta)) + \tilde{\mu}_1(\zeta - E_t[Z_{t,\tau}])]^r \\
= & \sum_{\kappa=0}^{r} \binom{r}{\kappa} [y_{t+\tau} - \tilde{\mu}(\zeta)]^{\kappa} [\tilde{\mu}_1(\zeta - E_t[Z_{t,\tau}])]^{r-\kappa}. \qquad (28)
\end{aligned}$$

Substituting (28) into (27) yields

$$\begin{aligned}
M_r(t) = & \sum_{\zeta=0}^{\tau}\sum_{\kappa=0}^{r} \binom{r}{\kappa} [\zeta(\tilde{\mu}_1 - E_t[Z_{t,\tau}])]^{r-\kappa} \times \\
& \int_{-\infty}^{\infty} [y_{t+\tau} - \tilde{\mu}(\zeta)]^{\kappa} f_y(y_{t+\tau}|\mathcal{H}_{t,\tau}(\zeta))\, dy_{t+\tau} \Pr[Z_{t,\tau} = \zeta|\mathcal{I}_t]. \qquad (29)
\end{aligned}$$

Notice that the integral in the last equation is the κ-th centered moment of a normal distribution, with mean $\tilde{\mu}(\zeta)$ and variance $\sigma(\zeta)^2$, i.e.

$$\int_{-\infty}^{\infty} [y_{t+\tau} - \tilde{\mu}(\zeta)]^{\kappa} f_y(y_{t+\tau}|\mathcal{H}_{t,\tau}(\zeta))\, dy_{t+\tau} = c_\kappa \sigma(\zeta)^{\kappa}. \qquad (30)$$

Substituting (30) into (29) proves the result of the proposition.

A.3 Proof of Proposition 2.3

The proposition can be proved by writing (20) using the assumption of independency between S_t and ϵ_t, as

$$
\begin{aligned}
& C_t(\tau) && (31)\\
= \;& e^{-\tau r_t(\tau)} \hat{E}_t \left[(Y_{t+\tau} - K)^+ \right] \\
= \;& e^{-\tau r_t(\tau)} \int_K^{+\infty} \frac{Y_{t+\tau} - K}{Y_{t+\tau}} \hat{f}_y \left(y_{t+\tau} | \mathcal{I}_t \right) dy_{t+\tau} \\
= \;& e^{-\tau r_t(\tau)} \sum_{\zeta=0}^{\tau} \left\{ \int_K^{+\infty} \frac{Y_{t+\tau} - K}{Y_{t+\tau}} \hat{f}_y \left(y_{t+\tau} | \mathcal{H}_{t,\tau}(\zeta) \right) dy_{t+\tau} \right\} \Pr \left[Z_{t+\tau} = \zeta | \mathcal{I}_t \right].
\end{aligned}
$$

Substituting

$$
\begin{aligned}
& \int_K^{+\infty} \frac{Y_{t+\tau} - K}{Y_{t+\tau}} f_y \left(\hat{y}_{t+\tau} | \mathcal{H}_{t,\tau}(\zeta) \right) dy_{t+\tau} \\
= \;& e^{\tau (r - \frac{1}{2}\sigma_0^2) + (\tilde{\mu}_1 \zeta - \mu_1 E_t[Z_{t,\tau}]) + \frac{\sigma(\zeta)^2}{2}} \Phi \left[d_1 \left(\zeta \right) \right] - K \Phi \left[d_2 \left(\zeta \right) \right] \qquad (32)
\end{aligned}
$$

into (31) proves the result of the proposition.

A.4 Derivation of the $g(x)$ function

The g function is chosen to satisfy two types of conditions: The first consists of the boundary conditions: $g(x) = 0, \forall x < 0$ and $g(x) = 1, \forall x > 1$. The second is the smoothness condition that $g(x)$ is continuous and has continuous first and second derivatives everywhere, i.e. $g(x) \in C^2$. The boundary conditions imply that $p_{ij,t}$ takes values in the closed interval $[0, 1]$. The smoothness conditions ensure that a linear, first order approximation of the conditional expectation or variance of (22), at time t, is feasible by applying a Taylor series expansion.

Consider that $g(x)$ has the following form

$$g(x) = \begin{cases} 0, & \text{if } x \leqslant 0 \\ g_s(x), & \text{if } 0 < x \leqslant 1 \\ 1, & \text{if } x > 1 \end{cases},$$

For the function $g_s(x)$, the boundary conditions imply

$$g_s(0) = 0, \text{ and } g_s(1) = 1, \tag{33}$$

while the smoothness conditions

$$g_s \in \mathcal{C}^2,\ g_s'(0) = g_s'(1) = 0, \text{ and } g_s''(0) = g_s''(1) = 0. \tag{34}$$

Given that polynomial functions are sufficiently smooth, a natural choice for $g_s(x)$ would be a polynomial spline which satisfies the above boundary and smoothness conditions. Based on these conditions, we can derive the parameters of a polynomial spline form of $g_s(x)$ by solving a system of differential equation with the method of undetermined coefficients. Note that the degree of the polynomial should be high enough so that all first and second order derivatives to exist and for a system that determines the argument values to be constructed. Since we have six conditions (two boundary and four smoothness), the most parsimonious spline $g_s(x)$ that we consider is of fifth order.

The solution of the system of equations (33) and (34) yields (23). The first and second derivatives of $g(x)$ are given by

$$g'(x) = \begin{cases} 0, & \text{if } x \leqslant 0 \\ 30x^2(x-1)^2, & \text{if } 0 < x \leqslant 1 \\ 0, & \text{if } x > 1 \end{cases}, \text{ and}$$

$$g''(x) = \begin{cases} 0, & \text{if } x \leqslant 0 \\ 60x(x-1)(2x-1), & \text{if } 0 < x \leqslant 1 \\ 0, & \text{if } x > 1 \end{cases}. \tag{35}$$

A.5 Monte Carlo simulation estimation method of the probability $\Pr[Z_{t,\tau} = \zeta|\mathcal{I}_t]$

The Monte Carlo procedure that we employ in the paper to calculate the distribution of the sojourn time, $\Pr[Z_{t,\tau} = \zeta|\mathcal{I}_t]$, for the case of time varying transition probabilities, consists of the following steps.

1. We draw values of S_t and ϵ_{t+h}, for $h = 1, 2, ..\tau$, from the Bernoulli and normal distributions, respectively, based on the ML estimates of the vector of parameters Θ and the starting values $\Pr[S_t = i|\mathcal{I}_t]$, $i \in \{0,1\}$ of the state variable S_t.

2. Given the values of S_t and ϵ_{t+i}, we calculate the value of the disturbance term u_t. The values of u_t and $p_{ij,t}$ are used to update the time-varying transition probabilities, $p_{ij,t+1}$, as equation (22) dictates. The obtained values of $p_{ij,t+1}$ together with those of S_t are consequently used to derive the sequence of the filter probabilities $\Pr[S_{t+h} = i|\mathcal{I}_t]$ of the MRS model and the values of the state variable sequence $\{S_{t+h}\}_{h=1}^{\tau}$.

3. From the values of the sequence $\{S_{t+h}\}_{h=1}^{\tau}$, we finally obtain the sojourn time variable value by $Z_{t,\tau} = \sum_{h=1}^{\tau} S_{t+h}$.

Repeating steps 1-3 N times, which is the number of the total Monte

Carlo simulation, enables us to calculate the distribution of $Z_{t,\tau}$ as

$$\Pr\left[Z_{t,\tau} = \zeta\right] = \frac{1}{N}\sum_{n=1}^{N} \aleph\left(Z_{t,\tau}(n) = \zeta\right), \tag{36}$$

where $\aleph$ is the indicator function. As N, we use $N = 10000$ replications.

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B Tables

B.1 Estimates of the MSR models

The table reports the ML estimates of the parameters of the MRS process: $\Delta \log Y_t = \tilde{\mu}_{t+1} + \sigma_{t+1}\varepsilon_{t+1}$, with constant (see Panel A) and time varying (see Panel B) transition probabilities using the S&P500 index prices for the period 1988-1999. Standard errors are in parentheses. In Panel A, all the reported values are in percentage terms. In Panel B, only μ's and σ's are in percentage terms.

Panel A: MSR model with constant transition probabilities

$\tilde{\mu}_0$	$\tilde{\mu}_0 + \mu_1$	σ_0	$\sqrt{\sigma_0^2 + \sigma_1^2}$	p_{01}	p_{10}
0.049	-0.003	0.648	1.514	2.207	14.07
(0.019)	(0.105)	(0.014)	(0.056)	(0.494)	(2.696)

Log-likelihood: 5399.41

Panel B: MSR model with time varying transition probabilities

$\tilde{\mu}_0$	$\tilde{\mu}_0 + \mu_1$	σ_0	$\sqrt{\sigma_0^2 + \sigma_1^2}$
0.051	0.000	0.525	1.249
(0.008)	(0.000)	(0.013)	(0.060)

α_{01}	κ_{01}	ϕ_{01}	α_{10}	κ_{10}	ϕ_{10}
0.174	0.656	-5.308	0.176	0.661	6.702
(0.003)	(0.002)	(0.281)	(0.000)	(0.000)	(0.070)

Log-likelihood: 5432.81

B.2 Percentage and absolute errors of the BS and MSR models

The table compares the relative performance of the MRS-CN and MRS-TV models with the Black-Scholes (BS) model and the Black-Scholes model, where the volatility is a deterministic function of the maturity and the moneyness level, BS-DVF.

Moneyness		Percentage				Absolute			
F/K	Model	<60	60-180	180>	All	<60	60-180	180>	All
<0.94	BS-DVF	-60.41	-36.99	106.9	8.60	0.143	0.254	1.105	0.529
	BS	-527.7	-668.0	-502.7	-582.2	1.029	2.682	5.273	3.256
	MSR-CN	-505.9	-599.9	-470.9	-536.0	1.077	2.654	5.152	3.211
	MSR-TV	-67.60	-156.7	-137.5	-132.1	0.220	0.730	1.381	0.855
0.94-0.97	BS-DVF	40.91	2.80	-5.21	20.15	0.253	0.418	1.200	0.374
	BS	-481.4	-244.9	-79.70	-347.7	2.137	4.076	6.192	3.265
	MSR-CN	-423.4	-218.5	-74.47	-307.6	1.934	3.815	5.934	3.031
	MSR-TV	-123.1	-73.39	-15.25	-94.02	0.588	1.200	1.211	0.916
0.97-1.00	BS-DVF	40.39	2.42	-7.57	22.07	0.691	0.591	1.170	0.680
	BS	-150.2	-71.80	-40.80	-111.7	2.610	4.201	5.615	3.440
	MSR-CN	-120.9	-65.64	-38.44	-93.43	2.200	3.817	5.292	3.046
	MSR-TV	-34.50	-17.44	-2.66	-25.63	0.702	0.997	0.795	0.827

Continued

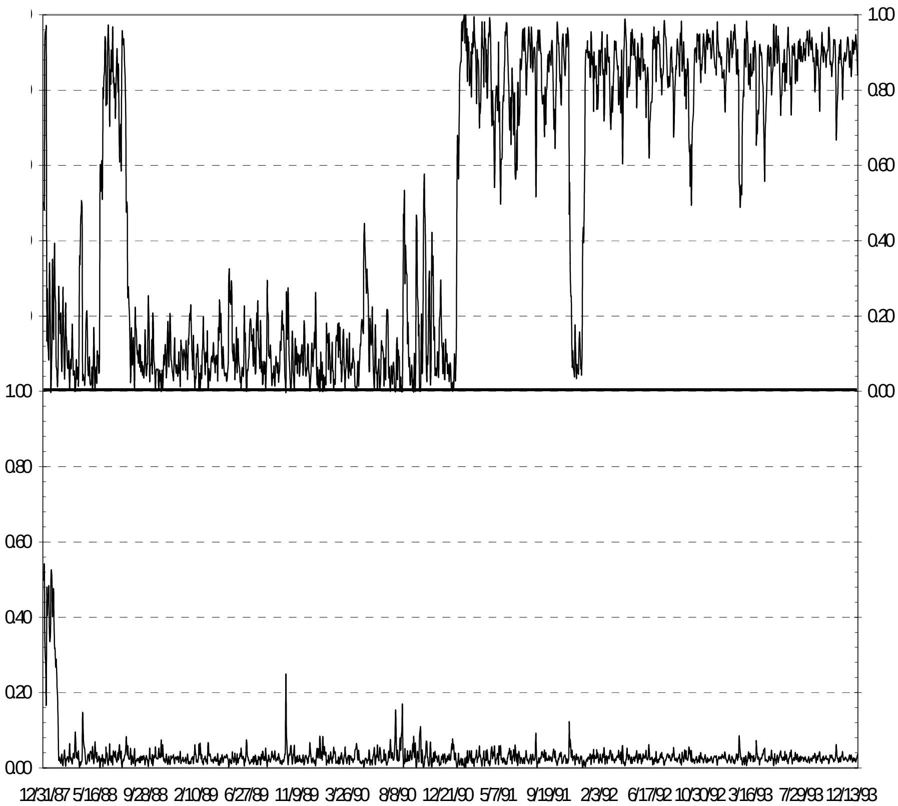
Moneyness F/K	Model	Percentage <60	Percentage 60-180	Percentage 180>	Percentage All	Absolute <60	Absolute 60-180	Absolute 180>	Absolute All
1.00-1.03	BS-DVF	6.50	0.02	-7.51	3.22	0.627	0.602	1.613	0.659
	BS	-20.20	-22.60	-19.60	-21.16	1.771	3.088	4.027	2.412
	MSR-CN	-15.11	-19.58	-17.86	-17.08	1.366	2.685	3.687	2.011
	MSR-TV	-0.10	0.02	2.93	0.08	0.519	0.671	0.994	0.602
1.03-1.06	BS-DVF	1.08	-0.35	-5.06	-1.11	0.368	0.490	1.419	0.705
	BS	-2.60	-7.00	-11.30	-6.43	0.612	1.607	3.293	1.682
	MSR-CN	-1.46	-5.12	-10.02	-5.02	0.442	1.232	2.938	1.397
	MSR-TV	2.41	3.84	2.63	2.91	0.613	1.055	1.837	1.098
>1.06	BS-DVF	-0.17	-0.10	-0.82	-0.16	0.179	0.277	0.598	0.248
	BS	0.40	-0.10	-0.20	0.10	0.266	0.485	0.626	0.399
	MSR-CN	-0.49	-0.23	0.03	-0.33	0.242	0.431	0.580	0.358
	MSR-TV	1.25	2.28	4.17	1.93	0.598	1.404	2.352	1.097
All	BS-DVF	14.49	0.66	-3.95	6.25	0.410	0.404	1.015	0.479
	BS	-110.9	-99.22	-124.9	-107.4	1.357	2.264	3.866	2.051
	MSR-CN	-96.27	-88.85	-116.7	-95.44	1.132	2.033	3.598	1.818
	MSR-TV	-21.14	-11.00	1.00	-14.09	0.584	0.974	1.228	0.831

C Figures

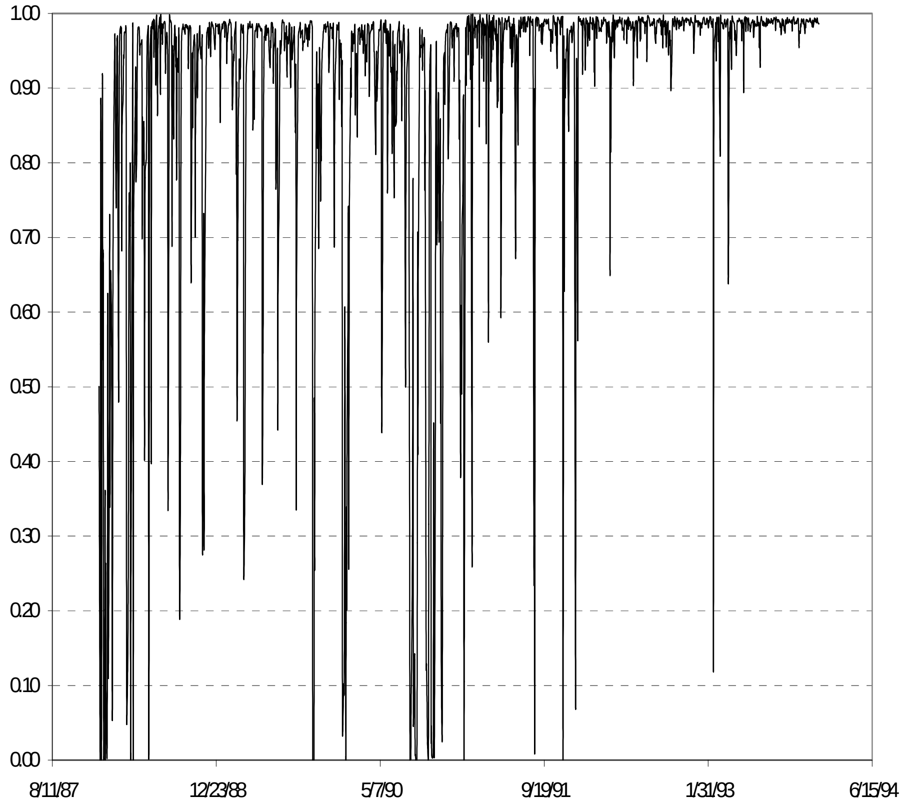
C.1 Transition and filter probabilities of the MSR-TV model

The figure illustrates the time varying transition Probabilities $p_{01,t}$ (see lower part of A) and $p_{10,t}$ (see upper part of A), and the filter probability that the stock market stays in regime "0", i.e. $\Pr[S_t = 0|\mathcal{I}_t]$ [see part B].

A. Transition probabilities $p_{ij,t}$



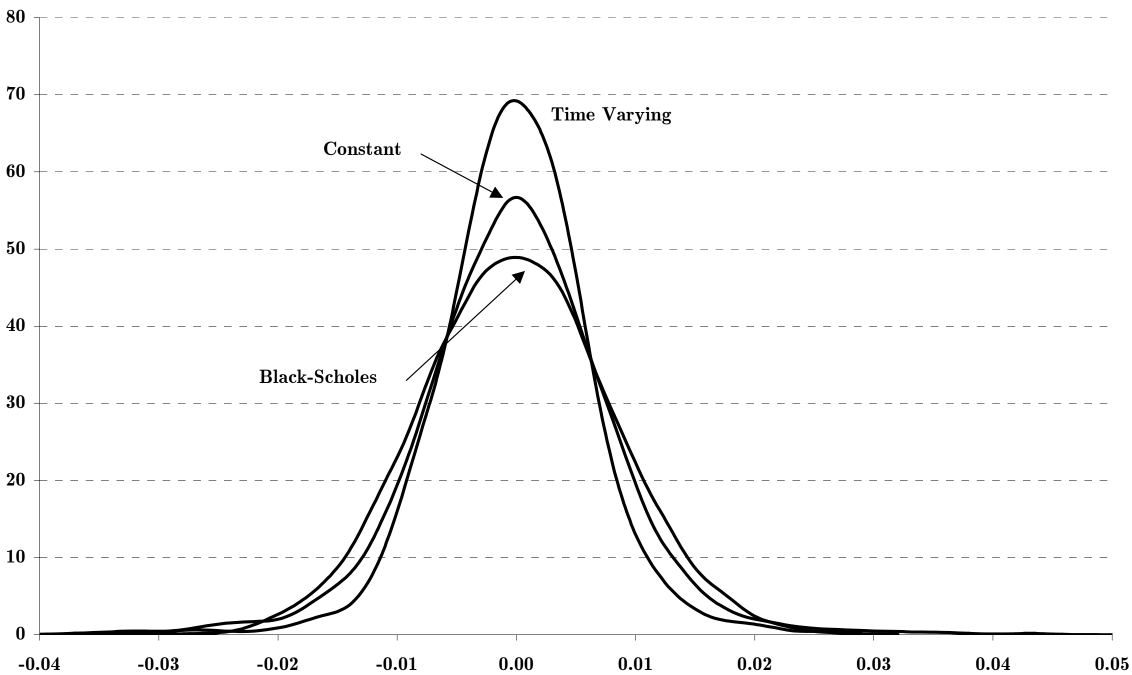
B: Filter probability $\Pr\left[S_t = 0 | \mathcal{I}_t\right]$



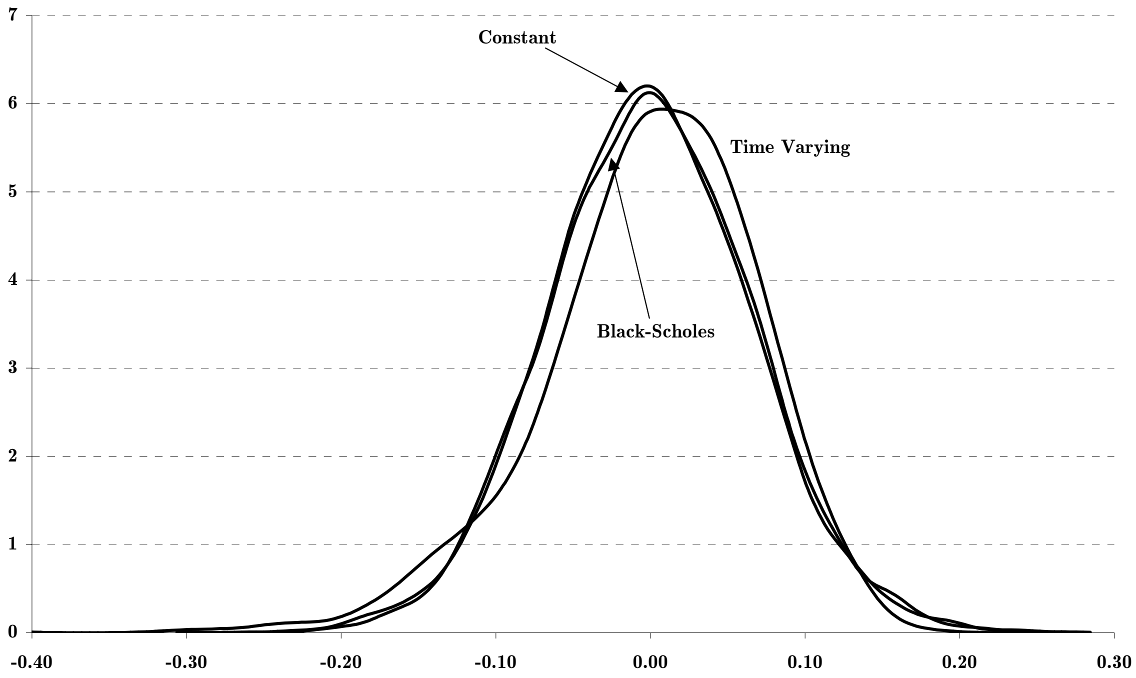
C.2 Probability Density functions of the BS and the MRS-CN and MRS-TV models

The figure presents graphs of the pdf of the MRS-CN and MRS-TV specifications, for $\tau = \{1, 66, 126, 252\}$, and the normal distribution (implied by the BS model) using as volatility parameter its sample estimate $\sigma_{BS} = 0.0085$. The interest rates are assumed to be zero, for convenience. The values of the sojourn time probabilities $\Pr\left[Z_{t,\tau} = \zeta|\mathcal{I}_t\right]$, used to estimate the pdf of the MRS-CN model, are calculated by Kedem's (1980) algorithm for the MSR-CN specification and based on Monte Carlo simulation for the MRS-TV specification.

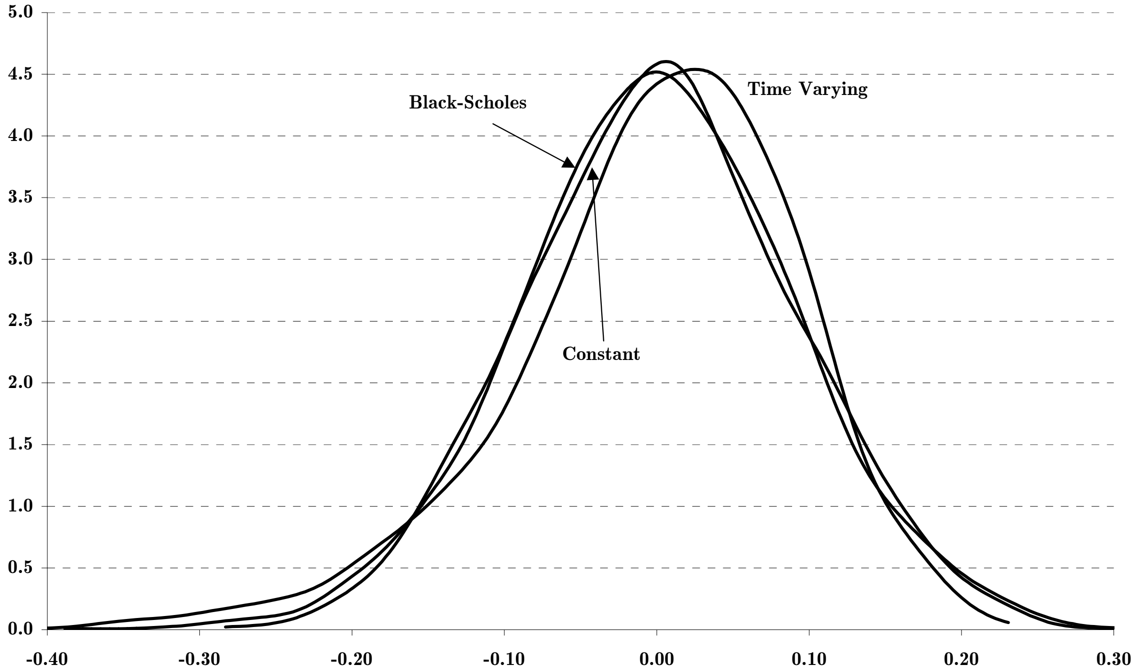
A. Probability Density function for $\tau = 1$



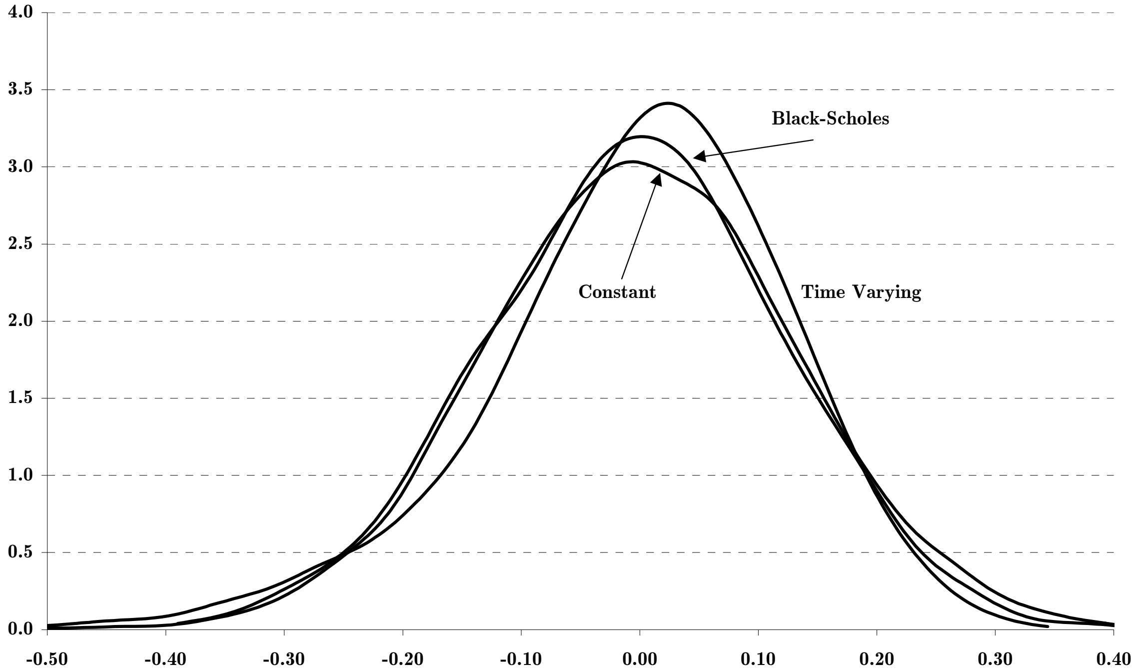
B. Probability Density function for $\tau = 66$



C. Probability Density function for $\tau = 126$



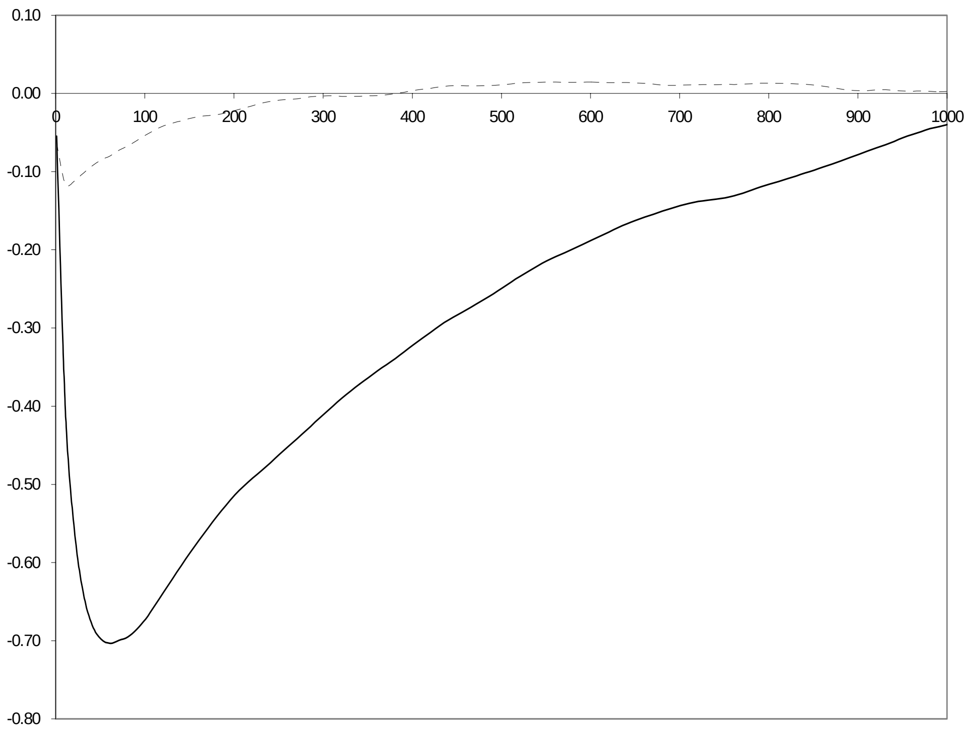
D. Probability Density function for $\tau = 252$



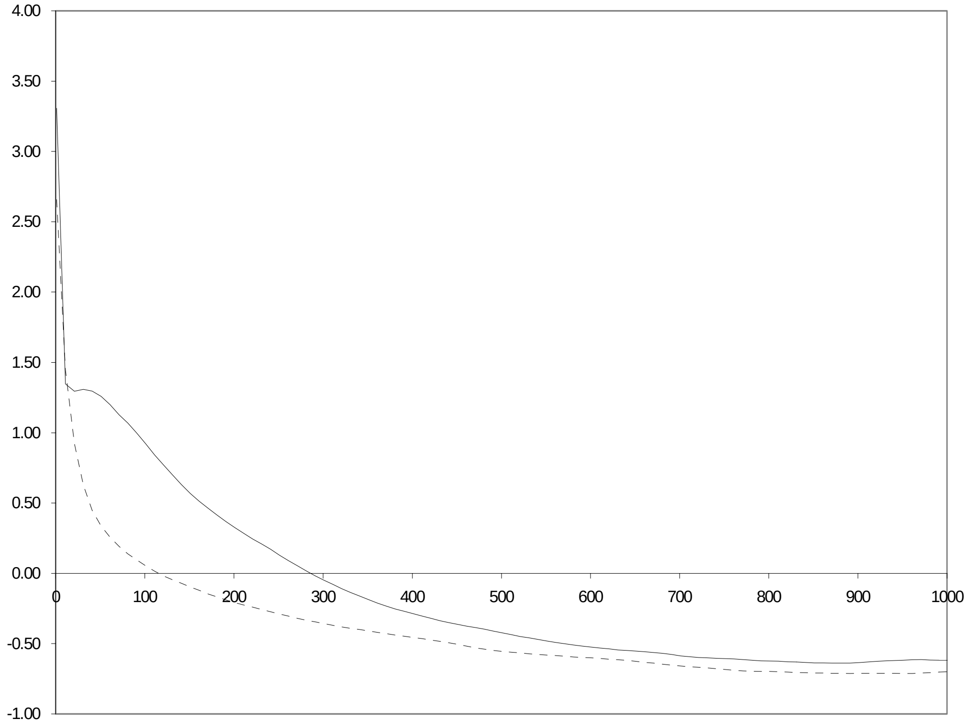
C.3 Skewness and kurtosis coefficients with τ

The figure presents graphs of the skewness (see A) and excess kurtosis (see B) coefficients of the stock returns implied by the MRS-CN (dashed line) and MRS-TV (solid line) models, respectively, over different investment horizons, τ. The sojourn time probabilities $\Pr\left[Z_{t,\tau} = \zeta | \mathcal{I}_t\right]$ are calculated as in Figure C.2.

A. Skewness coefficient with τ



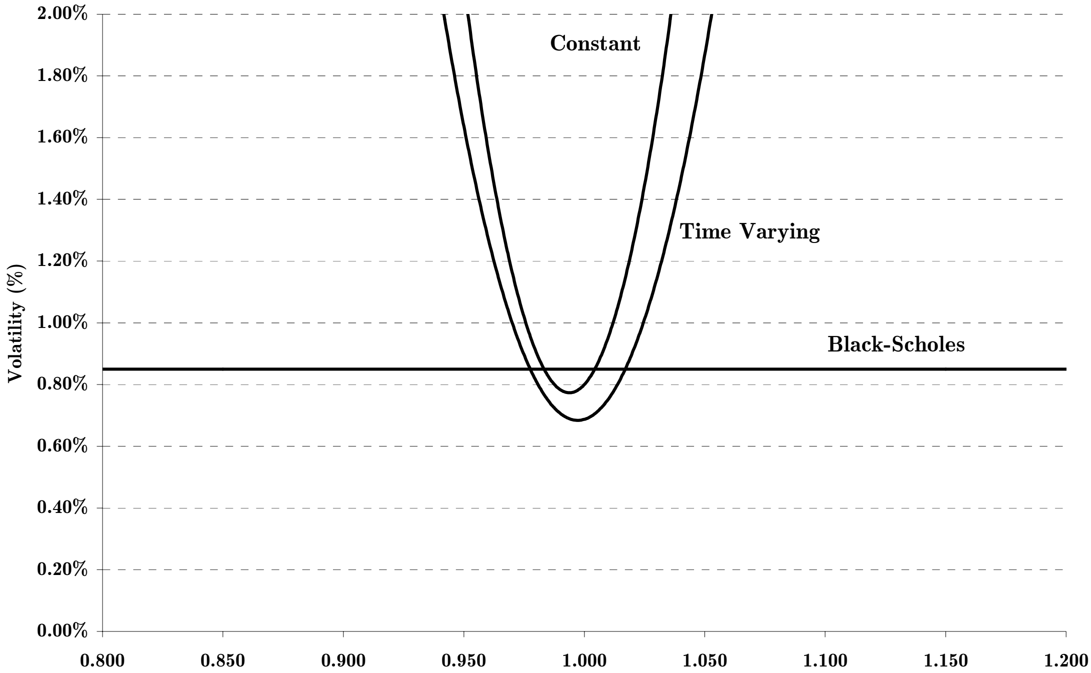
B. Excess Kurtosis coefficient with τ



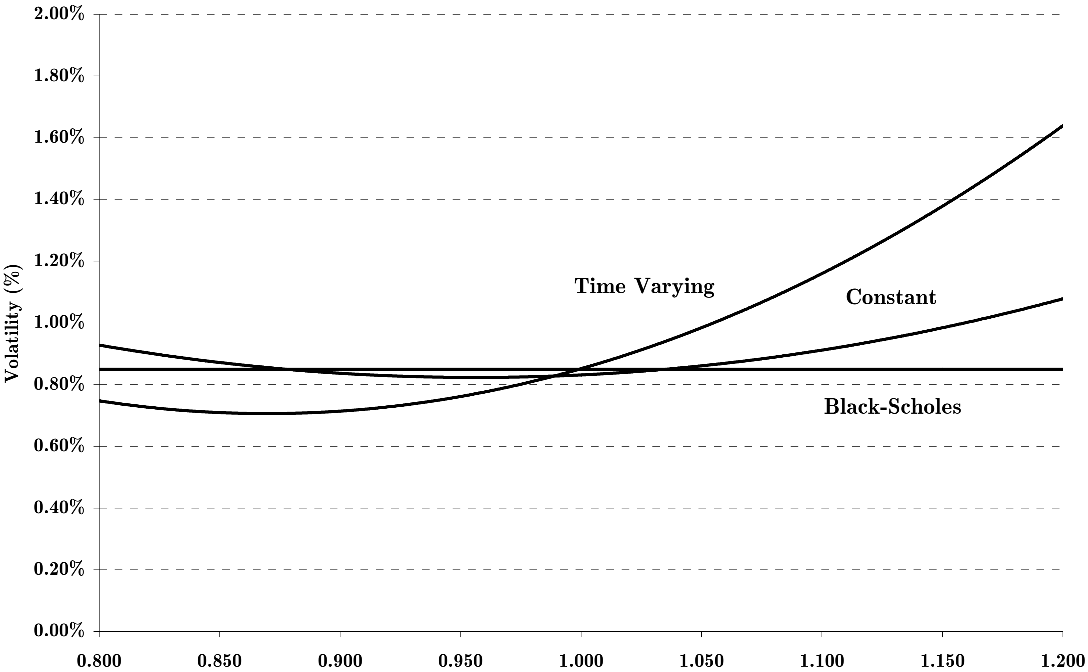
C.4 BS Implied Volatilities when data are generated according to the MRS-CN and MRS-TV models

The figure presents graphs of the BS implied volatility smiles, across different moneyness levels, when data are generated by the MRS-CN and MRS-TV versions of the MRS model based on the parameter estimates given in Table B.1. The volatility values are expressed in percentage terms. The volatility parameter associated with the BS model is 0.85%. The numbering of the graphs corresponds to that of Figure C.2. $\Pr\left[Z_{t,\tau} = \zeta | \mathcal{I}_t\right]$ are calculated as in Figure C.2

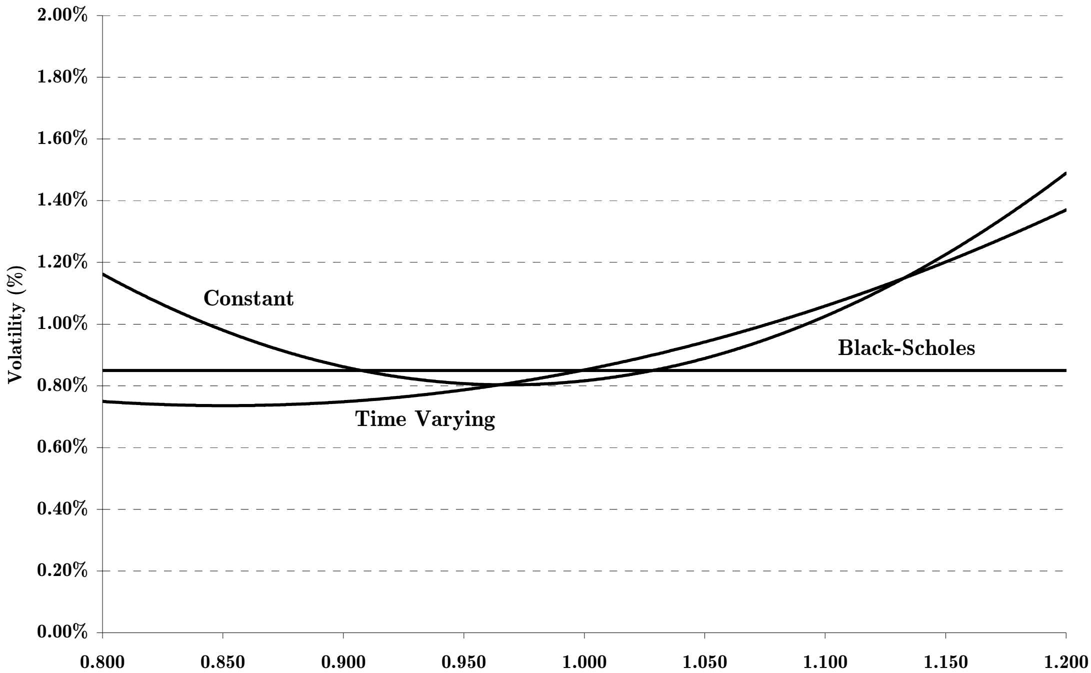
A. Implied BS volatility for $\tau = 1$



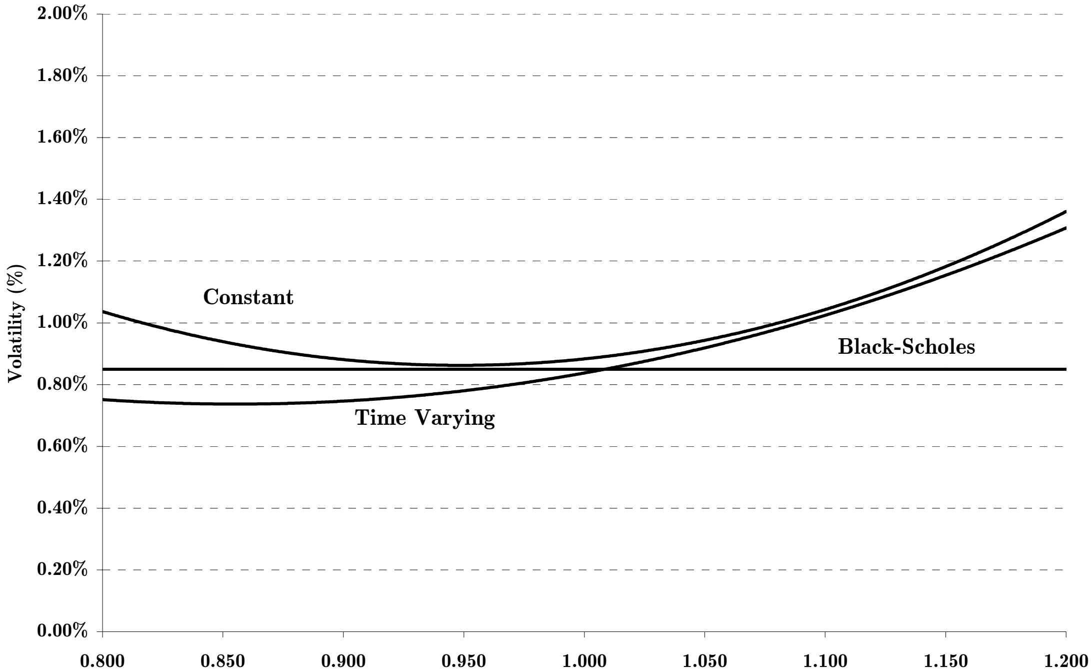
B. Implied BS volatility for $\tau = 66$



C. Implied BS volatility for $\tau = 126$



D. Implied BS volatility for $\tau = 252$



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