A GARCH Model of Inflation and Inflation Uncertainty with Simultaneous Feedback

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Abstract

We examine the relationship between inflation and inflation uncertainty using a GARCH model that allows for simultaneous feedback between the conditional mean and variance of inflation. We also derive a number of theoretical econometric results and illustrate the relevance of these results with an empirical example of the US monthly inflation process. Our results show that there is strong evidence in favour of a positive bi-directional relationship between inflation and inflation uncertainty in agreement with the predictions of economic theory.

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1. Introduction

The relationship between the inflation rate and inflation uncertainty has been the subject of considerable research in theoretical and empirical macroeconomics since the publication of Milton Friedman's (1977) Nobel lecture. Friedman (1977) analysed the causal effect of inflation on inflation uncertainty and output growth while subsequent theoretical research looked also at the opposite direction of causality, running from inflation uncertainty to the rate of inflation. Despite the considerable volume of primarily empirical research on the relationship between inflation and inflation uncertainty, the empirical literature to date has supplied scant evidence in support of the bidirectional causality between the two variables of interest that is implied by the theory. To this end, first, we purport to provide a number of theoretical econometric results from a general dynamic model of inflation with simultaneous feedback between the conditional mean and variance which nests many theoretical and empirical GARCH models of inflation. Then, we illustrate the relevance of our theoretical results with an empirical example of the US monthly inflation process and, therefore, we contribute also to the inflation-inflation uncertainty empirical literature.

Friedman (1977) and Ball (1992) have provided intuitive and formal arguments, respectively, that result in a positive influence of higher inflation on the uncertainty about inflation. The opposite type of causation between inflation and uncertainty has also been analysed in the theoretical macroeconomics literature. Cukierman and Meltzer (1986) employ the Barro-Gordon set up and show that an increase in uncertainty about money growth and inflation will increase the optimal average inflation rate because it provides an incentive to the policymaker to create an inflation surprise in order to stimulate output growth.

The use of the autoregressive conditional heteroskedasticity (ARCH) and generalised ARCH (GARCH) approaches introduced by Engle (1982) and Bollerslev (1986), respectively, allow us to proxy uncertainty using the conditional variance of unpredictable shocks to the inflation rate. In addition, the ARCH-in-mean (ARCH-M) model suggested by Engle et al. (1987) allows the econometric testing of the effect of a change in the variance of the series on the series itself. Engle (1983) and Bollerslev (1986), making use of the ARCH techniques, did not perform a statistical test of the Friedman-Ball hypothesis but only compared the estimated conditional variance series with the US average inflation rate over various time periods. Grier and Perry (1998) used the estimated conditional variance from a GARCH model and employed Granger-causality tests to test for the direction of causality between average inflation and inflation uncertainty. Baillie et al. (1996) performed these tests simultaneously in a single model by including lagged inflation in the conditional variance equation and the conditional standard deviation in the inflation equation. In particular, using US data, Grier and Perry (1998) find that inflation has a significant and positive effect on inflation uncertainty, but that increased inflation uncertainty dampens future inflation, a result that is opposite to the predictions of the Cukierman-Meltzer theory[1]. On the other hand, Baillie et al (1996) find no significant relationship between inflation and inflation uncertainty.

This study contributes to the literature on the inflation-uncertainty relationship in three ways: First, we provide a number of new theoretical econometric results on the univariate GARCH-in-mean (GARCH-M) model that includes also lagged values of inflation uncertainty in the mean equation and lagged inflation rates in the conditional variance equation. These results are: (i) We obtain the univariate ARMA representations of the inflation rate and inflation uncertainty. (ii) We use the canonical factorization of the auto/cross covariance generating functions to obtain the auto/cross covariances for the inflation rate and inflation uncertainty. (iii) We derive the condition for the existence of the second moment of the conditional variance and we give the kurtosis of the errors. We also illustrate the relevance of our results using monthly US inflation data. Second, in contrast to the majority of the existing literature (the exceptions being Baillie et al. (1996) and Grier and Perry (1998)), we test for the inflation-inflation uncertainty relationship by estimating a model of inflation with simultaneous feedback between the conditional mean and conditional variance. Third, in contrast to all previous studies mentioned in the empirical section and using US data, we provide strong evidence in favor of a positive effect of a change in inflation uncertainty on inflation, as predicted by the Cukierman-Meltzer theory. We also find strong evidence in support of the Friedman-Ball view.

The rest of the paper is outlined as follows: Section 2 provides a brief exposition of the theory behind the inflation-inflation uncertainty relationship, and our theoretical econometric model. Section 3 derives a number of theoretical results on the covariance structure of the inflation rate and inflation uncertainty. Section 4 presents our empirical approach, our results, and an interpretation. Finally, section 5 summarizes the major conclusions.

[1] Using a Component GARCH-M model of inflation that includes lagged inflation in the conditional variance, Grier and Perry (1998) estimate simultaneously the relationship between inflation and inflation uncertainty. They find that inflation has a positive effect on inflation uncertainty (the Friedman-Ball hypothesis), but uncertainty has no significant impact on inflation.

2. ARMA-GARCH-M-L Model

2.1. On the direction of causality between inflation and inflation uncertainty

Economists have appealed to the uncertainty about the future rate of inflation in order to account for the welfare loss that monetary economics has associated with inflation. Predictable inflation should not lead to welfare loss since indexation will allow agents to minimize the costs of inflation. However, uncertainty about future inflation distorts the efficient allocation of resources that is based on the price mechanism. This distortion, according to Friedman (1977) will lead to lower output. Furthermore, high inflation rates might result in more variable inflation and, hence, create more uncertainty about future inflation. As Friedman (1977, p. 466) wrote: "A burst of inflation produces strong pressure to counter it. Policy goes from one direction to another, encouraging wide variation in the actual and anticipated rate of inflation... Everyone recognises that there is great uncertainty about what actual inflation will turn out to be over any specific future interval." Combining the link of inflation to inflation uncertainty and the link of inflation uncertainty to output, we have the testable hypothesis that higher inflation leads to lower output, i.e. a positively-sloped Phillips curve.

Friedman's intuitive result has also been subsequently derived formally by Ball (1992) in an asymmetric information game where the public faces uncertainty about the type of the policymaker. The two types of policymaker differ in terms of their willingness to bear the economic costs of reducing inflation. In periods of low inflation, the tough type will apply contractionary monetary policy. Ball assumes that the two types of policymakers alternate in office in a stochastic manner. Therefore, a higher current inflation rate creates more uncertainty about the level of future inflation since it is not known whether the tough type will gain power and fight inflation.

Cukierman and Meltzer (1986) show that an increase in inflation uncertainty will raise the optimal inflation rate. However, a different outcome is derived under the stabilization motive suggested by Holland (1995). Under this scenario, if higher inflation raises inflation uncertainty, the policymaker responds by disinflating the economy in order to reduce uncertainty and the associated costs. In such a case, the effect of inflation uncertainty on the rate of inflation is negative. This is more likely to observe if, instead of examining the contemporaneous relationship between inflation and inflation uncertainty, we allow for a lag in policymaker's response and the change in the inflation rate.

2.2. The model

The simple AR(1)-GARCH(1,1)-M(0)-L(1) model is given by

$$\begin{aligned} y_t &= \alpha + \alpha_{yy}^1 y_{t-1} + \alpha_{yh}^0 h_t + \varepsilon_t,\ \varepsilon_t = h_t^{\frac{1}{2}} e_t,\ e_t/\Omega_{t-1} \sim IN(0,1) \quad (2.1) \\ h_t &= \omega + \widetilde{\beta}_{hv}^1 \varepsilon_{t-1}^2 + \widetilde{\alpha}_{hh}^1 h_{t-1} + \alpha_{hy}^1 y_{t-1}, \quad (2.2) \end{aligned}$$

where y_t stands for the rate of inflation, and h_t for its conditional variance. By including lagged inflation in the conditional variance equation, and the conditional variance in the inflation equation, we can simultaneously test the Friedman-Ball hypothesis and the Cukierman-Meltzer theory. Grier and Perry (1998) measured the conditional variance using a GARCH(2,2) specification. The GARCH(2,2) model has an ARMA(2,2) representation. In model (2.1) above, although the conditional variance has a GARCH(1,1) specification, due to the simultaneous feedback, it has an ARMA(2,2) representation (see Corollary 1).

Next we extend the simple model in equations (2.1)and (2.2) by increasing the order of the parameter polynomials in the conditional mean and the conditional variance of the inflation rate. First, we allow lagged values of the process (r lags), its conditional variance (n^* lags), and s^* lagged errors to affect the conditional mean:

$$\begin{aligned} A_{yy}(L)y_t &= \alpha + A_{yh}(L)h_t + \widetilde{B}_{y\varepsilon}(L)\varepsilon_t,\ \varepsilon_t = h_t^{\frac{1}{2}} e_t,\ e_t \sim IN(0,1) \quad (2.3a) \\ A_{yy}(L) &= -\sum_{j=0}^{r} \alpha_{yy}^j L^j,\ \alpha_{yy}^0 = -1,\ \widetilde{B}_{y\varepsilon}(L) = \sum_{j=0}^{s^*} \widetilde{\beta}_{y\varepsilon}^j L^j,\ \widetilde{\beta}_{y\varepsilon}^0 = 1, \quad (2.3b) \\ A_{yh}(L) &= \sum_{j=0}^{n^*} \alpha_{yh}^j L^j, \quad (2.3c) \end{aligned}$$

Second, we allow lagged values of the process (k^* lags), its conditional variance (p^* lags), and q^* lags of the squared errors to affect the conditional variance:

$$\begin{aligned} \widetilde{A}_{hh}(L)h_t &= \omega + A_{hy}(L)y_t + \widetilde{B}_{hv}(L)\varepsilon_t^2, \quad (2.4a) \\ \widetilde{A}_{hh}(L) &= -\sum_{j=0}^{p^*} \widetilde{\alpha}_{hh}^j L^j,\ \ \widetilde{\alpha}_{hh}^0 = -1,\ \ A_{hy}(L) = \sum_{j=1}^{k^*} \alpha_{hv}^j L^j, \quad (2.4b) \\ \widetilde{B}_{hv}(L) &= \sum_{j=1}^{q^*} \widetilde{\beta}_{hv}^j L^j \quad (2.4c) \end{aligned}$$

We refer to this model as the ARMA(r,s^*)-GARCH(p^*,q^*)-M(n^*)-L(k^*) model[2]. It is a general dynamic model with simultaneous feedback between the conditional

[2] To our knowledge the first paper on the GARCH-M-L model was written by Longstaff and Schwartz (1992).

variance and the conditional mean. This flexible framework nests the GARCH-L model if $A_{yh}(L) = 0$, the GARCH-M model[3] if $A_{hy}(L) = 0$, and the simple GARCH model if $A_{yh}(L) = A_{hy}(L) = 0$.

The use of monthly data has important implications for the order of the model and the possibility of lagged effects between inflation and its uncertainty. First, the use of monthly data in the estimation of the general model (2.3a-2.4a) above implies that the order of the AR and MA polynomials can be greater than one.[4] Therefore, we use an r-th order AR polynomial and a s-th order MA polynomial. Furthermore, the GARCH(1,1) specification of the conditional variance might be insufficient. For example, Grier and Perry (1998) used a GARCH(2,2) specification. Hence, our general model (2.4a) assumes a GARCH specification of order (p^*, q^*). In addition, the generalization of the model to allow for the contemporaneous and lagged effect in the relationship between the conditional variance of inflation and the average inflation rate is justified by the use of monthly data. Consider, for example, the case where higher inflation leads to more inflation uncertainty and associated real costs. If the Central bank responds by disinflating the economy, according to Holland's (1995) stabilization motive, it is more likely the case that the reduction in inflation will only appear with a lag in relation to the increase in uncertainty (Grier and Perry, 1998).

Note that including lagged inflation in the variance equation can cause problems with the nonnegativity of the variance. In contrast, the two-step method of Grier and Perry (1998) suffers from a contradiction: in the first step, the authors estimate the variance from a model that implies that there is no theoretical cross-correlation between the inflation rate and its variance and, in the second step, use this variance to check whether it Granger-causes the inflation rate.

The goal of this section is to provide a comprehensive methodology for the analysis of this model. First, we derive the bivariate ARMA representation of the process and its conditional variance. Second, we provide the univariate ARMA representations of the process and its conditional variance. Third, we give the general conditions for the stationarity, invertibility, and irreducibility of these

[3] The ARCH-in-mean model was introduced by Engle, Lilien, and Robins (1987). This model was used to investigate the existence of time varying term premia in the term structure of interest rates. Such time varying risk premia have beeen strongly supported by a huge body of empirical research, in interest rates (Hurn, McDonald and Moody, 1995), in forward and future prices of commodities (Hall, 1991, Moosa and Al-Loughani,1994), in industrial production (Caporale and McKierman, 1996), and especially in stock returns (Campbell and Hentscel, 1992, Glosten, Jagannathan and Runkle, 1993, Black and Fraser, 1995, Fraser, 1996, Hansson and Hordahl, 1997, Elyasiani and Mansur, 1998).

[4] For example, Grier and Perry (1998) used a 12-th order AR polynomial and Baillie et al (1996) used a 25-th order MA polynomial. In this paper, we use an autoregressive polynomial of order 24.

representations.

The GARCH(p^*, q^*)-L formulation in equation (2.4a) can readily be interpreted as an ARMA(p,q^*)-L model for the conditional variance. This ARMA representation is given in the following Corollary.

Corollary 1: *The ARMA representation of the conditional variance is given by*

$$\begin{aligned} A_{hh}(L)h_t &= \omega + A_{hy}(L)y_t + \widetilde{B}_{hv}(L)v_t, \quad v_t = \varepsilon_t^2 - h_t & (2.5a) \\ A_{hh}(L) &= \widetilde{A}_{hh}(L) - \widetilde{B}_{hv}(L) = -\sum_{j=0}^{p} \alpha_{hh}^j L^j, \quad p = \max(p^*, q^*), & (2.5b) \\ \alpha_{hh}^j &= \left\{ \begin{array}{ll} \widetilde{\alpha}_{hh}^j, & j, p^* > q^* \\ \widetilde{\beta}_{hv}^j, & j, q^* > p^* \\ \widetilde{\alpha}_{hh}^j + \widetilde{\beta}_{hv}^j, & p^*, q^* > j \end{array} \right\} & (2.5c) \end{aligned}$$

Note that v_t in equation (2.5a) is an uncorrelated term with expected value 0.

Proof: In (2.4a) we add and subtract $\widetilde{B}_{hv}(L)h_t$ and we get (2.5a). ■

In the Proposition that follows we will give the univariate ARMA representations for the process and its conditional variance. In other words, we will express y_t only as a function of lagged values of y_t, lagged values of the errors (ε_t) and lagged values of the v_t term.

Proposition 1: *The univariate ARMA representations of the process and its conditional variance are*

$$\begin{aligned} A(L)y_t &= \alpha^* + B_{y\varepsilon}(L)\varepsilon_t + B_{yv}(L)v_t, & (2.6a) \\ A(L) &= \sum_{j=0}^{f} \alpha^j L^j = \prod_{j=1}^{f}(1 - \lambda_j L), \quad f = \max(r + p, n^* + k^*) & (2.6b) \\ B_{y\varepsilon}(L) &= \sum_{j=0}^{s} \beta_{y\varepsilon}^j L^j, \ s = p + s^*, \ B_{yv}(L) = \sum_{j=1}^{n} \beta_{yv}^j L^j, \ n = n^* + q^* & (2.6c) \end{aligned}$$

$$\begin{aligned} A(L)h_t &= \omega^* + B_{h\varepsilon}(L)\varepsilon_t + B_{hv}(L)v_t & (2.7a) \\ B_{h\varepsilon}(L) &= \sum_{j=1}^{k} \beta_{h\varepsilon}^j L^j, \quad k = k^* + s^*, \quad B_{hv}(L) = \sum_{j=1}^{q} \beta_{hv}^j L^j, \ q = r + q^* & (2.7b) \end{aligned}$$

All the parameters α^*, ω^*, $\alpha^j, \beta_{y\varepsilon}^j, \beta_{yv}^j, \beta_{h\varepsilon}^j$,and β_{hv}^j together with the proof are given in Appendix A.

Thus, the ARMA-GARCH-M-L formulation in equations (2.3a) and (2.4a) is readily interpreted as an ARMA$[f, \max(n, s)]$ model for the process, and an

ARMA$[f, \max(k,q)]$ model for the conditional variance. From equations (2.6a) and (2.7a) it is apparent that the cross correlations between the process and its conditional variance are due to the fact that the error term and the v_t term are entering both equations (2.6a) and (2.7a). These cross correlations are given in the next Section. Note also that for the simple ARMA-GARCH model where there are no mean effects (from the conditional variance to the conditional mean) and no level effects (from the conditional mean to the conditional variance), the two polynomials $B_{yv}(L)$, and $B_{h\varepsilon}(L)$ are 0. This means that, since the error term is uncorrelated with the v_t term, there is no cross correlation between the process and its conditional variance. Finally, note that in the univariate representations (2.6a) and (2.7a), a measure of persistence for the inflation rate and the conditional variance is the highest root of the AR polynomial eq. (2.6b)[5].

- *Assumption 1* All the roots of the autoregressive polynomial $[A(L)]$ lie outside the unit circle (stationarity conditions for the univariate ARMA representations).

- *Assumption 2.* The polynomials $A(L)$, $B_{y\varepsilon}(L)$, and $B_{yv}(L)$ have no common left factors other than unimodular ones, i.e, if $A(L) = U(L)A_1(L)$, $B_{y\varepsilon}(L) = U(L)B^1_{y\varepsilon}(L)$ and $B_{yv}(L) = U(L)B^1_{yv}(L)$, then the common factor $U(L)$ must be unimodular (irreducibility condition for the univariate ARMA representation for the process).

- *Assumption 3.* The polynomials $A(L)$, $B_{h\varepsilon}(L)$, and $B_{hv}(L)$ have no common left factors other than unimodular ones, i.e, if $A(L) = U(L)A_1(L)$, $B_{h\varepsilon}(L) = U(L)B^1_{h\varepsilon}(L)$ and $B_{hv}(L) = U(L)B^1_{hv}(L)$, then the common factor $U(L)$ must be unimodular (irreducibility condition for the univariate ARMA representation for the conditional variance).

3. Covariance Structure

The moment structure of GARCH models is a topic that has recently attracted plenty of attention. Karanasos (1999) derived the autocovariances of the squared errors for the simple GARCH model. Karanasos (2000a) obtained the auto/cross covariances of the component variances and the aggregate variance for the Component GARCH model. Karanasos (2000b) gave the auto/cross covariances of the process and its conditional variance for the GARCH-in-mean model.

[5] In measuring the persistence of the inflation rate, Baillie et al. (1996) used the order of a fractional integrated process. In measuring the persistence of the conditional variance, Grier and Perry (1998) used the permanent component of a component GARCH model.

In this section we focus our attention on the second moment structure of the general ARMA-GARCH-M-L model. In what follows we use the canonical factorization (CF) of the autocovariance generating function (AGF) of the process and its conditional variance to derive the auto/cross correlations for the inflation rate and inflation uncertainty; we only examine the case where the roots of the AR polynomial are distinct. The goal of our method is not only theoretical purity but also the production of expressions intended for practical use.

Theorem 1: *The AGF and the autocorrelations of the inflation rate are given by*

$$g_{zy} = \frac{B_{y\varepsilon}(z)B_{y\varepsilon}(z^{-1})\sigma_\varepsilon^2 + B_{yv}(z)B_{yv}(z^{-1})\sigma_v^2}{A(z)A(z^{-1})} = \sum_{m=0}^{\infty} f_m \gamma_{ym}(z^m + z^{-m}), \quad (3.1a)$$

$$\sigma_\varepsilon^2 = E(h_t), \ \ \sigma_v^2 = 2E(h_t^2), \ \ \gamma_{ym} = cov_m(y_t) \quad (3.1b)$$

$$f_m = \left\{ \begin{array}{ll} .5 & \text{if} \ \ m = 0 \\ 1 & \text{if} \ \ m \neq 0 \end{array} \right\}, \quad \gamma_{ym} = \sum_{l=1}^{f} \zeta_{lm} \left[\xi_{lm}^{y\varepsilon}\sigma_\varepsilon^2 + \xi_{lm}^{yv}\sigma_v^2\right], \ \rho_{ym} = \frac{\gamma_{ym}}{\gamma_{y0}}, \quad (3.1c)$$

$$\zeta_{lm} = \frac{(\lambda_l)^{f+m-1}}{\prod_{k=1}^{f}(1-\lambda_l\lambda_k) \prod_{k=1,k\neq l}^{f}(\lambda_l - \lambda_k)}, \quad (3.1d)$$

$$\xi_{lm}^{y\varepsilon} = \sum_{j=0}^{s}(\beta_{y\varepsilon}^j)^2 + \sum_{d=1}^{m}\sum_{j=0}^{s-d}\beta_{y\varepsilon}^j\beta_{y\varepsilon}^{j+d}(\lambda_l^d + \lambda_l^{-d}) + \sum_{d=m+1}^{s}\sum_{j=0}^{s-d}\beta_{y\varepsilon}^j\beta_{y\varepsilon}^{j+d}(\lambda_l^d + \lambda_l^{d-2m}), \quad (3.1e)$$

$$\xi_{lm}^{yv} = \sum_{j=1}^{n}(\beta_{yv}^j)^2 + \sum_{d=1}^{m}\sum_{j=1}^{n-d}\beta_{yv}^j\beta_{yv}^{j+d}(\lambda_l^d + \lambda_l^{-d}) + \sum_{d=m+1}^{n}\sum_{j=1}^{n-d}\beta_{yv}^j\beta_{yv}^{j+d}(\lambda_l^d + \lambda_l^{d-2m}) \quad (3.1f)$$

The first and second moments of the conditional variance are given in Proposition 2 below. The proof of Theorem 1 is given in Appendix B.

Theorem 2: *The AGF and the autocorrelations of the conditional variance are*

$$g_{zh} = \frac{B_{h\varepsilon}(z)B_{h\varepsilon}(z^{-1})\sigma_\varepsilon^2 + B_{hv}(z)B_{hv}(z^{-1})\sigma_v^2}{A(z)A(z^{-1})} = \sum_{m=0}^{\infty} f_m \gamma_{hm}(z^m + z^{-m}), \quad (3.2a)$$

$$\gamma_{hm} = \sum_{l=1}^{f} \zeta_{lm} \left[\xi_{lm}^{h\varepsilon}\sigma_\varepsilon^2 + \xi_{lm}^{hv}\sigma_v^2\right], \ \rho_{hm} = \frac{\gamma_{hm}}{\gamma_{h0}} \quad (3.2b)$$

$$\xi_{lm}^{h\varepsilon} = \sum_{j=1}^{k}(\beta_{h\varepsilon}^{j})^2 + \sum_{d=1}^{m}\sum_{j=1}^{k-d}\beta_{h\varepsilon}^{j}\beta_{h\varepsilon}^{j+d}(\lambda_l^d + \lambda_l^{-d}) + \sum_{d=m+1}^{k}\sum_{j=1}^{k-d}\beta_{h\varepsilon}^{j}\beta_{h\varepsilon}^{j+d}(\lambda_l^d + \lambda_l^{d-2m}), \tag{3.2c}$$

$$\xi_{lm}^{hv} = \sum_{j=1}^{q}(\beta_{hv}^{j})^2 + \sum_{d=1}^{m}\sum_{j=1}^{q-d}\beta_{hv}^{j}\beta_{hv}^{j+d}(\lambda_l^d + \lambda_l^{-d}) + \sum_{d=m+1}^{q}\sum_{j=1}^{q-d}\beta_{hv}^{j}\beta_{hv}^{j+d}(\lambda_l^d + \lambda_l^{d-2m}) \tag{3.2d}$$

The proof of Theorem 2 is similar to that of Theorem 1.

Theorem 3: *The cross covariance generating function and the cross correlations between the inflation process and its conditional variance are*

$$g_{z,yh} = \frac{B_{y\varepsilon}(z)B_{h\varepsilon}(z^{-1})\sigma_\varepsilon^2 + B_{yv}(z)B_{hv}(z^{-1})\sigma_v^2}{A(z)A(z^{-1})} \tag{3.3a}$$

$$= \sum_{m=-\infty}^{\infty}\gamma_{yh,m}z^m,\ \gamma_{yh,m} = cov\left(y_t, h_{t-m}\right), \tag{3.3b}$$

$$\gamma_{yh,m} = \left\{\begin{array}{ll} \sum_{l=1}^{f}\zeta_{lm}\left[\xi_{lm}^{yh,\varepsilon}\sigma_\varepsilon^2 + \xi_{lm}^{yh,v}\sigma_v^2\right] & \text{if} \quad m>0 \\ \sum_{l=1}^{f}\zeta_{l|m|}\left[\xi_{l|m|}^{\varepsilon,hy}\sigma_\varepsilon^2 + \xi_{l|m|}^{v,hy}\sigma_v^2\right] & \text{if} \quad m\le 0 \end{array}\right\},\ \rho_{yh,m} = \frac{\gamma_{yh,m}}{\sqrt{\gamma_{y0}\gamma_{h0}}}, \tag{3.3c}$$

where

$$\xi_{lm}^{yh,\varepsilon} = \sum_{d=0}^{k}\sum_{j=0}^{s'}\beta_{y\varepsilon}^{j}\beta_{h\varepsilon}^{j+d}\lambda_l^d + \sum_{d=1}^{m^*}\sum_{j=1}^{k'}\beta_{h\varepsilon}^{j}\beta_{y\varepsilon}^{j+d}\lambda_l^{-d} + \sum_{d=m^*+1}^{s}\sum_{j=1}^{k'}\beta_{h\varepsilon}^{j}\beta_{y\varepsilon}^{j+d}\lambda_l^{(d-2m)},$$
$$\text{where } m^* = \min(m,s),$$

$$\xi_{lm}^{yh,v} = \sum_{d=0}^{q}\sum_{j=1}^{n'}\beta_{yv}^{j}\beta_{hv}^{j+d}\lambda_l^d + \sum_{d=1}^{m^*}\sum_{j=1}^{q'}\beta_{hv}^{j}\beta_{yv}^{j+d}\lambda_l^{-d} + \sum_{d=m^*+1}^{n}\sum_{j=1}^{q'}\beta_{hv}^{j}\beta_{yv}^{j+d}\lambda_l^{(d-2m)},$$
$$\text{where } m^* = \min(m,n),$$

$$\xi_{lm}^{\varepsilon,hy} = \sum_{d=0}^{s}\sum_{j=1}^{k'}\beta_{h\varepsilon}^{j}\beta_{y\varepsilon}^{j+d}\lambda_l^d + \sum_{d=1}^{m^*}\sum_{j=0}^{s'}\beta_{y\varepsilon}^{j}\beta_{h\varepsilon}^{j+d}\lambda_l^{-d} + \sum_{d=m^*+1}^{k}\sum_{j=0}^{s'}\beta_{y\varepsilon}^{j}\beta_{h\varepsilon}^{j+d}\lambda_l^{(d-2m)},$$
$$\text{where } m^* = \min(m,k),$$

$$\xi_{lm}^{v,hy} = \sum_{d=0}^{n}\sum_{j=1}^{q'}\beta_{hv}^{j}\beta_{yv}^{j+d}\lambda_l^d + \sum_{d=1}^{m^*}\sum_{j=1}^{n'}\beta_{yv}^{j}\beta_{hv}^{j+d}\lambda_l^{-d} + \sum_{d=m^*+1}^{q}\sum_{j=1}^{n'}\beta_{yv}^{j}\beta_{hv}^{j+d}\lambda_l^{(d-2m)},$$
$$\text{where } m^* = \min(m,q),$$

$$s' = \min(s,k-d),\ k' = \min(k,s-d),\ n' = \min(n,q-d),\ q' = \min(q,n-d)$$

The proof of Theorem 3 is given in Appendix C.

The structure of the cross covariances between the inflation series and its conditional variance given by equations (3.3a)-(3.3c) distinguishes ARMA-GARCH-M-L model from the ARMA-GARCH model, employed by Grier and Perry (1998), where the theoretical cross correlations are zero.

Several authors have studied conditions for the existence of higher order moments in GARCH models, see, for example, Ling and Li (1997), An and Chen (1998), Carrasco and Chen (1999), Giraitis, Kokoszka and Leipus (1999) and Ling (1999).

Proposition 2. *The first and second moment of the conditional variance, and the kurtosis* (k) *of the errors are given by*

$$\begin{aligned} E(h_t) &= \frac{\omega^*}{A(1)}, && (3.4a) \\ E\left(h_t^2\right) &= \frac{\left[E(h_t) + \gamma_{h0}^{\varepsilon}\right] E(h_t)}{1 - \gamma_{h0}^{v}}, && (3.4b) \\ k &= \frac{E(\varepsilon_t^4)}{[E(\varepsilon_t^2)]^2} = \frac{3E(h_t^2)}{[E(h_t)]^2}, && (3.4c) \\ \text{where } \gamma_{h0}^{\varepsilon} &= \sum_{l=1}^{f} \zeta_{l0} \xi_{l0}^{h\varepsilon},\ \gamma_{h0}^{v} = 2 \sum_{l=1}^{f} \zeta_{l0} \xi_{l0}^{hv}. \end{aligned}$$

The condition for the existence of the second moment is $1 - \gamma_{h0}^{v} > 0$. The proof of Proposition 2 follows from Theorem 2.

In this Section we presented a complete characterization of the moment structure of the inflation rate and its conditional variance for the general ARMA-GARCH-M-L model. The results in this section are useful, for example, if we want to compare the model with the simple ARMA-GARCH model. They reveal certain differences in the moment structure of both models. The coefficients in our formula are expressed in terms of the roots of the autoregressive polynomial $[A(L)]$ and the parameters of the moving average ones. We should mention that we only examine the case where the roots of the AR polynomial are distinct (the case of equal roots is left for future research). However, our methodology can be applied to even more complicated GARCH-M-L models like the Component and the Asymmetric Power GARCH-M-L models.

4. Empirical Application

4.1. The empirical evidence

Okun (1971) is one of the first studies to find that countries experiencing a high inflation rate are also countries where the standard deviation of inflation is large. The empirical approach to the inflation-uncertainty relationship faces the issue of measuring uncertainty. Two measures of uncertainty that have been used widely in empirical studies are the dispersion of survey-based individual forecasts and the moving standard deviation of inflation. The major disadvantage of these measures lies in their inability to distinguish between variability and uncertainty. In other words, they include both predictable and unpredictable variability, even though the former does not imply any uncertainty. Overall, the empirical evidence on the Friedman-Ball view is rather mixed[6]. Ball and Cecchetti (1990), Cukierman and Wachtel (1979), Evans (1991), and Grier and Perry (1998), among others, provide evidence in support of a positive influence of the average rate of inflation on inflation uncertainty. In particular, Grier and Perry (1998) find that in all G7 countries inflation has a significant and positive effect on inflation uncertainty. On the other hand, using US data, Baillie et al. (1996), Cosimano and Jansen (1988) and Fischer (1981), among others, find no significant relationship between inflation and inflation uncertainty. The opposite direction of causality, that is, from inflation uncertainty to the average rate of inflation has also been considered by the empirical literature. Baillie et al. (1996), using US data, find that a change in inflation uncertainty does not have a significant effect on the rate of inflation. However, Baillie et al. (1996) find some evidence in favor of a positive relationship using inflation data for the UK and some high-inflation countries (Argentina, Brazil, Israel). Grier and Perry (1998) obtain mixed results for the G7. In three countries (the US included), the authors find that an increase in inflation uncertainty lowers inflation, in sharp contrast to the predictions of the Cukierman-Meltzer theory. On the other hand, for Japan and France, they find that increased inflation uncertainty raises inflation.

4.2. Description of the data

In our empirical work we use seasonally adjusted time series on the US Consumer's Price Index which we obtained from the OECD Main Economic Indicators database. Our sample includes 470 monthly observations covering the period 1960M1-1999M2. Figure 1a presents the plot of the inflation rate (y_t) series (this is constructed as the first difference of the log of CPI), and Table 1a gives its

[6] Davis and Kanago (2000) provide a recent and detailed taxonomy of the results of cross-section and time-series studies on the subject.

descriptive statistics. The US inflation rate possesses significant autocorrelations, and according to the Jarque-Bera statistic it has a non-normal distribution (in particular, its distribution appears to be leptokurtic and skewed to the right). In addition, the significant $Q-statistics$ of the squared deviations of the inflation rate from its sample mean indicate the existence of ARCH effects.[7] Furthermore, application of standard unit root test shows that we can treat the inflation rate as a stationary process. The results of the Dickey-Fuller (DF) and Phillips-Perron (PP) tests are reported in Table 1b.[8]

4.3. Estimation results

We proceed with the estimation of models from the AR-GARCH-M-L family in order to take into account the serial correlation and the ARCH effects observed in our time series data, and to capture the possible simultaneous feedback between inflation and inflation uncertainty. Following a general to specific approach we estimated the following AR(24)-GARCH(1,1)-M(0)-L(1) model:

$$y_t = \underset{[0.00]}{0.7\,(10)^{-3}} + \underset{[0.00]}{0.31 y_{t-1}} - \underset{[0.55]}{0.03 y_{t-12}} - \underset{[0.28]}{0.04 y_{t-24}} + \underset{[0.00]}{460 \widehat{h}_t} + \widehat{\varepsilon}_t, \tag{4.1}$$

$$\widehat{h}_t = -\underset{[0.11]}{0.9\,(10)^{-7}} + \underset{[0.00]}{0.04 \widehat{\varepsilon}^2_{t-1}} + \underset{[0.00]}{0.84 \widehat{h}_{t-1}} + \underset{[0.00]}{0.2\,(10)^{-3}\, y_{t-1}}\,, \tag{4.2}$$

where probabilities are given in brackets. Table 2 presents some of our estimations. The above model, which is given as Model 1 in Table 2, was selected on the basis of the Akaike Information (AIC) and Schwarz (SC) criteria. According to the above estimates, the "in-mean" effect is stronger than the "level" effect: a one unit increase in the inflation rate will increase next period's inflation uncertainty by 0.2 units, while a unit increase in inflation uncertainty will increase the inflation rate by 0.46 units.[9] The positive relationship between inflation and

[7] When we use the LM test we cannot reject the presence of ARCH effects at any conventional significance levels.

[8] To check the sensitivity of our results to the order of augmentation of the unit root tests, we include both a "small" and a "large" number of lagged differenced terms in the DF regressions. Similarly, we use both a "low" and a "high" truncation lag for the Bartlett kernel in the PP tests.

[9] The unit of measurement of our monthly inflation rate series is 0.1 % , i.e. 0.001. Consequently, the unit of measurement of its variance is $(0.1\%)^2$, i.e. 0.000001. So the estimated "in-mean" and "level" effects are given by

$$\begin{aligned} 460\,(0.000001) &= (0.46)\,(0.001)\,, \\ (0.2)\,10^{-3}\,(0.001) &= 0.2\,(0.000001)\,, \end{aligned}$$

inflation uncertainty is depicted in Figure 1b which plots the inflation rate and its corresponding conditional standard deviation (eq. (4.2)).

Table 3 gives the diagnostics of the standardized residuals of the above estimated model (Model 1 in Table 2). We believe that the significant first order autocorrelation of the squared standardized residuals is due, not to the inadequacy of the model to capture the autoregressive conditional heteroskedasticity of the inflation rate, but to the presence of some outliers in the data. In fact, when we add to our model two dummies for the 8/1973 and the 3/1986 data points (see Figure 1 and Model 2 in Table 2), we obtain uncorrelated squared standardized residuals. Also note that the Jarque-Bera test for normality is very sensitive to outliers. For the model with the dummies we obtain a value for the test equal to 8.66 [prob=0.01], much lower than the one computed for the selected model. The reason that we finally prefer Model 1 to Model 2 is that, while both models exhibit similar persistence, Model 1 is better according to the *AIC* and *SC* model selection criteria (see Table 2).

4.4. Stability conditions of the estimated model

Now consider the above AR(24)-GARCH(1,1)-M(0)-L(1) model in terms of its theoretical parameters:

$$\begin{aligned} \left(1-\phi_1 L-\phi_{12}L^{12}-\phi_{24}L^{24}\right) y_t &= b+\varepsilon_t+\delta h_t, && (4.1') \\ (1-\beta^* L)\, h_t &= \omega+\alpha L v_t+\gamma L y_t, && (4.2') \end{aligned}$$

where $\beta^*=\alpha+\beta$, and $v_t=\varepsilon_t^2-h_t$. The univariate ARMA representations of the inflation rate (y_t) and its conditional variance (h_t) are

$$A(L)\, y_t = b^* + B_{y\varepsilon}(L)\,\varepsilon_t + B_{yv}(L)\, v_t, \qquad (4.3a)$$

where

$$\begin{aligned} A(L) &= 1-(\phi_1+\beta^*+\delta\gamma)\,L+\phi_1\beta^* L^2-\phi_{12}L^{12}+\beta^*\phi_{12}L^{13}-\phi_{24}L^{24}+\beta^*\phi_{24}L^{25} \\ &= \Pi_{i=1}^{25}\left(1-\lambda_i L\right), && (4.3b) \\ B_{y\varepsilon}(L) &= 1-\beta^* L, \quad B_{yv}(L)=\delta\alpha L, \quad b^*=(1-\beta^*)\, b+\delta\omega && (4.3c) \end{aligned}$$

and

$$A(L)\, h_t = \omega^* + B_{h\varepsilon}(L)\,\varepsilon_t + B_{hv}(L)\, v_t, \qquad (4.4a)$$

where

$$\begin{aligned} B_{h\varepsilon}(L) &= \gamma L,\ B_{hv}(L)=a\left(L-\phi_1 L^2-\phi_{12}L^{13}-\phi_{24}L^{25}\right), && (4.4b) \\ \omega^* &= \omega(1-\phi)+\gamma b,\ \phi=\phi_1+\phi_{12}+\phi_{24} && (4.4c) \end{aligned}$$

respectively.

We use eq.(4.3b) to check the stability of our estimated model (4.1)-(4.2). Note that the λ's we calculate in (4.3b) denote the reciprocals of the roots of the $A(L)$ polynomial. Using our estimated parameters we find one real root and twelve pairs of conjugate complex roots, all with modulus less than one (see Table 4). Therefore our selected model satisfies the stability conditions.

4.5. Autocorrelation structure of the estimated model

The autocorrelation function of the inflation rate (ρ_{ym}) is

$$\rho_{ym} = \frac{\gamma_{ym}}{\gamma_{y0}}, \ \gamma_{ym} = \gamma_{ym}^{\varepsilon} E(h_t) + \gamma_{ym}^{v} E\left(h_t^2\right), \ m \geq 0, \tag{4.5a}$$

where γ_{y0} and γ_{ym} denote the variance and mth autocovariance of y_t, respectively, and

$$\gamma_{ym}^{\varepsilon} = \sum_{i=1}^{25} \frac{\lambda_i^{24}}{\prod_{\substack{k=1 \\ k \neq i}}^{25} (\lambda_i - \lambda_k) \prod_{k=1}^{25} (1 - \lambda_i \lambda_k)} [1 + (\beta^*)^2 - 2\beta^* \lambda_i], \ m = 0, \tag{4.5b}$$

$$\gamma_{ym}^{\varepsilon} = \sum_{i=1}^{25} \frac{\lambda_i^{24+m}}{\prod_{\substack{k=1 \\ k \neq i}}^{25} (\lambda_i - \lambda_k) \prod_{k=1}^{25} (1 - \lambda_i \lambda_k)} [1 + (\beta^*)^2 - \beta^*(\lambda_i + \lambda_i^{-1})], \ m \geq 1, \tag{4.5c}$$

$$\gamma_{ym}^{v} = \sum_{i=1}^{25} \frac{\lambda_i^{24+m}}{\prod_{\substack{k=1 \\ k \neq i}}^{25} (\lambda_i - \lambda_k) \prod_{k=1}^{25} (1 - \lambda_i \lambda_k)} 2\delta^2 a^2. \tag{4.5d}$$

The first and second moments of the conditional variance h_t, $E(h_t)$ and $E\left(h_t^2\right)$, and the kurtosis coefficient (k) of the errors are

$$E(h_t) = \frac{(1-\phi)\,\omega + \gamma b}{1 - (\phi_1 + \beta^* + \delta\gamma) + \phi_1 \beta^* - \phi_{12} + \beta^* \phi_{12} - \phi_{24} + \beta^* \phi_{24}}, \tag{4.6a}$$

$$E\left(h_t^2\right) = \frac{[E(h_t) + \gamma_{h0}^{\varepsilon}]\, E(h_t)}{1 - \gamma_{h0}^{v}}, \ var(h_t) = E\left(h_t^2\right) - [E(h_t)]^2, \tag{4.6b}$$

$$k = \frac{E(\varepsilon_t^4)}{[E(\varepsilon_t^2)]^2} = \frac{3E(h_t^2)}{[E(h_t)]^2}, \tag{4.6c}$$

where $\gamma_{h0}^{\varepsilon}$ and γ_{h0}^{v} are given below by eq. (4.7b) and (4.7d), respectively.

The autocorrelation function of the conditional variance (ρ_{hm}) for $m \geq 24$ is

$$\rho_{hm} = \frac{\gamma_{hm}}{\gamma_{h0}}, \ \gamma_{hm} = \gamma_{hm}^{\varepsilon} E(h_t) + \gamma_{hm}^{v} E(h_t^2), \tag{4.7a}$$

where γ_{h0} and γ_{hm} denote the variance and mth autocovariance of h_t, respectively, and

$$\gamma_{hm}^{\varepsilon} = \sum_{i=1}^{25} \frac{\lambda_i^{24+m}}{\prod\limits_{\substack{k=1\\k\neq i}}^{25}(\lambda_i - \lambda_k) \prod\limits_{k=1}^{25}(1-\lambda_i\lambda_k)} \gamma^2, \ m \geq 0, \tag{4.7b}$$

$$\begin{aligned}\gamma_{hm}^{v} = {} & \sum_{i=1}^{25} \frac{\lambda_i^{24+m}}{\prod\limits_{\substack{k=1\\k\neq i}}^{25}(\lambda_i - \lambda_k) \prod\limits_{k=1}^{25}(1-\lambda_i\lambda_k)} 2\alpha^2 [1 + \phi_1^2 + \phi_{12}^2 + \phi_{24}^2 \\ & -\phi_1(\lambda_i + \lambda_i^{-1}) + \phi_1\phi_{12}(\lambda_i^{11} + \lambda_i^{-11}) + (\phi_{12}\phi_{24} - \phi_{12})(\lambda_i^{12} + \lambda_i^{-12}) \\ & +\phi_1\phi_{24}(\lambda_i^{23} + \lambda_i^{-23}) - \phi_{24}(\lambda_i^{24} + \lambda_i^{-24})], \ m \geq 24,\end{aligned} \tag{4.7c}$$

$$\begin{aligned}\gamma_{hm}^{v} = {} & \sum_{i=1}^{25} \frac{\lambda_i^{24}}{\prod\limits_{\substack{k=1\\k\neq i}}^{25}(\lambda_i - \lambda_k) \prod\limits_{k=1}^{25}(1-\lambda_i\lambda_k)} 2\alpha^2 \\ & [1 + \phi_1^2 + \phi_{12}^2 + \phi_{24}^2 - 2\phi_1\lambda_i + 2\phi_1\phi_{12}\lambda_i^{11} + \\ & 2(\phi_{12}\phi_{24} - \phi_{12})\lambda_i^{12} + 2\phi_1\phi_{24}\lambda_i^{23} - 2\phi_{24}\lambda_i^{24}], \ m = 0.\end{aligned} \tag{4.7d}$$

Recall that the condition for the existence of the second moment of the conditional variance is that the denominator of eq. (4.6b) is positive, i.e. $1 - \gamma_{h0}^{v} > 0$. Inserting our estimated parameters in eq. (4.7d) we get $\gamma_{h0}^{v} = 0.125$ which satisfies the above condition. Next, we use eq. (4.6a)-(4.6b) to compute the theoretical variance of h_t : $var\,(h_t) = 1.2\,(10)^{-11}$; note that the sample variance of the estimated conditional variance h_t is $1.1\,(10)^{-11}$. To compute the theoretical variance of the inflation rate $\left(\gamma_{y0}\right)$ we use eq. (4.5a), (4.5b), and (4.5d), for $m = 0$, and obtain $\gamma_{y0} = 9.82\,(10)^{-6}$. This value is very close to $9.27\,(10)^{-6}$, the sample variance of the inflation rate for the estimation period.

Furthermore, we use eq. (4.5a)-(4.5d) to compute the first 132 autocorrelations of the inflation rate which are presented in Figure 2a. We should note that the theoretical autocorrelations move quite closely with the sample autocorrelations of the inflation rate (see Figure 3). This shows that our AR-GARCH-M-L model can approximate reality quite well. We then use eq. (4.7a)-(4.7c) to compute the autocorrelations (of order 24 to 132) of h_t, ploted in Figure 2b. Observe the high correlations that characterize the uncertainty of the inflation rate process.

Finally, for the computation of the cross correlations between the inflation rate and its conditional variance we use the following results:

$$\rho_{yh,m} = \frac{\gamma_{yh,m}}{\sqrt{\gamma_{y0}\gamma_{h0}}}, \; \gamma_{hy,m} = cov(h_t, y_{t-m}) = \gamma^{\varepsilon}_{hy,m} E(h_t) + \gamma^{v}_{hy,m} E(h_t^2), \qquad (4.8a)$$

where

$$\gamma^{v}_{hy,m} = \left\{ \begin{array}{ll} \sum\limits_{i=1}^{25} \frac{\lambda_i^{24+m}}{\prod\limits_{\substack{k=1\\k\neq i}}^{25}(\lambda_i-\lambda_k)\prod\limits_{k=1}^{25}(1-\lambda_i\lambda_k)} 2\delta\alpha^2[1-\phi_1\lambda_i^{-1}-\phi_{12}\lambda_i^{-12}-\phi_{24}\lambda_i^{-24}] & m>0 \\ \sum\limits_{i=1}^{25} \frac{\lambda_i^{24+|m|}}{\prod\limits_{\substack{k=1\\k\neq i}}^{25}(\lambda_i-\lambda_k)\prod\limits_{k=1}^{25}(1-\lambda_i\lambda_k)} 2\delta\alpha^2[1-\phi_1\lambda_i-\phi_{12}\lambda_i^{12}-\phi_{24}\lambda_i^{24}] & m\leq 0 \end{array} \right\}, \qquad (4.8b)$$

$$\gamma^{\varepsilon}_{hy,m} = \left\{ \begin{array}{ll} \sum\limits_{i=1}^{25} \frac{\lambda_i^{24+m}}{\prod\limits_{\substack{k=1\\k\neq i}}^{25}(\lambda_i-\lambda_k)\prod\limits_{k=1}^{25}(1-\lambda_i\lambda_k)} [-\beta^*\gamma+\gamma\lambda_i^{-1}] & m>0 \\ \sum\limits_{i=1}^{25} \frac{\lambda_i^{24+|m|}}{\prod\limits_{\substack{k=1\\k\neq i}}^{25}(\lambda_i-\lambda_k)\prod\limits_{k=1}^{25}(1-\lambda_i\lambda_k)} [-\beta^*\gamma+\gamma\lambda_i] & m\leq 0 \end{array} \right\}. \qquad (4.8c)$$

When $m = 0$, eq. (4.8a)-(4.8c) give the theoretical instantaneous cross correlation between inflation and its conditional variance: $\rho_{hy,0} = 0.703$. This is very close to the corresponding sample value of 0.682. Figures 2c-2d present the theoretical cross correlation functions between inflation and its conditional variance. Observe the slowly decaying pattern characterizes the correlation structure in Figures 2a-2d.

5. Conclusions

We have examined the relationship between inflation and inflation uncertainty using a GARCH model that allows for simultaneous feedback between the conditional mean and variance of inflation. We have also derived a number of theoretical econometric results and illustrated the relevance of these results with an empirical example of the US monthly inflation process. Our empirical analysis, in sharp contrast with existing evidence, shows that there is a strong positive bi-directional relationship between inflation and inflation uncertainty, in agreement with the predictions of economic theory expressed by the Cukierman-Meltzer theory and the Friedman-Ball view. It would be interesting to examine the robustness of this result using data from a number of countries.

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Table 1a: Inflation rate (y_t), 2/1960 - 2/1999, 469 obs.							
Mean= 0.0037							
Maximum= 0.0179							
Minimum= -0.0055							
Standard deviation= 0.003							
Skewness= 1.0403							
Kurtosis= 4.5885							
Jarque-Bera= 133.90 [0.00]							
	Correlations of $(y_t - \overline{y})$				*Correlations of* $(y_t - \overline{y})^2$		
m	AC_m	PAC_m	$Q-Statistic$	m	AC_m	PAC_m	$Q-Statistic$
1	0.677	0.677	216.54 [0.000]	1	0.477	0.477	107.41 [0.000]
2	0.636	0.328	407.97 [0.000]	2	0.403	0.228	184.35 [0.000]
3	0.564	0.108	558.78 [0.000]	3	0.314	0.077	231.06 [0.000]
4	0.548	0.127	701.33 [0.000]	4	0.314	0.119	277.89 [0.000]
5	0.557	0.164	849.25 [0.000]	5	0.355	0.169	337.75 [0.000]
6	0.555	0.119	996.35 [0.000]	6	0.334	0.085	390.89 [0.000]
7	0.545	0.076	1138.6 [0.000]	7	0.330	0.079	442.83 [0.000]
8	0.544	0.087	1280.5 [0.000]	8	0.294	0.038	484.34 [0.000]
9	0.594	0.200	1450.1 [0.000]	9	0.336	0.118	538.53 [0.000]
10	0.550	0.012	1595.8 [0.000]	10	0.301	0.030	582.26 [0.000]
11	0.536	0.006	1734.2 [0.000]	11	0.234	-0.059	608.68 [0.000]
12	0.464	-0.094	1838.2 [0.000]	12	0.241	0.024	636.73 [0.000]
Notes:	Probabilities are given in brackets						
	The Aymptotic standard error is $1/\sqrt{T} = 0.046$						

Table 1b:Unit root tests		
$\underrightarrow{Dickey-Fuller:}$	DF(4) $= -3.69$	DF(24) $= -2.91$
$\underrightarrow{Phillips-Perron:}$	PP(4) $= -9.08$	PP(24) $= -13.92$
Notes:	The tests include a constant	
	Order of augmentation and lag truncation in parentheses	
	Critical values: -3.45 (1%), -2.87 (5%), -2.57 (10%)	

Table 2: AR-GARCH-M-L estimation, 2/62 - 2/99
Dependent variable is the inflation rate

	★Model 1	Model 2	Model 3	Model 4	Model 5
$\widehat{b}$	$0.7(10)^{-3}$ [0.00]	$0.6(10)^{-3}$ [0.00]	$0.8(10)^{-3}$ [0.00]	$0.8(10)^{-3}$ [0.00]	$0.4(10)^{-3}$ [0.00]
$\widehat{\phi}_1$	0.31 [0.00]	0.57 [0.00]	0.61 [0.00]	0.60 [0.00]	0.33 [0.00]
$\widehat{\phi}_2$					0.18 [0.00]
$\widehat{\phi}_3$					-0.01 [0.89]
$\widehat{\phi}_4$					0.02 [0.59]
$\widehat{\phi}_5$					0.10 [0.00]
$\widehat{\phi}_6$					0.09 [0.02]
$\widehat{\phi}_7$					0.01 [0.79]
$\widehat{\phi}_8$					0.03 [0.50]
$\widehat{\phi}_9$					0.21 [0.00]
$\widehat{\phi}_{10}$					0.03 [0.43]
$\widehat{\phi}_{11}$					0.05 [0.13]
$\widehat{\phi}_{12}$	-0.03 [0.54]	0.15 [0.00]	0.11 [0.00]	0.13 [0.00]	-0.07 [0.03]
$\widehat{\phi}_{24}$	-0.04 [0.28]	-0.003 [0.89]	0.03 [0.30]		-0.11 [0.00]
$\widehat{\delta}$	459.9 [0.00]	103.3 [0.02]		34.3 [0.36]	0.87 [0.98]
$d1$		0.02 [0.00]			
$d2$		-0.004 [0.00]			
$\widehat{\omega}$	$-0.9(10)^{-7}$ [0.11]	$0.6(10)^{-6}$ [0.03]	$0.8(10)^{-7}$ [0.62]	$0.5(10)^{-7}$ [0.62]	$-0.1(10)^{-7}$[0.91]
$\widehat{\alpha}$	0.04 [0.00]	0.10 [0.00]	0.21 [0.00]	0.17 [0.00]	0.13 [0.00]
$\widehat{\beta}$	0.84 [0.00]	0.84 [0.00]	0.63 [0.07]	0.80 [0.00]	0.84 [0.00]
$\widehat{\gamma}$	$0.2(10)^{-3}$ [0.00]	$0.1(10)^{-4}$ [0.27]	$0.2(10)^{-3}$ [0.00]	$0.4(10)^{-4}$ [0.00]	$0.7(10)^{-5}$ [0.40]
SC	-9.55	-9.53	-9.48	-9.46	-9.39
AIC	-9.63	-9.62	-9.55	-9.53	-9.57
$\overline{R}^2$	0.51	0.55	0.46	0.47	0.56

Notes: b is the constant term in the conditional mean of the process
the ϕ's denote the autoregressive parameters; δ is the in-mean effect
$d1$ and $d2$ are 0, 1 dummies: $d1 = 1$ for 8/73, $d2 = 1$ for 2/86-4/86
ω is the constant term in the conditional variance of the process
α and β denote the GARCH parameters
γ captures the effect of lagged inflation on its conditional variance
Probabilities are given in brackets; the ★ indicates the selected model

Table 3: Diagnostics of Model 1

Autocorrelations of standardized residuals

1	2	3	4	5	6	7	8	9	10	11	12
-0.04	0.04	-0.02	-0.04	0.06	0.08	0.01	0.01	0.12	0.03	0.07	-0.01

Autocorrelations of standardized residuals squared

1	2	3	4	5	6	7	8	9	10	11	12
0.21	-0.01	-0.001	-0.01	0.09	0.05	0.01	-0.04	0.01	0.01	0.01	0.02

Asymptotic standard error= 0.05

Jarque-Bera test= 150.7 [0.00]

Table 4: Roots of the ARMA representation of Model 1

Roots	Modulus
$\lambda_1 = -0.854\,39 - 0.118\,26i$	0.86254
$\lambda_2 = -0.854\,39 + 0.118\,26i$	0.86254
$\lambda_3 = -0.797\,89 - 0.329\,74i$	0.86334
$\lambda_4 = -0.797\,89 + 0.329\,74i$	0.86334
$\lambda_5 = -0.678\,55 - 0.534\,19i$	0.86359
$\lambda_6 = -0.678\,55 + 0.534\,19i$	0.86359
$\lambda_7 = -0.524\,11 - 0.689\,42i$	0.86602
$\lambda_8 = -0.524\,11 + 0.689\,42i$	0.86602
$\lambda_9 = -0.317\,48 - 0.806\,26i$	0.86652
$\lambda_{10} = -0.317\,48 + 0.806\,26i$	0.86652
$\lambda_{11} = -0.106\,24 - 0.864\,05i$	0.87056
$\lambda_{12} = -0.106\,24 + 0.864\,05i$	0.87056
$\lambda_{13} = 0.132\,11 - 0.86117i$	0.87124
$\lambda_{14} = 0.132\,11 + 0.86117i$	0.87124
$\lambda_{15} = 0.343\,99 - 0.806\,4i$	0.8767
$\lambda_{16} = 0.343\,99 + 0.806\,4i$	0.8767
$\lambda_{17} = 0.549\,66 - 0.683\,54i$	0.87713
$\lambda_{18} = 0.549\,66 + 0.683\,54i$	0.87713
$\lambda_{19} = 0.706\,06 - 0.530\,63i$	0.88323
$\lambda_{20} = 0.706\,06 + 0.530\,63i$	0.88323
$\lambda_{21} = 0.821\,24 - 0.317\,76i$	0.88057
$\lambda_{22} = 0.821\,24 + 0.317\,76i$	0.88057
$\lambda_{23} = 0.870\,9 - 9.276\,7 \times 10^{-2}i$	0.87583
$\lambda_{24} = 0.870\,9 + 9.276\,7 \times 10^{-2}i$	0.87583
$\lambda_{25} = 0.979\,36$	$0.979\,36$

Figure 1a
US Inflation Rate, 2/1960 - 2/1999

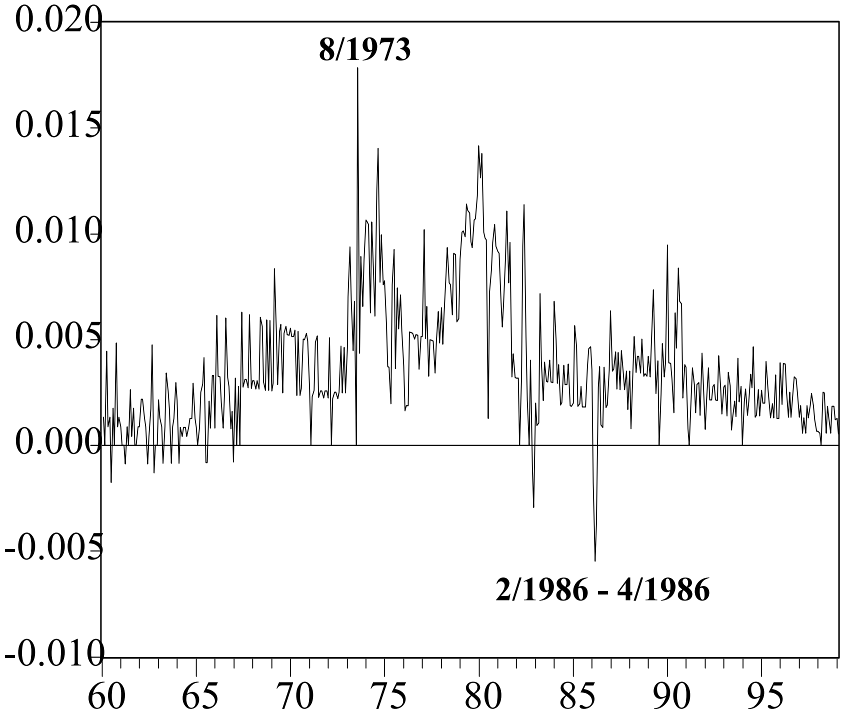


Figure 1b
Inflation Rate and its Conditional Standard Deviation
Estimation Period: 2/1962 - 2/1999

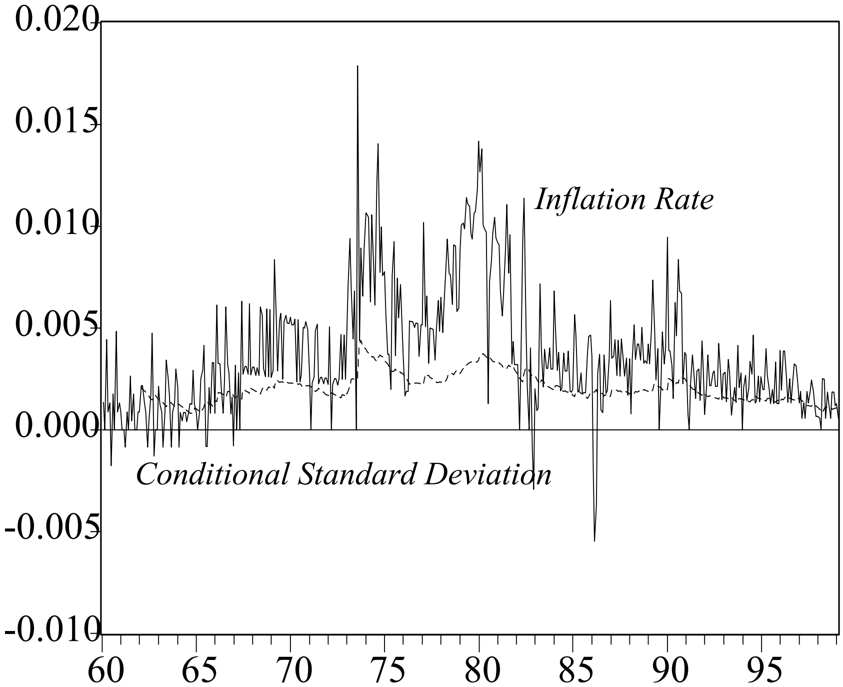


Figure 2a
Autocorrelations of Inflation

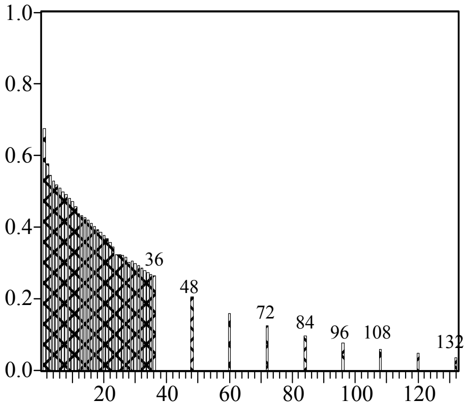


Figure 2b
Autocorrelations of Conditional Variance

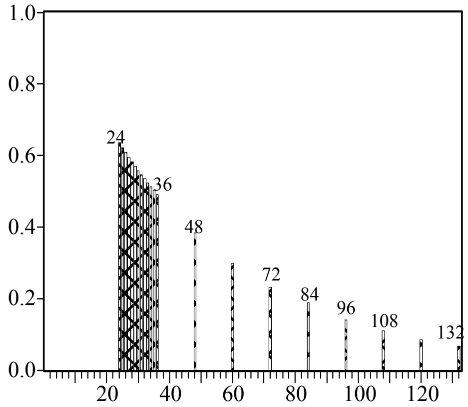


Figure 2c
Cross Correlations between
Inflation and its lagged Conditional Variance

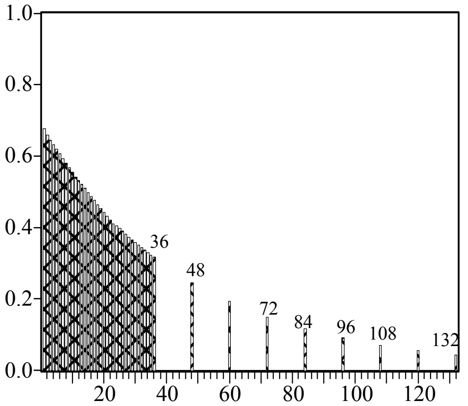


Figure 2d
Cross Correlations between
lagged Inflation and its Conditional Variance

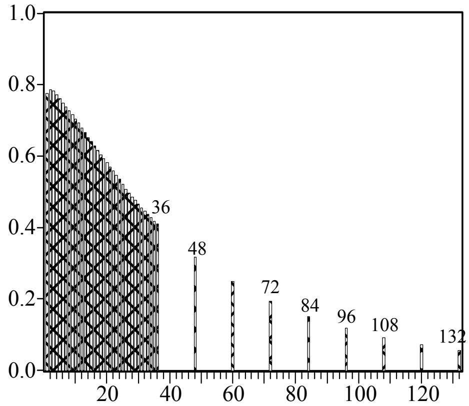
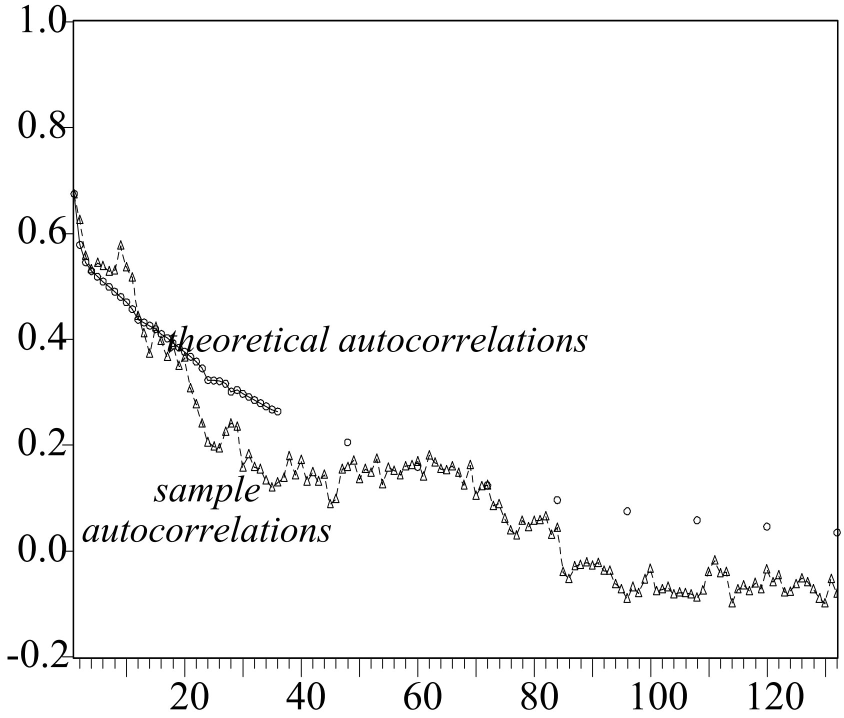


Figure 3
Theoretical and Sample Autocorrelations of Inflation



Appendix A: Proof of Proposition 1

Multiply (2.3a) by $A_{hh}(L)$ and substitute (2.5a) into (2.3a) to get

$$A_{hh}(L)A_{yy}(L)y_t = \alpha A_{hh}(1) - A_{yh}(L)\left[\omega + A_{hy}(L)y_t + \widetilde{B}_{hv}(L)v_t\right] + \widetilde{B}_{y\varepsilon}(L)A_{hh}(L)\varepsilon_t \Rightarrow$$

$$\begin{aligned}\left[A_{hh}(L)A_{yy}(L) - A_{yh}(L)A_{hy}(L)\right]y_t &= \left[\alpha A_{hh}(1) + \omega A_{yh}(1)\right] + A_{yh}(L)\widetilde{B}_{hv}(L)v_t \\ &\quad + \widetilde{B}_{y\varepsilon}(L)A_{hh}(L)\varepsilon_t \qquad \text{(A.1)}\end{aligned}$$

Multiply (2.5a) by $A_{yy}(L)$ and substitute (2.3a) into (2.5a) to get

$$A_{hh}(L)A_{yy}(L)h_t = \omega A_{yy}(1) + A_{hy}(L)\left[\alpha + A_{yh}(L)h_t + \widetilde{B}_{y\varepsilon}(L)\varepsilon_t\right] + \widetilde{B}_{hv}(L)A_{yy}(L)v_t \Rightarrow$$

$$\begin{aligned}\left[A_{hh}(L)A_{yy}(L) - A_{hy}(L)A_{yh}(L)\right]h_t &= \left[\omega A_{yy}(1) + \alpha A_{hy}(1) + A_{hy}(L)\widetilde{B}_{y\varepsilon}(L)\varepsilon_t \right.\\ &\quad + \widetilde{B}_{hv}(L)A_{yy}(L)v_t \qquad \text{(A.2)}\end{aligned}$$

Using

$$\begin{aligned}
A_{yy}(L)A_{hh}(L) &= \sum_{j=0}^{r+p}\sum_{i=\max(0,j-p)}^{\min(r,j)} \alpha_{yy}^{i}\alpha_{hh}^{j-i}L^j = \sum_{j=0}^{r+p}\alpha_{yy,hh}^{j}L^j, \\
A_{yh}(L)A_{hy}(L) &= \sum_{j=1}^{n^*+k^*}\sum_{i=\max(0,j-k^*)}^{\min(n^*,j-1)} \alpha_{yh}^{i}\alpha_{hy}^{j-i}L^j = \sum_{j=1}^{n^*+k^*}\alpha_{yh,hy}^{j}L^j, \\
A(L) &= A_{yy}(L)A_{hh}(L) - A_{yh}(L)A_{hy}(L) = \\
&= \sum_{j=0}^{r+p}\alpha_{yy,hh}^{j}L^j - \sum_{j=1}^{n^*+k^*}\alpha_{yh,hy}^{j}L^j = \sum_{j=0}^{f}\alpha^j L^j \\
&= \prod_{j=1}^{f}(1-\lambda_j L), \quad \alpha^j = \begin{cases}\alpha_{yy,hh}^{j} - \alpha_{yh,hy}^{j} & \text{if } \; r+p, n^*+k^* > j \\ \alpha_{yy,hh}^{j} & \text{if } \; j, r+p > n^*+k^* \\ -\alpha_{yh,hy}^{j} & \text{if } \; j, n^*+k^* > r+p\end{cases}, \\
f &= \max(r+p, n^*+k^*)
\end{aligned}$$

and

$$
\begin{aligned}
B_{y\varepsilon}(L) &= A_{hh}(L)\widetilde{B}_{y\varepsilon}(L) = -\sum_{j=0}^{p+s^*}\sum_{i=\max(0,j-s^*)}^{\min(p,j)} \alpha_{hh}^i \widetilde{\beta}_{y\varepsilon}^{j-i} L^j = \sum_{j=0}^{s} \beta_{y\varepsilon}^j L^j, \ s = p + s^*, \\
B_{yv}(L) &= A_{yh}(L)\widetilde{B}_{hv}(L) = \sum_{j=1}^{n^*+q^*}\sum_{i=\max(0,j-q^*)}^{\min(n^*,j-1)} \alpha_{yh}^i \widetilde{\beta}_{hv}^{j-i} L^j = \sum_{j=1}^{n} \beta_{yv}^j L^j, \ n = n^* + q^*, \\
B_{h\varepsilon}(L) &= A_{hy}(L)\widetilde{B}_{y\varepsilon}(L) = \sum_{j=1}^{k^*+s^*}\sum_{i=\max(0,j-k^*)}^{\min(s^*,j-1)} \widetilde{\beta}_{y\varepsilon}^i \alpha_{hy}^{j-i} L^j = \sum_{j=1}^{k} \beta_{h\varepsilon}^j L^j, \ k = k^* + s^*, \\
B_{hv}(L) &= A_{yy}(L)\widetilde{B}_{hv}(L) = -\sum_{j=1}^{r+q^*}\sum_{i=\max(0,j-q^*)}^{\min(r,j-1)} \alpha_{yy}^i \widetilde{\beta}_{hv}^{j-i} L^j = \sum_{j=1}^{q} \beta_{hv}^j L^j, \ q = r + q^*
\end{aligned}
$$

Equations (A.1), (A.2) give equations (2.6a) and (2.7a) where

$$
\begin{aligned}
\alpha^* &= A_{hh}(1)\alpha + A_{yh}(1)\omega \\
\omega^* &= A_{yy}(1)\omega + A_{hy}(1)\alpha
\end{aligned}
$$

Appendix B: Proof of Theorem 1

The CF of the AGF for the inflation rate (y_t) (3.1a) follows immediately from the univariate ARMA representation of y_t (2.6a) and the CF of the AGF of an ARMA model given in Nerlove, Grether and Carvalho (1979). Moreover, using

$$B_{y\varepsilon}(z)B_{y\varepsilon}(z^{-1}) = \left(\sum_{l=0}^{s}\beta_{y\varepsilon}^{l}z^{l}\right)\left(\sum_{l=0}^{s}\beta_{y\varepsilon}^{l}z^{-l}\right) = \sum_{k=0}^{s}\sum_{l=0}^{s-k} f_k\beta_{y\varepsilon}^{l}\beta_{y\varepsilon}^{l+k}(z^k+z^{-k}), \tag{B.1}$$

$$B_{yv}(z)B_{yv}(z^{-1}) = \left(\sum_{l=1}^{n}\beta_{yv}^{l}z^{l}\right)\left(\sum_{l=1}^{n}\beta_{yv}^{l}z^{-l}\right) = \sum_{k=0}^{n}\sum_{l=1}^{n-k} f_k\beta_{yv}^{l}\beta_{yv}^{l+k}(z^k+z^{-k}) \tag{B.2}$$

$$f_k = \left\{\begin{array}{lll} .5 & \text{if} & k=0 \\ 1 & \text{if} & k\neq 0 \end{array}\right\},$$

$$\begin{aligned}
\frac{1}{A(z)A(z^{-1})} &= \sum_{l=1}^{f}\frac{1}{(1-\lambda_l z)(1-\lambda_l z^{-1})}\times\frac{\lambda_l^{f-1}}{\prod\limits_{k=1,k\neq l}^{f}(\lambda_l-\lambda_k)(1-\lambda_l\lambda_k)} \\
&= \sum_{l=1}^{f}\frac{\lambda_l^{f-1}}{\prod\limits_{k=1,k\neq l}^{f}(\lambda_l-\lambda_k)(1-\lambda_l\lambda_k)}\left[\sum_{k=0}^{\infty}(\lambda_l z)^k\right]\left[\sum_{k=0}^{\infty}(\lambda_l z^{-1})^k\right] \\
&= \sum_{m=0}^{\infty}\sum_{l=1}^{f} f_m\zeta_{lm}(z^m+z^{-m})
\end{aligned} \tag{B.3}$$

into (3.1a) we get (3.1b)-(3.1f).

Appendix C: Proof of Theorem 3

The CF of the cross covariance generating function between the inflation rate (y_t) and its conditional variance (h_t), eq (3.3a) follows immediately from the univariate ARMA representations of y_t and h_t (2.6a, 2.7a) and the CF of the AGF of ARMA processes given in Sargent (1979, p. 228). Next, rewriting (B.3) we have

$$\frac{1}{A(z)A(z^{-1})} = \sum_{l=1}^{f} \frac{1}{(1-\lambda_l z)(1-\lambda_l z^{-1})} \times \frac{\lambda_l^{f-1}}{\prod\limits_{k=1,k\neq l}^{f} (\lambda_l - \lambda_k)(1-\lambda_l \lambda_k)} \tag{C.1}$$

Moreover,

$$\frac{B_{y\varepsilon}(z)B_{h\varepsilon}(z^{-1})}{(1-\lambda_l z)(1-\lambda_l z^{-1})} = \frac{1}{1-\lambda_l^2} \sum_{m=0}^{\infty} \left(\xi_{lm}^{yh,\varepsilon} z^m + \xi_{lm}^{\varepsilon,hy} z^{-m} \right) \lambda_l^m \tag{C.2}$$

From (C.1) and (C.2) we have

$$\frac{B_{y\varepsilon}(z)B_{h\varepsilon}(z^{-1})}{A(z)A(z^{-1})} = \sum_{l=1}^{f} \sum_{m=0}^{\infty} f_m \zeta_{lm} \left(\xi_{lm}^{yh,\varepsilon} z^m + \xi_{lm}^{\varepsilon,hy} z^{-m} \right) \tag{C.3}$$

Similarly

$$\frac{B_{yv}(z)B_{hv}(z^{-1})}{A(z)A(z^{-1})} = \sum_{l=1}^{f} \sum_{m=0}^{\infty} f_m \zeta_{lm} \left(\xi_{lm}^{yh,v} z^m + \xi_{lm}^{v,hy} z^{-m} \right) \tag{C.4}$$

Using equations (C.3) and (C.4) into (3.3a) we get (3.3b)-(3.3c).