PROPERTY RIGHTS AND LONG-RUN CAPITAL

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Abstract. The fact that some proprietary capital gradually falls into the public domain (*e.g.* patents) or is taxed to fund productive public spending (*e.g.* public infrastructures and the institutional framework) inefficiently decreases capital accumulation, impacting households' consumption. Specifically, for a neoclassical infinitely-lived agents economy with *constant returns to scale* the planner's steady state consumption is 4.6%-9.1% higher than the market one —for standard empirically supported parameters. For a similarly parametrised overlapping generations economy it is around 10.5%. A tax and subsidy balanced policy able to decentralise the planner's steady consists of (i) subsidising the rental rate of private capital by an amount equal to its depreciation by (ii) taxing households' net position between, on the one hand, firm and depreciated capital ownership and, on the other, borrowing against future dividends and its resale value. From standard functions and parameterisations of the OG setup it follows that the savings rate decentralising the planner's steady state is close to 61.5% —of which 1/3 in loans to firms and 2/3 in real monetary balances and assets ownership net of borrowing against the latter— and that the tax rate on household net debt is smaller the bigger are monetary real balances and debt.

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1. Introduction

Each generation passes on to subsequent generations the results of its achievements, both tangible —infrastructures, facilities, networks— and intangible —technology, know-how, institutions, organisations, culture.[1] Since their creation has required the use of resources and are moreover productive,[2] they constitute capital in a broad sense. Some of this capital is passed on through the trade or bequest of individual property rights —*e.g.* real estate and production facilities, or intellectual property rights to the extent they have not expired yet— but also another part just slides eventually into the public domain —namely expired intellectual property rights,[3] but also any physical infrastructures publicly built out of taxed private capital or savings,[4] and institutional and organisational schemes (*e.g.* governmental agencies and services, judiciary, law enforcement, urban planning,...) for the running of which output is used up. This paper explores thus the consequences for capital accumulation and consumption of at least some capital eventually escaping individual property rights.

[1]Some of them actually fall into both categories like, for instance, cities, metropolitan areas or regions —with their combined nature of public infrastructures and organisations— or public education —with its combined nature of material means, organisation, and knowledge.

[2]It suffices to think of the counterfactual of their absence or diminished level.

[3]Property rights over the technologies resulting from R+D investments are temporarily protected by law to allow the investor to get a return from the investment and, hence, supposedly to incentivise growth-enhancing R+D activities. There is a heated debate about whether the current patent system implementing those property rights is actually the most adequate for spurring innovation. For instance, Boldrin and Levine (2013) point that the evidence shows no correlation between the number of patents and productivity, and highlight that the rent-seeking nature of *patenting aims rather at preventing further innovation* from competitors, which typically builds on previous innovations. On the other hand, Gould and Gruben (1996) find that intellectual property protection is an important determinant of growth, although this seems to hinge on the openness of the country to international competition, without which it can be detrimental to growth. Also, in a Romer-style endogenous growth model Saint-Paul (2003) argues the crowding out effects of free blueprints on proprietary innovation and, hence, its negative impact on growth and welfare. At any rate, besides the issue of what drives innovation and what incentives are at play, there is the fact that technology is the result of investment, and is hence capital, but one whose property rights are protected only temporarily and thus eventually falls into the public domain.

[4]Strictly speaking, some commonly held physical capital is actually subject to property rights of state institutions —the *res publicae* and *res universitatis* of municipalities in Roman law, as opposed to things not subject to property rights at all by their nature (*res communes*) or by lack of appropriation (*res nullius*). Notwithstanding, for all purposes, I will consider it to be freely available in the public domain —at least for residents or citizens, depending on the case at hand— since the relevant feature characterising it is the fact of *not* being subject to *individual* property rights.

What I show in this paper is that the progressive drift of proprietary investments into the public domain distorts significantly capital accumulation away from the optimal level that would be chosen by a utilitarian planner unconstrained by property rights. More specifically, the model first (i) shows why the gradual drift of capital into the public domain prevents the markets to deliver, under laissez-faire, the optimal level of capital accumulation; then (ii) gives a rough assessment of the size of the distortion; and finally (iii) provides a balanced fiscal policy that allows to decentralise through the market the planner's steady state allocation. For reasons that will become clear below, this requires both to subsidise the return to savings *and to tax households' net position between firm and capital ownership and borrowing against future dividends and resale value.*

The mechanism behind the result is simple enough. Some of the capital saved by households either eventually falls into the public domain (*e.g.* patents) or is taxed and invested as capital in the public domain[5] (*e.g.* taxed savings used to fund public infrastructures, as well as the organisational and legal framework in which the economy operates).[6] As a result, firms effectively operate using not only the capital they borrow, but also the capital coming from prior private investments that has fallen, one way or another, into the public domain. Since only capital on which property rights can be enforced is remunerated,[7] savers do not take into account the impact that their loans to firms have on the future capital in the public domain, and hence on the future productivity of factors and on output through this channel.

The idea that firms may operate with more capital than the one they have to remunerate might remind of the mechanism at the heart of the contribution made in Romer (1986) regarding technology as intangible capital. Notwithstanding, the

[5]Some savings are even deliberately privately invested in capital intended, from the start, to be in the public domain, as it is the case for open source and shareware software. This paper is nonetheless not about such instances of capital deliberately accumulated to be freely available, but rather on the consequences of proprietary capital eventually sliding into the public domain.

[6]A related problem is that of firms' investment in the human capital of their employees. Such investments can be substantial, but they stop being "proprietary" for the firm if the employees quit for another job (since embodied in them). In a sense, such human capital investments by firms, while being proprietary to the employees, have the potentiality of becoming a *de facto* (excludable) public good provided by each firm to the industry.

[7]This is not to say that the productivity of public domain capital is not appropriated by anyone. It accrues firms' profits and eventually feeds into firms' owners wealth as distributed dividends. In effect, with proprietary capital eventually sliding into the public domain even constant returns to scale firms make profits. Nonetheless, even if these profits are distributed to households, the latter will still fail to internalise the effect in their saving decision —since they do not manage the firms themselves.

mechanism introduced in this paper is distinct from the one underpinning Romer (1986). Indeed, increasing returns to scale are —among others elements— a *key* ingredient to the results in Romer (1986), while the misallocation of resources that depresses capital accumulation in this paper takes place even with a neoclassical, constant returns to scale technology. In a nutshell, Romer (1986) is driven by a technological assumption that is not needed here, while the results of this paper are driven by vanishing property rights —and the impossibility to restore them, as it will be seen below. Moreover —although admittedly not as much a definitive conceptual difference as the previous one— the positive externality that investments have on everybody else's productivity is contemporaneous at every period in Romer (1986), while in the case of capital falling into the public domain the externality exerted by the latter takes place across time, and is therefore intertemporal. More substantially, there is still another crucial difference with the mechanism in Romer (1986) explained next.

By sliding into the public domain, the productivity of the capital doing so is not directly remunerated to its investors, but is instead fed into the profits of the firms operating with it for free.[8] Even in an aggregate model, where the representative household is both the lender to the firm —so that it receives the returns to privately held capital— and the owner of the firm —so that it receives the distributed profits too— and therefore receives the entirety of the productivity of the capital used by the firm (whether privately or publicly held) the channel through which this productivity is received matters for the saving decision of the household. Namely, the productivity of capital in the public domain does not incentivise savings, while that of capital privately held does. This differentiated impact would be even more obvious with heterogeneous agents of which some are lenders and others owners, or all are both but to different extents. Thus, in another crucial difference with Romer (1986), since technology is in the latter linearly homogeneous *"with respect to the factors that receive compensation"* firms do not make profits in Romer (1986) and therefore this differentiated impact of the remuneration of capital —whether as return to loans or distributed dividends— on households' saving decision cannot be captured by the framework in Romer (1986).

As a consequence, private investments differ from those that a planner able to take into account the effect of public domain capital would choose. Specifically, in the

[8]As it will be shown below, while with free entry positive profits allow firms to enter the market driving *each firms'* profits down to zero, *aggregate* profits —which amount to the unremunerated productivity of the capital in the public domain— will remain constant at a positive level and will be distributed as dividends to the owners of the firms, i.e. the households.

case of infinitely-lived agents I explicitly show below that the market accumulates too little capital —leading to miss a 4.6%-9.1% higher (!) steady state consumption that, for standard empirically supported values for the parameters, the planner would deliver. Interestingly enough, in the overlapping generations case this is, in general, only seen indirectly through the subsidy to capital returns required by the policy decentralising the planner's allocation. Nonetheless, with additional assumptions on the production and utility functions the gap can be explicitly computed in the overlapping generations setup too, pointing to the market missing a 10.5% higher steady state aggregate consumption delivered by the planner.

In order to address the problem, I provide a policy allowing to steer decentralised choices towards the planner's allocation of choice. Since at the heart of the problem there lays an expiration of property rights,[9] it might seem that a simple all-encompassing extension of property rights would be enough. Nonetheless, since this is clearly impracticable —some of this capital cannot be appropriated (institutions, organisations,...) or is not advisable to be so because, for instance, of the perverse incentives on innovation of extending indefinitely intellectual property rights, see Boldrin and Levine (2013)— it is important to devise an implementable policy that avoids running into generating additional inefficiencies. For that purpose, the policy put forward in this paper, in an overlapping generations setup, requires instead (i) to subsidise the rental rate of capital by an amount equal to the depreciation factor of the capital sliding into the public domain, and (ii) to tax households' net position between firms and depreciated capital ownership and borrowing against future dividends and resale value. While the first element of this policy —*i.e.* the subsidisation of capital returns— may be expected (although probably not the exact rate at which this needs to be done), its second element —i.e. taxing households' net position between ownership and borrowing— only makes full sense once one understands the differentiated impact on the incentives to save of, on the one hand, the return to privately held capital and, on the other, the dividends received from the productivity of the unremunerated capital in the public domain.

More specifically, while for standard functions and parameterisations in the infinitely-lived agents setup the capital returns subsidy needed to implement an initial state that the planner would keep as a steady state ranges from 30%-50% —even more to implement the best steady state. In the (richer) overlapping generations setup, it results that the subsidy needed to decentralise the planner's steady state when capital slides into the public domain must match the depreciation factor, and the savings rate is close to 61.5% of output—of which 1/3 in loans to firms and 2/3

[9]Of which there is none in Romer (1986), which is driven by increasing returns instead.

in real monetary balances and assets ownership net of borrowing against the latter. Moreover, it results too that the tax rate on household net debt needed to decentralise the planner's steady state is smaller —and converges to zero— for bigger amounts of monetary real balances and debt, allowing for a trade-off between monetary and fiscal instruments to address the public domain inefficiency.

In what follows, Section 2 presents the model with two variants for its demographics —infinitely-lived agents and overlapping generations respectively. Section 3, addressing the issue first in the infinitely-lived agents economy, establishes that the market necessarily under-accumulates capital due to part of it falling into the public domain. It provides also an assessment of the size of the inefficiency for standard, empirically supported parameter values of the model. Section 4 addresses then the question for an overlapping generations setup —along with a new assessment of the size of the inefficiency— which provides additional insight on the way the externality operates and allows to provide a policy decentralising the planner's choice as a market outcome. This is done through a subsidy on capital returns, and a tax on households' net position between firm and depreciated capital ownership, and borrowing against future dividends and resale value. Section 5 concludes.

2. The model

Consider an economy in which firms use, on top of labor, both privately owned capital in exchange of a rental rate, as well as productive inputs that they can use freely —like e.g. expired patents, public infrastructures, facilities, urban planning, or even state services like the judiciary, law enforcement, the regulatory framework, etc. Such inputs, insofar they result from uses of previous output and are productive, constitute capital, although a non-proprietary one. This capital —even if rival and susceptible of congestion— is in the public domain for any firm to benefit from.

Specifically, let N_t be the (possibly constant) population of households at t, and k_t be the *per* household amount of savings lent to firms by the representative household at period t, so that the aggregate private investment at t used in production at $t+1$ is $N_t k_t$, which depreciates each period by a factor $\delta \in (0,1)$. Proprietary capital falls in the public domain in two ways: (1) every period a share $\phi \in [0,1]$ of the accumulated private capital slides into the public domain, and (2) any remaining part of proprietary capital resulting from a private investment falls entirely into the public domain after T periods, in order to allow for capturing, for instance, the expiration of patents.[10] Output invested as capital, on the other hand, is not reversible into consumption good.

Aggregate capital K_t available for production at any given period t is therefore the aggregate saving at $t-1$ in physical capital $k_{t-1}N_{t-1}$, plus the stock δK_{t-1} of depreciated capital available at $t-1$, i.e.

$$K_t = k_{t-1}N_{t-1} + \sum_{i=2}^{+\infty} \delta^{i-1} k_{t-i} N_{t-i} \tag{1}$$

but firms need, nonetheless, to remunerate only the proprietary part K_t^p of K_t consisting of

(1) the capital resulting from the investment $N_{t-1}k_{t-1}$ of savings made at $t-1$
(2) and the depreciated capital that is still proprietary $(1-\phi)\delta K_{t-1}^p$ resulting from previous investments

that is to say,

$$\begin{aligned} K_t^p &= k_{t-1}N_{t-1} + (1-\phi)\delta K_{t-1}^p \\ &= k_{t-1}N_{t-1} + \sum_{i=2}^{T} (1-\phi)^{i-1} \delta^{i-1} k_{t-i} N_{t-i} \end{aligned} \tag{2}$$

but, crucially, firms do not need to remunerate capital resulting from prior investments that has already fallen into the public domain.

The production function $F(K_t, N_t)$ is neoclassical, *i.e.* returns to scale are constant in labor and available capital —both proprietary and in the public domain— as opposed to the overall increasing returns to scale that are the keystone of the setup considered in Romer (1986).

Finally, as for the demographics, I will consider next —for the sake of the generality of the point being made— both an infinitely-lived agents setup and an overlapping generations setup. Specifically, a normalised unit of labor is supplied inelastically by each household

(1) each period, if agents are infinitely-lived, with a constant population (*i.e.* $n = 1$) normalised to 1, so that $N_t = 1$, for all t

[10] Note that $\phi \in [0, 1]$ and $T \in \mathbb{N}$ allow for all possible patterns of this sliding in to the public domain to happen, or not, from the case of a negligible or inexistent one (if $\phi \approx 0$ and $T \approx +\infty$) to that of a virtually immediate and complete loss of property rights (if $\phi \approx 1$ and $T \approx 1$).

(2) when young, if the economy consists of 2-period-lived overlapping generations, and the cohort sizes change each period by a constant factor n, so that, for all $i = 1, 2, \ldots$

$$N_t = n^i N_{t-i} \tag{3}$$

3. Market under-accumulation: the infinitely-lived agents case

In the next subsections I will characterise the allocations of market equilibria as well as those chosen by a planner given some initial state, and then I will compare the resulting steady states.

3.1 The market allocation for the infinitely-lived agents economy.

A household behaving competitively aims at maximising its increasing and concave utility u —discounted by a factor β— under its budget constraint, choosing the sequences of consumptions c_t and savings k_t lent to firms as capital that solve

$$\begin{gathered} \max_{c_t, k_t} \sum_{t=1}^{+\infty} \beta^{t-1} u(c_t) \\ c_t + k_t \leq w_t + r_t \sum_{i=1}^{T} (1-\phi)^{i-1} \delta^{i-1} k_{t-i} + \pi_t \end{gathered} \tag{4}$$

—where ϕ is the fraction of proprietary capital that falls into the public domain each period (so that the proprietary share of any investment decreases geometrically by a factor $1-\phi$ each period), and T is the number of periods that even a reduced share of any given investment remains proprietary (so that the entirety of any remaining depreciated investment falls into the public domain after T periods)— given the sequence of aggregate profits π_t it receives as owner of the firm,[11] and the

[11] The productivity of the capital in the public domain is not remunerated to any owner of production factors, and hence feeds aggregate profits, which are distributed as dividends to firm owners —that is to say $\pi_t = F_K\left(\sum_{i=1}^{+\infty} \delta^{i-1} k_{t-i}, 1\right) \cdot \left[\sum_{i=2}^{T} \phi^{i-1} \delta^{i-1} k_{t-i} + \sum_{i=T+1}^{+\infty} \delta^{i-1} k_{t-i}\right]$. Note that even though free entry in the industry might drive *per firm* profits down to zero if an unbounded number of firms enter the market —and hence the aggregate profits above are reaped

sequences of factor prices w_t and r_t, determined at equilibrium by their marginal productivities, *i.e.*

$$
\begin{aligned}
w_t &= F_L\Big(\sum_{i=1}^{+\infty}\delta^{i-1}k_{t-i}, 1\Big)\\
r_t &= F_K\Big(\sum_{i=1}^{+\infty}\delta^{i-1}k_{t-i}, 1\Big)
\end{aligned}
\tag{5}
$$

where, for all $t = 1, 2, \dots,$ trivially $k_{t-i} = 0$ for all $i > t$, given some initial endowment $k_0 > 0$.

The household's choice therefore necessarily satisfies at equilibrium, for all t and some multipliers $\lambda_t, \lambda_{t+1} > 0$

$$
\begin{pmatrix}\beta^{t-1}u'(c_t)\\ 0\end{pmatrix} = \lambda_t\begin{pmatrix}1\\1\end{pmatrix} + \lambda_{t+1}\begin{pmatrix}0\\ -F_K^{t+1}\end{pmatrix} + \lambda_{t+2}\begin{pmatrix}0\\ -F_K^{t+2}\cdot(1-\phi)\delta\end{pmatrix} + \dots \\
+ \lambda_{t+T}\begin{pmatrix}0\\ -F_K^{t+T}\cdot(1-\phi)^{T-1}\delta^{T-1}\end{pmatrix}
\tag{6}
$$

where F_K^{t+j} stands for the marginal productivity of capital at $t+j$, *i.e.*

$$
F_K^{t+j} = F_K\Big(\sum_{i=1}^{+\infty}\delta^{i-1}k_{t+j-i}, 1\Big) \tag{7}
$$

and from where the next characterisation easily follows.

Proposition 1. *In the infinitely-lived agents economy in Section 2, in which a fraction ϕ of private capital falls into the public domain each period, and entirely after T periods, a market allocation is characterised by consumptions c_t and capital savings k_t, for all $t = 1, 2, \dots$, such that*

$$
1 = \sum_{j=1}^{T}\beta^j\frac{u'(c_{t+1})}{u'(c_t)}F_K\Big(\sum_{i=1}^{+\infty}\delta^{i-1}k_{t+j-i}, 1\Big)(1-\phi)^{j-1}\delta^{j-1} \tag{8}
$$

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by an increasing number of firms— *aggregate* profits remain nonetheless constant at a positive level regardless the number of firms, given the linear homogeneity of F. It should be noticed that, accordingly, capital in the public domain is implicitly here nonproprietary but excludable (*e.g.* commons), which applies to any capital whose use may suffer from congestion (*e.g.* infrastructure, urban networks, judiciary, police forces,... but not technology).

and the budget constraint[12] *hold for all* $t = 1, 2, \ldots,$ *given some initial endowment* $k_0 > 0$ *—and trivially* $k_{t-i} = 0$ *for all* $i > t$.

It follows from Proposition 1 above that a market steady state level of capital savings $\bar{k}$ is characterised by

$$1 = \beta F_K\Big(\frac{\bar{k}}{1-\delta}, 1\Big)\frac{1-(\beta\delta(1-\phi))^T}{1-\beta\delta(1-\phi)} \tag{9}$$

We will now compare this necessary characterisation of the market equilibria with that of the planner's choice next.

3.2 The planner's allocation for the infinitely-lived agents economy.

A planner is not constrained by property rights. It just aims at maximising the discounted utility that a representative household derives from a sequence of consumptions c_t while satisfying, at each period t, the feasibility constraint. Specifically, the planner chooses the sequence of nonnegative c_t and k_t solving

$$\begin{aligned} &\max_{c_t, k_t} \sum_{t=1}^{+\infty} \beta^{t-1} u(c_t) \\ &c_t + k_t \leq F\Big(\sum_{i=1}^{+\infty} \delta^{i-1} k_{t-i}, 1\Big) \end{aligned} \tag{10}$$

where, in each constraint, *i.e.* for all $t = 1, 2, \ldots,$ trivially $k_{t-i} = 0$ for all $i > t$, given some initial endowment $k_0 > 0$. Note that the feasibility constraint conveys the assumption of irreversibility of capital.

The planner's choice must, therefore, necessarily satisfy, for each $t = 1, 2, \ldots,$ and some positive multipliers $\lambda_t, \lambda_{t+1}, \ldots,$ the condition

$$\begin{pmatrix} \beta^{t-1} u'(c_t) \\ 0 \end{pmatrix} = \lambda_t \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \lambda_{t+1} \begin{pmatrix} 0 \\ -F_K^{t+1} \end{pmatrix} + \lambda_{t+2} \begin{pmatrix} 0 \\ -F_K^{t+2} \cdot \delta \end{pmatrix} + \ldots \tag{11}$$

where F_K^{t+j} stands for the marginal productivity of capital at $t+j$ and from which the next characterisation easily follows.

[12] Which is equivalent to the feasibility constraint.

Proposition 2. *In the infinitely-lived agents economy in Section 2, a planner's allocation is characterised by consumptions c_t and capital savings k_t, for all $t = 1, 2, \dots$, such that*

$$1 = \sum_{j=1}^{+\infty} \beta^j \frac{u'(c_{t+j})}{u'(c_t)} F_K \Big(\sum_{i=1}^{+\infty} \delta^{i-1} k_{t+j-i}, 1 \Big) \delta^{j-1} \tag{12}$$

and the feasibility constraint binding hold for all $t = 1, 2, \dots,$ given some initial $k_0 > 0$ —and trivially $k_{t-i} = 0$ for all $i > t$.

From the characterisation of Proposition 2, an initial level of capital k^* would be kept as a steady state by the planner if it satisfies

$$1 = \beta F_K \Big(\frac{k^*}{1-\delta}, 1 \Big) \frac{1}{1-\beta\delta} \tag{13}$$

Note however that k^* is not the best possible steady state, i.e. the steady state k^{**} that the planner would choose *if it was free to choose the initial state*,[13] since the latter is characterised by the first-order condition for the maximisation of steady state output net of investment —i.e. the maximisation of steady state consumption— that is to say

$$1 = F_K \Big(\frac{k^{**}}{1-\delta} \Big) \frac{1}{1-\delta} \tag{14}$$

The characterisations provided in Propositions 1 and 2 allow to compare the planner's and market possible steady state allocations next.

3.3. The planner *vs* the market steady states in the infinitely-lived agents economy.

When it comes to comparing the steady state allocations that the market and the planner would deliver for the economy, in the framework of the infinitely-lived agents economy of Section 2, the previous characterisations point to a clear-cut result: at a steady state the market accumulates less capital than the planner would —whether it is the best possible steady state or not— as the next proposition establishes.

[13] Allowing different initial capital levels is unavoidable when it comes to compare steady states.

Proposition 3. *In the infinitely-lived agents economy in Section 2, in which a fraction ϕ of private capital eventually falls into the public domain each period, and entirely after T periods, the market steady state level of capital $\bar{k}$ is smaller than the initial level k^* that the planner would keep as a steady state, which is itself smaller than the best steady state level of capital k^{**} that the planner would choose, i.e.*

$$\bar{k} < k^* < k^{**} \tag{15}$$

Proof. From the characterisations (9), (13), and (14) above and the decreasing marginal productivity of capital it follows that all three $\bar{k}$, k^*, and k^{**} are unique. Also, given that the last factor in the right-hand side of (9) —which is equal to $\sum_{j=1}^{T}\left(\beta\delta(1-\phi)\right)^{j-1}$— is smaller than the last factor in (13) —which is equal to $\sum_{j=1}^{+\infty}(\beta\delta)^{j-1}$, term by term bigger than $\sum_{j=1}^{+\infty}\left(\beta\delta(1-\phi)\right)^{j-1}$, and hence bigger than $\sum_{j=1}^{T}\left(\beta\delta(1-\phi)\right)^{j-1}$— they imply straightforwardly that $\bar{k} < k^*$ —since $\beta, \delta, \phi \in (0,1)$. Also, from (13) and (14), $\beta < 1$ implies that $k^* < k^{**}$. □

The message of Proposition 3 on the impact, in the infinitely-lived agents economy, of private capital sliding into the public domain is clear: the market accumulates too little capital. But in order to have an idea of the quantitative importance of the phenomenon under consideration, the following should shed some light on the issue.

Note first that, since all investments remaining proprietary corresponds to the case $\phi = 0$ and $T = +\infty$, one expects a gap between the planner's and the market levels of capital accumulation to appear as soon as ϕ becomes positive or T finite. Also, that in the opposite extreme case in which $\phi = 1$ or $T = 1$ —i.e. the admittedly unrealistic case in which all property rights vanish after just one period— this gap becomes huge may come at not surprise either.[14] What is surprising, nonetheless, is that even just a 1% per period slide of private capital into the public domain and a 100 periods before full disappearance of whichever remaining property rights were left still translates into a whopping 30.26% market under-accumulation compared

[14]Indeed, if the entirety of an investment ($\phi = 1$) falls into the public domain after just one period ($T = 1$) —so that savers get to be remunerated the productivity of their savings just once— then the ratio $k^{**}/\bar{k}$ reaches the value 17.74 (and even that of $k^*/\bar{k}$ reaches 14.65), i.e. the best steady state level of capital accumulation is almost 1,700% higher than the market's ! (and the initial level of capital that the planner would keep as a steady state is almost 1,400% higher).

to the best steady state level k^{**} (see Table 1a) —or even a quite considerable over 7.57% of market under-accumulation of capital relative to the initial level k^* that the planner would sustain as a steady state (see Table 1b).[15]

As a matter of fact, given that the government gross fixed capital formation can be deemed to be taxed proprietary capital forced to fall into the public domain, and given that between 1995 and 2016 it has actually hovered around 2.25% (Germany) and 3.50% (US and Eurozone excluding Germany, with a peak of 4% in 2009 and a decline towards 3% afterwards),[16] and assuming the optimal rate of investment (for a logarithmic utility) of 32.66% of GDP for the values of the parameters α et β considered, then the relevant range for ϕ is rather $6.89\% - 10.71\%$.[17] As a consequence, the size of the gap is actually substantially bigger, going from the market capital investment being between about over a half and two thirds of the initial capital that the planner would sustain as a steady state —see $\frac{k^*}{\bar{k}} - 1$ in percentage terms at Table 1b— and even more off target compared to its level at the best possible steady state —see $\frac{k^{**}}{\bar{k}} - 1$ in percentage terms at Table 1a.

[15]This values are computed for a standard share of income remunerating capital for a Cobb-Douglas production function with normalised total factor productivity $F(K,L) = K^\alpha L^{1-\alpha}$, *i.e.* $\alpha = 1/3$, and a value of $\beta = .98$ corresponding approximately to a discounting by a rate of 2%, as well as a value of $\delta = .85$ corresponding to a 15% consumption of fixed capital —or CFC, that captures in national accounts the depreciation of aggregate capital stock as the difference between the gross investment (aggregate gross fixed capital formation) and net investment (net fixed capital formation) or between the Gross National Product and Net National Product. The CFC has remained in the vicinity of 15% for major economies like the US and Germany, for instance, since the 80's (source: AMECO annual macro-economic database of the European Commission's Directorate General for Economic and Financial Affairs).

[16]Source: European Commission, AMECO database.

[17]Indeed, letting g be the share of government gross fixed capital formation, then $g = \frac{\phi k}{c+k}$ —if one assumed, quite conservatively, that this is the only way for proprietary capital to fall in the public domain— with $c = \frac{1-\alpha\beta}{\alpha\beta}k$, from which $\phi = \frac{g}{\alpha\beta}$.

Table 1a. Share in % of the best steady state capital in excess of the market steady state

columns: No. of periods that capital remains proprietary
rows: per period share of capital sliding into the public domain

	⋯ 20	30	40	50	60	70 ⋯
1%	34.51	30.86	30.35	30.27	30.26	30.26
⋮						
6%	81.37	79.51	79.35	79.33	79.33	79.33
7%	91.49	89.89	89.77	89.76	89.76	89.76
8%	101.85	100.48	100.38	100.38	100.38	100.38
9%	112.44	111.26	111.19	111.19	111.19	111.19
10%	123.24	122.24	122.19	122.19	122.19	122.19
11%	134.25	133.41	133.37	133.37	133.37	133.37
⋮						

Table 1b. Share in % of the initial capital sustained by the planner as a steady state in excess of the market steady state

columns: No. of periods that capital remains proprietary
rows: per period share of capital sliding into the public domain

	⋯ 20	30	40	50	60	70 ⋯
1%	11.08	8.07	7.65	7.59	7.58	7.57
⋮						
6%	49.78	48.24	48.11	48.10	48.10	48.10
7%	58.15	56.82	56.72	56.71	56.71	56.71
8%	66.70	65.56	65.49	65.48	65.48	65.48
9%	75.44	74.47	74.41	74.41	74.41	74.41
10%	84.36	83.54	83.49	83.49	83.49	83.49
11%	93.46	92.76	92.73	92.73	92.73	92.73
⋮						

Table 2a. Share in % of the best steady state consumption in excess of the market steady state

columns: No. of periods that capital remains proprietary
rows: per period share of capital sliding into the public domain

	⋯	20	30	40	50	60	70	⋯
⋮								
6%		4.79	4.64	4.62	4.62	4.62	4.62	
7%		5.62	5.48	5.47	5.47	5.47	5.47	
8%		6.47	6.36	6.35	6.35	6.35	6.35	
9%		7.36	7.26	7.25	7.25	7.25	7.25	
10%		8.26	8.17	8.17	8.17	8.17	8.17	
11%		9.17	9.10	9.10	9.10	9.10	9.10	
⋮								

Table 2b. Share in % of the initial consumption sustained by the planner as a steady state in excess of the market steady state

columns: No. of periods that capital remains proprietary
rows: per period share of capital sliding into the public domain

	⋯	20	30	40	50	60	70	⋯
⋮								
6%		4.20	4.05	4.04	4.03	4.03	4.03	
7%		5.02	4.89	4.88	4.88	4.88	4.88	
8%		5.88	5.76	5.76	5.76	5.76	5.76	
9%		6.75	6.66	6.65	6.65	6.65	6.65	
10%		7.65	7.57	7.56	7.56	7.56	7.56	
11%		8.56	8.49	8.49	8.49	8.49	8.49	
⋮								

More informative of the potential impact on the households' well-being, in terms of consumption, is the loss of consumption implied by the market steady state[18] given in Tables 2a (when compared to the best steady state) and 2b (when compared to the initial state sustained by the planner as a steady state). There it is shown that —for empirically reasonable values of basic parameters— the market fails to deliver the around 4%-9% additional consumption that households would be allocated by a hypothetical planner.

This simple assessment points to the existence of a huge market inefficiency, even if the figures provided are just illustrative. The goal of the exercise is clearly not precision, but rather to show that the inefficiency is substantial, pointing to a far from negligible order of magnitude. It would nonetheless be interesting to have an estimate of the actual size of the inefficiency that followed from empirical data. It is nonetheless interesting too to see that the distortions converge quite rapidly —in the number of periods T some investment remains proprietary— to a constant level. Accordingly, in the next section we will dispose of T and will allow it to be infinity, since it does not make a difference in realistic horizons.

Having now an idea of the order of magnitude and relevance of the market inefficiency, what would be necessary for the market to decentralise the best steady state? From the conditions in (9) and (13) characterising the market steady state and the initial capital that the planner would keep as steady state, it follows that subsidising the market return to capital by a factor τ such that the two conditions next are equivalent

$$\begin{aligned} 1 &= \beta F_K\Big(\frac{\bar{k}}{1-\delta},1\Big)\frac{1-(\beta\delta(1-\phi))^T}{1-\beta\delta(1-\phi)}\cdot\tau \\ 1 &= \beta F_K(\frac{k^*}{1-\delta},1)\frac{1}{1-\beta\delta} \end{aligned} \tag{16}$$

would make $\bar{k}$ and k^* coincide. That is to say, the decentralisation of the initial capital that the planner would keep as steady state requires subsidising the return

[18]That is to say, the percentage excess over 1 of

$$\frac{c^{**}}{\bar{c}} = \frac{\left(\frac{k^{**}}{1-\delta}\right)^{\frac{1}{3}} - k^{**}}{\left(\frac{\bar{k}}{1-\delta}\right)^{\frac{1}{3}} - \bar{k}} \text{ and } \frac{c^{*}}{\bar{c}} = \frac{\left(\frac{k^{*}}{1-\delta}\right)^{\frac{1}{3}} - k^{*}}{\left(\frac{\bar{k}}{1-\delta}\right)^{\frac{1}{3}} - \bar{k}}$$

respectively.

to capital by a factor

$$\tau = \frac{1}{1-\beta\delta}\frac{1-\beta\delta(1-\phi)}{1-(\beta\delta(1-\phi))^T} \tag{17}$$

which for the values of the parameters considered ($\beta = .98$ and $\delta = .85$) requires the subsidy rate $\tau - 1$ to be in percentage terms as shown in Table 3a —depending on the value of ϕ within the empirically relevant range 6%-11% argued above (see footnote 17). It can be seen in the table that such a decentralisation requires to distort the representative household's choice subsidising the return to savings by about 30%-55% (!), and pay for it by means of some non-distortionary lump-sum tax on income —Table 3b shows the even higher range (45%-72%!!) of subsidies necessary to decentralise the best steady state.

The enormity of the intervention is obviously due to the minimalist character of the framework, which asks from the savings in physical capital —the only possible means of saving here— to do all the heavy-lifting. Addressing, in Section 4 next, this same question in an overlapping generations setup instead —which allows for the introduction of other means of saving and even borrowing— the policy allowing to offset in the market allocation the inefficiency arising from capital sliding into the public domain will take a less extreme form as for the tax rate needed to finance the subsidy which will need to match the depreciation factor. The richer model will provide, too, more insight about why households interacting through the market miss so spectacularly so much surplus.

Table 3a. Subsidy in % to the return to savings decentralising the initial capital k^* sustained by the planner as a steady state

columns: No. of periods that capital remains proprietary
rows: per period share of capital sliding into the public domain

	⋯	20	30	40	50	60	70	⋯
⋮								
6%		30.91	30.01	29.94	29.93	29.93	29.93	
7%		35.74	34.98	34.92	34.92	34.92	34.92	
8%		40.59	39.95	39.91	39.90	39.90	39.90	
9%		45.46	44.93	44.89	44.89	44.89	44.89	
10%		50.35	49.91	49.88	49.88	49.88	49.88	
11%		55.26	54.89	54.87	54.87	54.87	54.87	
⋮								

Table 3b. Subsidy in % to the return to savings decentralising the best steady state k^{**}

columns: No. of periods that capital remains proprietary
rows: per period share of capital sliding into the public domain

	⋯	20	30	40	50	60	70	⋯
⋮								
6%		45.75	44.75	44.66	44.65	44.65	44.65	
7%		51.12	50.28	50.21	50.21	50.21	50.21	
8%		56.52	55.81	55.76	55.76	55.76	55.76	
9%		61.95	61.35	61.32	61.31	61.31	61.31	
10%		67.39	66.90	66.87	66.87	66.87	66.87	
11%		72.85	72.44	72.42	72.42	72.42	72.42	
⋮								

Because of the need to distinguish —in the 2-period-lived (say, young and old) representative agent overlapping generations economy in Section 2— variables relating to different generations as well as time periods, we will use superscript t to identify t's generation choice variables like, among others, the intertemporal profile of consumption c_0^t, c_1^t when young and old respectively, or the amount k^t lent to firms by generation t's representative household for production at $t+1$.

As previously, I will characterise next the best steady state that the planner —unconstrained by property rights— would choose,[19] and I will compare it to the market steady state of the economy. From the systems of equations characterising the optimal and the market steady states will follow the policy that is necessary to correct the depressing effect on capital accumulation of capital sliding into the public domain in an overlapping generations setup.

4.1 The planner's problem in the overlapping generations economy.

A utilitarian planner would choose an allocation of each period's output —between consumption for the agents alive in that period and investment for future production— that maximises a weighted sum of the utilities of all households under each period's feasibility constraint, expressed below in *per* young terms, for a population growth factor $n > 1$,[20]

$$\begin{gathered} \max_{c_0^t, c_1^t, k^t \geq 0} \sum_{t=1}^{+\infty} \eta^{t-1} \big(u(c_0^t) + \beta u(c_1^t) \big) \\ c_0^t + \frac{c_1^{t-1}}{n} + k^t \leq F\Big(\frac{1}{\delta} \sum_{i=1}^{+\infty} (\frac{\delta}{n})^i k^{t-i}, 1 \Big) \end{gathered} \tag{18}$$

—with $k^{t-i} = 0$ for $i > t$ trivially— given some initial c_1^0, k^0, a discount factor η for future generations, the households' own discounting β of old age utility, and the depreciation/obsolescence factor δ for capital in the public domain.

[19] As opposed to what happened in the infinitely-live agents economy, for the overlapping generations setup the best steady state coincides with the initial state that the planner would keep as a steady state (see the proof of Proposition 5 below).

[20] As a matter of fact, everything next holds true for a constant or even decreasing population as long as the latter does not decrease too fast or, more specifically, if $n > \delta$.

From the problem above follows the next characterisation of the planner's choice linking, on the one hand, the contribution of savings k^t at any given period t to the marginal productivity of capital at all future periods $t+j$, for all $j = 1, 2, \ldots$ to, on the other hand, the marginal rates of intertemporal substitution of consumption for all agents between t and each $t+j$.

Proposition 4. *In the overlapping generations economy in Section 2, a planner's allocation is characterised by intertemporal consumption profiles c_0^t, c_1^t and capital savings k^t, for all $t = 1, 2, \ldots$, such that*

$$1 = \frac{1}{\delta} \sum_{j=1}^{+\infty} \left[F_K \Big(\frac{1}{\delta} \sum_{i=1}^{+\infty} (\frac{\delta}{n})^i k^{t+j-i}, 1 \Big) (\delta\beta)^j \prod_{h=0}^{j-1} \frac{u(c_1^{t+h})}{u(c_0^{t+h})} \right] \tag{19}$$

and the feasibility constraint binding hold for all $t = 1, 2, \ldots$ given some c_1^0, k^0 —and trivially $k^{t-i} = 0$ for $i > t$.

Proof. The solution to the planner's problem is necessarily characterised, for some $\lambda^{t+i} > 0$ with $i = 0, 1, 2, \ldots$, by

$$\begin{aligned} \begin{pmatrix} \eta^{t-1} u'(c_0^t) \\ \eta^{t-1} \beta u'(c_1^t) \\ 0 \end{pmatrix} = \lambda^t \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} &+ \lambda^{t+1} \begin{pmatrix} 0 \\ \frac{1}{n} \\ -\frac{1}{\delta} F_K \big(\frac{1}{\delta} \sum_{i=1}^{+\infty} (\frac{\delta}{n})^i k^{t+1-i}, 1 \big) \frac{\delta}{n} \end{pmatrix} \\ &+ \lambda^{t+2} \begin{pmatrix} 0 \\ 0 \\ -\frac{1}{\delta} F_K \big(\frac{1}{\delta} \sum_{i=1}^{+\infty} (\frac{\delta}{n})^i k^{t+2-i}, 1 \big) (\frac{\delta}{n})^2 \end{pmatrix} \\ &+ \ldots \end{aligned} \tag{20}$$

for all $t = 1, 2, \ldots$, that is to say, by the following conditions on the marginal rates of substitution

(1) within generations

$$\frac{1}{\beta} \frac{u'(c_0^t)}{u'(c_1^t)} = \frac{\lambda^t}{\lambda^{t+1}} \cdot n \tag{21}$$

(2) across generations

$$\frac{u'(c_0^t)}{u'(c_0^{t+i})} = \eta^i \frac{\lambda^t}{\lambda^{t+i}} \tag{22}$$

and

$$\frac{1}{\beta} \frac{u'(c_0^t)}{u'(c_1^{t+i})} = \eta^i \frac{\lambda^t}{\lambda^{t+i}} n \tag{23}$$

on top of

$$1 = \frac{1}{\delta} \sum_{j=1}^{+\infty} \left[F_K\Big(\frac{1}{\delta} \sum_{i=1}^{+\infty} (\frac{\delta}{n})^i k^{t+j-i}, 1 \Big) (\frac{\delta}{n})^j \frac{\lambda^{t+j}}{\lambda^t} \right] \tag{24}$$

and

$$c_0^t + \frac{c_1^{t-1}}{n} + k^t = F\Big(\frac{1}{\delta} \sum_{i=1}^{+\infty} (\frac{\delta}{n})^i k^{t-i}, 1 \Big). \tag{25}$$

The necessary condition obtains then from repeated direct substitutions of the intra-generational intertemporal marginal rate of substitution (21) into (24). □

From the previous characterisation, we can obtain in Proposition 5 next the characterisation of the allocation that a planner treating equally all generations would optimally choose —which in this case, for the overlapping generations economy, coincides with the initial level of capital and intertemporal consumption profile that the planner would choose to keep as a steady state (see the proof of Proposition 5).

Proposition 5. *In the overlapping generations economy in Section 2, the egalitarian planner's steady state is characterised by the unique intertemporal consumptions profile and savings c_0^*, c_1^*, k^* solving*

$$\begin{aligned} \frac{1}{\beta} \frac{u'(c_0)}{u'(c_1)} &= n = F_K(\frac{k}{n-\delta}, 1) + \delta \\ c_0 + \frac{c_1}{n} + k &= F(\frac{k}{n-\delta}, 1) \end{aligned} \tag{26}$$

given n, δ.

Proof. We will see first that the system above characterises any steady state, and then we will see that there is a solution to the system and only one.

From Proposition 4 and its proof, the symmetric limit allocation resulting from the planner treating all generations increasingly equally —so that $\frac{u'(c_0^t)}{u'(c_0^{t+i})} \to 1$ for all t and all i as $\eta \to 1$,[21] and hence so that $\frac{\lambda^t}{\lambda^{t+i}} \to 1$— is necessarily characterised by[22]

[21]Obtaining this characterisation from that in Proposition 4 requires indeed the argument in the limit as $\eta \to 1$, since for an equal weight for all generations in the planner's problem, i.e. $\eta = 1$, the planner's objective is not well defined.

[22]This implies $\delta\beta\frac{u'(c_1)}{u'(c_0)} < 1$ whenever $n > 1$. Indeed, equivalently $\beta\frac{u'(c_1)}{u'(c_0)} = \frac{1}{n} < 1$, since $n > 1$,

$$\frac{1}{\beta}\frac{u'(c_0)}{u'(c_1)} = n \tag{27}$$

and[23]

$$1 = \frac{1}{\delta}F_K\Big(\frac{k}{n-\delta},1\Big)\sum_{j=1}^{+\infty}\Big(\delta\beta\frac{u'(c_1)}{u'(c_0)}\Big)^j \tag{28}$$

that is to say —replacing the series by its value, since $\delta\beta\frac{u'(c_1)}{u'(c_0)} < 1$, and rearranging terms—

$$\frac{1}{\beta}\frac{u'(c_0)}{u'(c_1)} = F_K\Big(\frac{k}{n-\delta},1\Big) + \delta \tag{29}$$

holds, along with the feasibility constraint.

Note that the solution to the system (26) is locally unique since it is a regular zero of the left-hand side of the planner's steady state equations

$$\begin{aligned} u'(c_0) - n\beta u'(c_1) &= 0 \\ F_K(\frac{k}{n-\delta},1) + \delta - n &= 0 \\ c_0 + \frac{c_1}{n} + k - F(\frac{k}{n-\delta},1) &= 0 \end{aligned} \tag{30}$$

In effect,

$$\begin{vmatrix} u''(c_0) & -n\beta u''(c_1) & 0 \\ 0 & 0 & F_{KK}(\frac{k}{n-\delta},1)\frac{1}{n-\delta} \\ 1 & \frac{1}{n} & 1 - F_K(\frac{k}{n-\delta},1)\frac{1}{n-\delta} \end{vmatrix} = \\ -F_{KK}(\frac{k}{n-\delta},1)\frac{1}{n-\delta}\Big[n\beta u''(c_1) + \frac{1}{n}u''(c_0)\Big] < 0. \tag{31}$$

But it is globally unique too, since if c_0, c_1, k and c_0', c_1', k' were two distinct steady states for the planner, then necessarily $k = k'$ —since the (injective) marginal productivity of capital must match $n - \delta$ for both of them— and $c_0 < c_0'$ would imply $c_1 < c_1'$, which cannot be —since the *per* young aggregate consumption each

from which the inequality follows given that $\delta < 1$ too. Therefore, the series next in (28) is convergent.

[23]Since $K_t = \sum_{i=1}^{+\infty}\delta^{i-1}k^{t-i}N_{t-i}$ and $N_t = n^i N_{t-i}$, then $\frac{K_t}{N_t} = \frac{1}{\delta}\sum_{i=1}^{+\infty}\left(\frac{\delta}{n}\right)^i k^{t-i}$ which at a steady state becomes $\frac{k}{n-\delta}$ since $\delta < n$.

period must match the common $F(\frac{k}{n-\delta},1)-k$. Therefore $c_0 = c_0'$ and from the feasibility constraint $c_1 = c_1'$ too. As a consequence of its uniqueness, according to the problem (18), it is also the initial level of capital and intertemporal profile of consumptions that the planner would choose to keep as steady state for $\eta \to 1$.

The existence follows from the fact that the equations (26) are also the first-order conditions of the problem characterising the optimal steady state chosen by a planner maximising the representative agent's utility —and, therefore, treating all generations equally—

$$\max_{c_0,c_1,k\geq 0} u(c_0)+\beta u(c_1)$$
$$c_0+\frac{c_1}{n}+k\leq F(\frac{k}{n-\delta},1) \tag{32}$$

for which the existence of a solution is guaranteed by the usual differentiably strict concavity of u —which ensures the continuity of the planner's objective and the strict convexity of the utility upper contour sets— and the strict concavity of F with respect to capital —which makes the constrained set of the planner to be compact. □

In the next sections I will characterise now, from the firms' and households' optimising behavior, the market steady state.

4.2. The competitive firms problem.

The stock K_t of capital available for production at t is the aggregate savings of generation $t-1$ in physical capital, $K^{t-1} = k^{t-1}N_{t-1}$, plus the stock δK_{t-1} of depreciated capital available for production at $t-1$, *i.e.*

$$\begin{aligned} K_t &= K^{t-1}+\delta K_{t-1} \\ &= k^{t-1}N_{t-1}+\sum_{i=1}^{+\infty}\delta^i k^{t-1-i}N_{t-1-i} \end{aligned} \tag{33}$$

or, in *per* young terms,

$$\frac{K_t}{N_t}=\frac{k^{t-1}}{n}+\frac{1}{n}\sum_{i=1}^{+\infty}(\frac{\delta}{n})^i k^{t-1-i} \tag{34}$$

Only the proprietary part of the stock of capital available for production at t

$$
\begin{aligned}
K_t^p &= K^{t-1} + (1-\phi)\delta K_{t-1}^p \\
&= k^{t-1}N_{t-1} + \sum_{i=1}^{+\infty}(1-\phi)^i\delta^i k^{t-1-i}N_{t-1-i}
\end{aligned} \tag{35}
$$

is remunerated by firms, or in *per* young terms

$$
\frac{K_t^p}{N_t} = \frac{k^{t-1}}{n} + \frac{1}{n}\sum_{i=1}^{+\infty}(1-\phi)^i(\frac{\delta}{n})^i k^{t-1-i} \tag{36}
$$

—i.e. the entirety of the first term $\frac{k^{t-1}}{n}$, saved at $t-1$ by the old at t, and the fractions of depreciated capital (accumulated through all previous loans from households to firms) in each of the terms of the sum in the second term in (34) that remain proprietary and were bought at $t-1$ from the previous generation by the old at t— while the remaining fractions of depreciated capital now in the public domain is not, so that the firm does not have to remunerate the fraction of capital

$$
K_t - K_t^p = \sum_{i=1}^{+\infty}[1-(1-\phi)^i]\delta^i k^{t-1-i}N_{t-1-i} \tag{37}
$$

in the public domain that it uses. At equilibrium, factor prices are hence determined by the marginal productivities

$$
\begin{aligned}
r_{t+1} &= F_K(K_{t+1}, N_{t+1}) \\
w_t &= F_L(K_t, N_t)
\end{aligned} \tag{38}
$$

As a consequence, firms make at t the following aggregate profits

$$
\pi_t = F_K(K_t, N_t)(K_t - K_t^p) \tag{39}
$$

or, equivalently, the *per* young aggregate profits

$$
\frac{\pi_t}{N_t} = F_K\Big(\frac{k^{t-1}}{n} + \frac{1}{n}\sum_{i=1}^{+\infty}(\frac{\delta}{n})^i k^{t-1-i}, 1\Big)\,\frac{1}{n}\sum_{i=1}^{+\infty}[1-(1-\phi)^i](\frac{\delta}{n})^i k^{t-1-i} \tag{40}
$$

according to (34) and (37).

It is worth reminding now[24] that —because of the linear homogeneity of the production function— since *aggregate* profits are positive at every period, free entry of firms in the market drives, at any given t, the level of *each firm*'s profits to zero, but not the level of *aggregate* profits π_t, which remains positive as the product of the positive marginal productivity of capital and the positive amount of capital in the public domain. As a result, the ownership of the firms —which entitles to being distributed the positive dividends— is traded across generations.

4.3 Households' problem in the overlapping generations economy.

As a consequence of the firms distributing as dividends the profits obtained from the non remunerated productivity of the capital in the public domain, firm ownership has a return, and can therefore be used by households as a means of saving. The representative household born at t can therefore now transfer wealth *from its first period into the second* in three ways: (i) lending to firms to get at $t+1$ the return r_{t+1} *per* unit of capital lent, (ii) holding real monetary balances[25] and, *moreover*, (iii) taking a stock in the ownership of firms —in order to be distributed an equal share d_{t+1} of the aggregate profits π_{t+1} made by firms at $t+1$ so that $d_{t+1} = \pi_{t+1}/N_t$ and obtain the resale value of the firms at $t+1$— as well as in the ownership of the fraction of depreciated capital that remains proprietary. Besides, we are going to assume that households can also transfer wealth *from their second period into the first* by (iv) borrowing from perfectly competitive financial intermediaries operating through the lives of all generations and that lend back to every generation the funds reimbursed by the contemporaneous old generation.

Thus, let k^t be the amount lent by the representative household born at t to firms, m^t be the household real balances, and s^t be the net saving in assets other than these two, that is to say the *net position* resulting from, on the one hand, investing in the ownership of the firm and the depreciated capital already accumulated and, on the other hand, borrowing from the competitive financial intermediaries against the future income from firm ownership, i.e. distributed dividends and resale value.

[24] See footnote 11 above.

[25] Money is introduced in the model for the benchmark equilibrium to be optimal. In effect, in the Diamond (1965) setup that is at the foundation of the current one, in the absence of a bubbly asset —i.e. without fundamental value— in which to be able to save, there is no hope for the market to implement the planner's choice, independently of whether the additional effect of public domain capital studied here is included or not. The reason is that it is the presence of such an asset which allows the market allocation to replicate the planner's link between the return to capital *and the population growth factor*. In more precise terms, generically, no non-monetary equilibrium can decentralise the planner's allocation, neither in Diamond (1965) nor in this paper's setup.

If $s^t > 0$, household t is therefore investing in the ownership of the firm and accumulated capital more than it may be borrowing from its second period income. If $s^t < 0$ instead, household t is rather borrowing more from its second period income than it is investing in the ownership of the firm and accumulated capital, and hence, effectively, transferring wealth from the second period to the first, which capital or money does not allow.[26]

Household t's choice must therefore satisfy the budget constraints

$$\begin{aligned} c_0^t + k^t + s^t + m^t &\leq w_t \\ c_1^t \leq r_{t+1}\left[k^t + \sum_{i=1}^{+\infty}(1-\phi)^i(\frac{\delta}{n})^i k^{t-i}\right] &+ d_{t+1} + s^{t+1}n + \frac{p_t}{p_{t+1}}m^t \end{aligned} \tag{41}$$

—where d_{t+1} is the *per* owner distributed profits— given the wage w_t, the rental rate of capital r_{t+1}, the level of prices during the household's lifetime p_t, p_{t+1}, the profits made by firms when old π_{t+1}, the *per* young net position s^{t+1} of generation $t+1$,[27] and the population growth factor n. Moreover, the household's net position of savings in firm ownership and depreciated cumulated capital minus borrowing against the future profits and resale value, s^t, must —for the household to invest in the firm and capital at all— entitle to a present value of the net revenue from firm and depreciated cumulated capital ownership that matches the return it would have as a loan to firms instead, *i.e.*

$$r_{t+1}s^t = r_{t+1}\sum_{i=1}^{+\infty}(1-\phi)^i(\frac{\delta}{n})^i k^{t-i} + d_{t+1} + s^{t+1}n \tag{42}$$

[26]For the ease of its interpretation, a *positive* s^t can be thought of as the amount payed by each household born at t to the households born at $t-1$ for the firm ownership, i.e. for the right to receive its dividends and the value of its resale to its n children paying each s^{t+1}. The claim on future dividends, and hence the possibility to borrow against them, is what allows to extend the interpretation of s^t to that of a net position that can be negative as well as positive.

[27]Note that, according to the interpretation of s^t as a net position, the amount received or reimbursed at $t+1$ by household t will match at equilibrium the aggregate of the (positive or negative, respectively) net positions of its n children.

Therefore, household t solves the problem

$$\max_{c_0^t, c_1^t, k^t, m^t \geq 0, s^t \in \mathbb{R}} u(c_0^t) + \beta u(c_1^t)$$
$$c_0^t + k^t + s^t + m^t \leq w_t$$
$$c_1^t \leq r_{t+1}\left[k^t + \sum_{i=1}^{+\infty}(1-\phi)^i(\frac{\delta}{n})^i k^{t-i}\right] + d_{t+1} + s^{t+1}n + \frac{p_t}{p_{t+1}}m^t \tag{43}$$
$$r_{t+1}s^t = r_{t+1}\sum_{i=1}^{+\infty}(1-\phi)^i(\frac{\delta}{n})^i k^{t-i} + d_{t+1} + s^{t+1}n$$

the solution of which is necessarily characterised by the first-order conditions

$$\begin{pmatrix} u'(c_0^t) \\ \beta u'(c_1^t) \\ 0 \\ 0 \\ 0 \end{pmatrix} = \lambda_0^t \begin{pmatrix} 1 \\ 0 \\ 1 \\ 1 \\ 1 \end{pmatrix} + \lambda_1^t \begin{pmatrix} 0 \\ 1 \\ -r_{t+1} \\ -\frac{p_t}{p_{t+1}} \\ 0 \end{pmatrix} + \mu^t \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ -r_{t+1} \end{pmatrix} \tag{44}$$

for some multipliers $\lambda_0^t, \lambda_1^t, \mu^t > 0$, along with the constraints binding, or equivalently

$$\frac{1}{\beta}\frac{u'(c_0^t)}{u'(c_1^t)} = \frac{p_t}{p_{t+1}} = r_{t+1}$$
$$c_0^t + k^t + s^t + m^t = w_t$$
$$c_1^t = r_{t+1}\left[k^t + \sum_{i=1}^{+\infty}(1-\phi)^i(\frac{\delta}{n})^i k^{t-i}\right] + d_{t+1} + s^{t+1}n + \frac{p_t}{p_{t+1}}m^t \tag{45}$$
$$r_{t+1}s^t = r_{t+1}\sum_{i=1}^{+\infty}(1-\phi)^i(\frac{\delta}{n})^i k^{t-i} + d_{t+1} + s^{t+1}n$$

Note that it follows from the first-order conditions above that the value of firm and depreciated capital ownership for the household is $\mu^t = u'(c_0^t)/r_{t+1} > 0$.

The optimising behaviour of households in (45) and firms in (38) determines, when compatible, a competitive equilibrium of this economy, as stated in the next section.

4.4. Competitive equilibria of the overlapping generations economy.

A competitive equilibrium of the overlapping generations economy is therefore characterised by the conditions provided in Proposition 6 next.

Proposition 6. *In the overlapping generations economy in Section 2, in which a fraction ϕ of private capital falls into the public domain each period, a competitive equilibrium is characterised by sequences of consumption profiles c_0^t, c_1^t, loans to firms k^t, net positions between firms and capital ownership and borrowing s^t, real balances m^t, and distributed profits d_{t+1}, for each agent born in each period t, as well as prices p_t, for all t, such that*

$$\begin{aligned}
\frac{1}{\beta}\frac{u'(c_0^t)}{u'(c_1^t)} = \frac{p_t}{p_{t+1}} &= F_K\Big(\frac{1}{\delta}\sum_{i=1}^{+\infty}\big(\frac{\delta}{n}\big)^i k^{t+1-i}, 1\Big) \\
c_0^t + k^t + s^t + m^t &= F_L\Big(\frac{1}{\delta}\sum_{i=1}^{+\infty}\big(\frac{\delta}{n}\big)^i k^{t-i}, 1\Big) \\
\frac{c_1^t}{n} = F_K\Big(\frac{1}{\delta}\sum_{i=1}^{+\infty}\big(\frac{\delta}{n}\big)^i k^{t+1-i}, 1\Big)\Big[\frac{k^t}{n} + \frac{1}{n}\sum_{i=1}^{+\infty}(1-\phi)^i\big(\frac{\delta}{n}\big)^i k^{t-i}\Big] &+ \frac{d_{t+1}}{n} + s^{t+1} + \frac{p_t}{p_{t+1}}\frac{m^t}{n} \\
F_K\Big(\frac{1}{\delta}\sum_{i=1}^{+\infty}\big(\frac{\delta}{n}\big)^i k^{t+1-i}, 1\Big)\Big[s^t - \sum_{i=1}^{+\infty}(1-\phi)^i\big(\frac{\delta}{n}\big)^i k^{t-i}\Big] &= d_{t+1} + s^{t+1}n \\
\frac{d_{t+1}}{n} = F_K\Big(\frac{k^t}{n} + \frac{1}{n}\sum_{i=1}^{+\infty}\big(\frac{\delta}{n}\big)^i k^{t-i}, 1\Big)\frac{1}{n}\sum_{i=1}^{+\infty}&[1-(1-\phi)^i]\big(\frac{\delta}{n}\big)^i k^{t-i} \\
\frac{p_t}{p_{t+1}}\frac{m^t}{m^{t+1}} &= n
\end{aligned} \tag{46}$$

Proof. The first four lines follow from the household choice in (45) with the factor prices replaced by the marginal productivities of factors according to the firms' optimal behavior in (38). The fifth line is the equilibrium *per* young profits in (40) distributed at each $t+1$, since

$$d_{t+1} = \frac{\pi_{t+1}}{N_t} \tag{47}$$

The sixth line is equivalent to the feasibility of the allocation of resources for this economy, and can be obtained in the usual way adding up the budget constraints of the agents alive at any given period t —of which there are n young agents *per* old one— and taking into account the homogeneity of degree 1 of the production function, *i.e.* adding up

$$c_0^t + k^t + s^t + m^t = w_t \tag{48}$$

and

$$\frac{c_1^{t-1}}{n} = r_t\left[\frac{k^{t-1}}{n} + \frac{1}{n}\sum_{i=1}^{+\infty}(1-\phi)^i(\frac{\delta}{n})^i k^{t-1-i}\right] + \frac{d_t}{n} + s^t + \frac{p_{t-1}}{p_t}\frac{m^{t-1}}{n} \tag{49}$$

which, with the per young dividends

$$\frac{d_t}{n} = F_K\Big(\frac{k^{t-1}}{n} + \frac{1}{n}\sum_{i=1}^{+\infty}(\frac{\delta}{n})^i k^{t-1-i}, 1\Big)\frac{1}{n}\sum_{i=1}^{+\infty}[1-(1-\phi)^i](\frac{\delta}{n})^i k^{t-1-i} \tag{50}$$

and the feasibility constraint

$$c_0^t + \frac{c_1^{t-1}}{n} + k^t = F\Big(\frac{1}{\delta}\sum_{i=1}^{+\infty}(\frac{\delta}{n})^i k^{t-i}, 1\Big) \tag{51}$$

amounts to

$$\frac{p_t}{p_{t+1}}\frac{m^t}{m^{t+1}} = n \tag{52}$$

at any given t. □

Therefore, a competitive equilibrium steady state will be an allocation where the consumption profile, the loans to firms and the *total* (but not necessarily the composition) of *savings* in instruments *other than loans to firms* will stay constant at some level $\tilde{s}$, as shown in the Proposition 7 next. It is shown there too that, in the case in which savings in firm and depreciated capital ownership net of borrowing do *not* explode over time, (i) the share of real balances m^t within savings $\tilde{s}$ in instruments other than loans to firms converges to zero so that, in the limit, the net position s^t of *firm and depreciated capital ownership minus borrowing asymptotically replaces money* as the bubbly asset in the economy; and (ii) $\tilde{s} < 0$, meaning that, in the limit, *each generation borrows when young against future income from firm and depreciated capital ownership more than it pays for it*, the funds of the loan being provided by the repayments to the financial intermediaries made by the previous generation.

Proposition 7. *In the overlapping generations economy in Section 2, in which some fraction of private capital falls into the public domain each period, a competitive equilibrium steady state is characterised by a constant intertemporal profile of* *consumptions $\bar{c}_0, \bar{c}_1$ and a constant loan to firms $\bar{k}$, for all generations, as well as a constant growth factor of real balances m^{t+1}/m^t solving*

$$\begin{aligned} \frac{1}{\beta}\frac{u'(c_0)}{u'(c_1)} &= n\frac{m^{t+1}}{m^t} = F_K\Big(\frac{k}{n-\delta}, 1\Big) \\ c_0 + k + \tilde{s} &= F_L\Big(\frac{k}{n-\delta}, 1\Big) \\ \frac{c_1}{n} &= F_K\Big(\frac{k}{n-\delta}, 1\Big)\frac{k}{n-\delta} + \tilde{s} \end{aligned} \tag{53}$$

—where $\tilde{s} = \frac{r\delta}{r-n} \cdot \frac{k}{n-\delta}$, with $r = F_K\big(\frac{k}{n-\delta}, 1\big)$. Moreover, for all t,

$$s^t + m^t = \tilde{s} \tag{54}$$

and, the household net position s^t of savings in ownership of the firm and depreciated capital minus borrowing converges if, and only if, $r < n$, and its limit is $\tilde{s}$, while positive real balances m^t converge to zero,[28] *i.e.*

$$\begin{aligned} \lim_{t\to+\infty} s^t &= \tilde{s} < 0 \\ \lim_{t\to+\infty} m^t &= 0. \end{aligned} \tag{55}$$

Proof. From Proposition 6, a competitive equilibrium steady state is characterised by the conditions next, where consumptions c_0^t, c_1^t, capital savings k^t, and distributed profits d_{t+1} —but not necessarily s^t or m^t (nor p_t, *a fortiori*)— stay

[28]If $r \geq n$, savings invested in the net position of firm and depreciated capital ownership minus borrowing diverge.

constant at levels c_0, c_1, k, and d,

$$
\begin{aligned}
\frac{1}{\beta}\frac{u'(c_0)}{u'(c_1)} = \frac{p_t}{p_{t+1}} &= F_K\Big(\frac{1}{\delta}\sum_{i=1}^{+\infty}\big(\frac{\delta}{n}\big)^i k, 1\Big) \\
c_0 + k + s^t + m^t &= F_L\Big(\frac{1}{\delta}\sum_{i=1}^{+\infty}\big(\frac{\delta}{n}\big)^i k, 1\Big) \\
\frac{c_1}{n} = F_K\Big(\frac{1}{\delta}\sum_{i=1}^{+\infty}\big(\frac{\delta}{n}\big)^i k, 1\Big)\left[\frac{k}{n} + \frac{1}{n}\sum_{i=1}^{+\infty}(1-\phi)^i\big(\frac{\delta}{n}\big)^i k\right] &+ \frac{d}{n} + s^{t+1} + \frac{p_t}{p_{t+1}}\frac{m^t}{n} \\
F_K\Big(\frac{1}{\delta}\sum_{i=1}^{+\infty}\big(\frac{\delta}{n}\big)^i k, 1\Big)\Big[s^t - \sum_{i=1}^{+\infty}(1-\phi)^i\big(\frac{\delta}{n}\big)^i k\Big] &= d + s^{t+1}n \\
\frac{d}{n} = F_K\Big(\frac{k}{n} + \frac{1}{n}\sum_{i=1}^{+\infty}\big(\frac{\delta}{n}\big)^i k, 1\Big)\frac{1}{n}\sum_{i=1}^{+\infty}&[1-(1-\phi)^i]\big(\frac{\delta}{n}\big)^i k \\
\frac{p_t}{p_{t+1}}\frac{m^t}{m^{t+1}} &= n
\end{aligned}
\tag{56}
$$

Note that, nonetheless, from the second line above, the aggregate $s^t + m^t$ has necessarily to be some constant $\tilde{s}$ at a competitive equilibrium steady state, even though s^t and m^t might not.

Thus, after substituting the sixth equation into the first line and replacing the series by their values, a competitive equilibrium steady state is characterised by

$$
\begin{aligned}
\frac{1}{\beta}\frac{u'(c_0)}{u'(c_1)} = n\frac{m^{t+1}}{m^t} &= F_K\Big(\frac{k}{n-\delta}, 1\Big) \\
c_0 + k + s^t + m^t &= F_L\Big(\frac{k}{n-\delta}, 1\Big) \\
\frac{c_1}{n} = F_K\Big(\frac{k}{n-\delta}, 1\Big)\Big[\frac{k}{n} + \frac{1}{n}\frac{(1-\phi)\delta}{n-(1-\phi)\delta}k\Big] &+ \frac{d}{n} + s^{t+1} + \frac{p_t}{p_{t+1}}\frac{m^t}{n} \\
F_K\Big(\frac{k}{n-\delta}, 1\Big)\Big[s^t - \frac{(1-\phi)\delta}{n-(1-\phi)\delta}k\Big] &= d + s^{t+1}n \\
\frac{d}{n} = F_K\Big(\frac{k}{n-\delta}, 1\Big)\frac{1}{n}\Big[\frac{\delta}{n-\delta} &- \frac{(1-\phi)\delta}{n-(1-\phi)\delta}\Big]k \\
\frac{p_t}{p_{t+1}}\frac{m^t}{m^{t+1}} &= n
\end{aligned}
\tag{57}
$$

or equivalently, after substituting the last two equations into the third line,

$$\begin{aligned}
\frac{1}{\beta}\frac{u'(c_0)}{u'(c_1)} = n\frac{m^{t+1}}{m^t} &= F_K\Big(\frac{k}{n-\delta},1\Big) \\
c_0 + k + s^t + m^t &= F_L\Big(\frac{k}{n-\delta},1\Big) \\
\frac{c_1}{n} = F_K\Big(\frac{k}{n-\delta},1\Big)\frac{k}{n-\delta} &+ s^{t+1} + m^{t+1} \\
F_K\Big(\frac{k}{n-\delta},1\Big)\Big[s^t - \frac{(1-\phi)\delta}{n-(1-\phi)\delta}k\Big] &= d + s^{t+1}n
\end{aligned} \tag{58}$$

whose first three lines are those in (53), with $s^t + m^t = \tilde{s}$ as requested. It remains to be checked that the no-arbitrage condition at the bottom of (58) implies that $\tilde{s}$ is the value claimed and that the convergence of s^t to that $\tilde{s}$ obtains. Indeed, from the no-arbitrage condition at the steady state in (58), which can be rewritten as

$$s^{t+1} = \frac{r}{n}\Big(s^t - \frac{(1-\phi)\delta}{n-(1-\phi)\delta}k\Big) - \frac{d}{n} \tag{59}$$

it follows that convergence obtains only if $r < n$ and, straightforwardly, the limit is

$$\tilde{s} = \frac{r\delta}{r-n}\frac{k}{n-\delta} < 0 \tag{60}$$

where $r = F_K\big(\frac{k}{n-\delta},1\big)$.[29] □

A few remarks are now in order. Firstly, from conditions (26) and (53) it follows that the planner's steady state cannot be decentralized through markets under laissez-faire. In particular, the market leads the agents to consume too early at the steady state, in the sense of choosing an intertemporal marginal rate of substitution smaller than the planner's, as made precise in the Proposition 8 next.

Proposition 8. *In the overlapping generations economy in Section 2, in which some fraction of private capital falls into the public domain each period, the planner's steady state cannot be decentralised as a laissez-faire competitive markets outcome. Specifically, the market makes households choose a profile of consumptions whose intertemporal marginal rate of substitution is smaller than the planner's.*

[29] It is worth noting that for the overlapping generations economy, a competitive equilibrium steady state does not depend on the actual fraction of proprietary capital that may fall in the public domain, as long as it is positive.

Proof. Indeed, note first that, since $s^t + m^t = \tilde{s}$, for $m^t > 0$, it must be that $s^t < \tilde{s}$, so that s^t converges to $\tilde{s} < 0$ from the left —as illustrated in Figure 1 below— decreasing in absolute value. Therefore, $m^t = \tilde{s} - s^t$ is decreasing, so that $m^t/m^{t+1} > 1$ which, from the equilibrium condition

$$\frac{p_t}{p_{t+1}} \frac{m^t}{m^{t+1}} = n \tag{61}$$

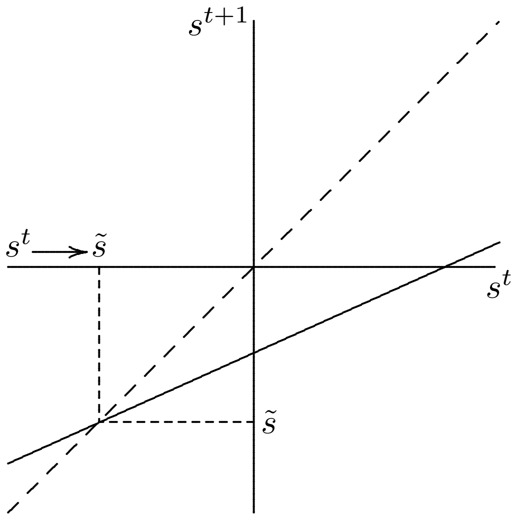
implies $p_t/p_{t+1} < n$.

Now, if $\bar{c}_0, \bar{c}_1$ is the competitive equilibrium steady state profile of consumption, while the steady state profile chosen by the planner is c_0^*, c_1^*, it follows from the respective characterisations in (53) and (26) that

$$\frac{1}{\beta} \frac{u'(\bar{c}_0)}{u'(\bar{c}_1)} = \frac{p_t}{p_{t+1}} < n = \frac{1}{\beta} \frac{u'(c_0^*)}{u'(c_1^*)} \tag{62}$$

as claimed. □

Figure 1



Interestingly enough, it is not immediate in this overlapping generations setup —as opposed to what happened in the infinitely-lived agents case— whether the market lends too few or too much capital to firms, compared to what the planner would choose. Indeed, since at the competitive equilibrium steady state $p_t/p_{t+1} < n$, it follows from (53) and (26) that

$$F_K(\frac{\bar{k}}{n-\delta}, 1) = \frac{p_t}{p_{t+1}} < n = F_K(\frac{k^*}{n-\delta}, 1) + \delta \tag{63}$$

where $\bar{k}$ is the steady state *per* young market level of capital, and k^* is the planner's. As a consequence, it holds that

$$F_K(\frac{\bar{k}}{n-\delta}, 1) < n > F_K(\frac{k^*}{n-\delta}, 1) \tag{64}$$

so that the *per* young market level of capital $\bar{k}$ could, in principle, be smaller or bigger than the planner's k^*.

Note however that, while the planner's steady state *per* young level of capital k^* needs to be such that

$$F_K(\frac{k^*}{n-\delta}, 1) = n - \delta \tag{65}$$

the competitive equilibrium steady state equations (53) pin down the market steady state *per* young level of capital to be $\bar{k}$ such that

$$F_K(\frac{\bar{k}}{n-\delta}, 1) = \frac{1}{\beta}\frac{u'(F_L(\frac{\bar{k}}{n-\delta}, 1) - \bar{k} - s)}{u'(F_K(\frac{\bar{k}}{n-\delta}, 1)\frac{\bar{k}}{n-\delta}n + sn)} \tag{66}$$

which —for the sake of assessing the wedge between the market *per* young steady state capital accumulation and the planner's— in the case of $u(c) = \ln c$ and $F(K, N) = K^\alpha N^{1-\alpha}$ takes (after some algebra) respectively the form

$$\alpha\frac{1-\alpha}{n-\delta} \cdot x^2 - \left(\left[n\left(1 - \frac{1}{\beta}\right) - \delta\right]\frac{1-\alpha}{n-\delta} + \alpha\right) \cdot x + n\left(1 + \frac{1}{\beta}\right) = 0 \tag{67}$$

where $x = \frac{\bar{k}}{n-\delta}$.

For the profile of parameters we considered in Section 3.3, i.e. $\beta = .98$, $\delta = .85$, a population growth factor $n = 1.007$ —the empirical value for the US in 2016—[30] and α slighlty below 1/3,[31] the latter has a single root at (approximately)

$$\frac{\bar{k}}{n-\delta} = 1.2015 \tag{68}$$

[30]Source https://data.worldbank.org/indicator/SP.POP.GROW.

[31]Specifically, $\alpha = 0.32996710445$, this being the bifurcating value of the root of (67). Needless to say, precision is not the point, but rather the order of magnitude of the inefficiency.

bifurcating into two roots for smaller values of α, while from (65) the same value is for the planner (approximately)

$$\frac{k^*}{n-\delta} = \left(\frac{n-\delta}{\alpha}\right)^{\frac{1}{\alpha-1}} = 3.0299 \tag{69}$$

so that the planner to market steady state *per* young capital ratio is (approximately)

$$\frac{k^*}{\bar{k}} = 2.5218 \tag{70}$$

In other words, the market's steady state level of capital accumulation is once more way too low: the planner would choose to save/invest about 150% more (!), which is even bigger than the 50-100% more found for the infinitely-lived agents economy (see Table 1b).

While the size of this gap is shocking, the implied gap in terms of aggregate consumption —a more informative estimate of the potential impact on the households' well-being— might be more relevant. Specifically, the ratio of the planner's to the market steady state *per* young aggregate consumption— is approximately[32]

$$\frac{c_0^* + \frac{c_1^*}{n}}{\bar{c}_0 + \frac{\bar{c}_1}{n}} = \frac{(\frac{k^*}{n-\delta})^\alpha - k^*}{(\frac{k^*}{n-\delta})^\alpha - \bar{k}} = 1.1054 \tag{71}$$

since, from (68) and (69),

$$\begin{aligned} k^* &= 3.0299 \cdot (1.007 - 0.85) = 0.4757 \\ \bar{k} &= 1.2015 \cdot (1.007 - 0.85) = 0.1886 \end{aligned} \tag{72}$$

That is to say, for empirically reasonable values of basic parameters, the market fails to deliver the 10.5% more *per* young aggregate consumption that households would be allocated by a hypothetical planner. This is of the same order of magnitude as the inefficiency due to the slide into public domain of private capital found in the infinitely-lived agents setup —and in the upper end of the range found then. As a consequence, whatever the shortcomings of the quantitative assessment, whose point is clearly not precision, this robustness unequivocally points to a significant cost of not addressing the problem by means of some offsetting policy. What that policy should be in this setup is presented next.

[32]For α slightly below the value $\alpha = 0.32996710445$ in footnote 30.

4.5. Market implementation of the planner's steady state.

If households see their returns from loans to firms subsidised at a rate τ and their net position between (i) firm and depreciated capital ownership and (ii) borrowing against it taxed at a rate $\sigma - 1$, then the household born at t would face instead the problem

$$\max_{c_0^t, c_1^t, k^t, m^t \geq 0, s^t} u(c_0^t) + \beta u(c_1^t)$$

$$c_0^t + k^t + s^t + m^t \leq w_t$$

$$c_1^t \leq (r_{t+1} + \tau)\left[k^t + \sum_{i=1}^{+\infty}(1-\phi)^i(\frac{\delta}{n})^i k^{t-i}\right] + d_{t+1} + \sigma s^{t+1} n + \frac{p_t}{p_{t+1}} m^t \quad (73)$$

$$(r_{t+1} + \tau)s^t = (r_{t+1} + \tau)\sum_{i=1}^{+\infty}(1-\phi)^i(\frac{\delta}{n})^i k^{t-i} + d_{t+1} + \sigma s^{t+1} n$$

given the wage w_t, the rental rate of capital r_{t+1}, the level of prices during his lifetime p_t, p_{t+1}, the profits received as dividends when owner d_{t+1}, the household $t+1$'s net positions s^{t+1}, the population growth factor n, and the policy τ and σ.

As a consequence, the choice of a household born at t necessarily satisfies

$$\begin{pmatrix} u'(c_0^t) \\ \beta u'(c_1^t) \\ 0 \\ 0 \\ 0 \end{pmatrix} = \lambda_0^t \begin{pmatrix} 1 \\ 0 \\ 1 \\ 1 \\ 1 \end{pmatrix} + \lambda_1^t \begin{pmatrix} 0 \\ 1 \\ -(r_{t+1} + \tau) \\ -\frac{p_t}{p_{t+1}} \\ 0 \end{pmatrix} + \mu^t \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ -(r_{t+1} + \tau) \end{pmatrix} \quad (74)$$

for some $\lambda_0^t, \lambda_1^t, \mu^t > 0$, along with the binding constraints, or equivalently

$$\frac{1}{\beta}\frac{u'(c_0^t)}{u'(c_1^t)} = \frac{p_t}{p_{t+1}} = r_{t+1} + \tau$$

$$c_0^t + k^t + s^t + m^t = w_t$$

$$c_1^t = (r_{t+1} + \tau)\left[k^t + \sum_{i=1}^{+\infty}(1-\phi)^i(\frac{\delta}{n})^i k^{t-i}\right] + d_{t+1} + \sigma s^{t+1} n + \frac{p_t}{p_{t+1}} m^t \quad (75)$$

$$(r_{t+1} + \tau)s^t = (r_{t+1} + \tau)\sum_{i=1}^{+\infty}(1-\phi)^i(\frac{\delta}{n})^i k^{t-i} + d_{t+1} + \sigma s^{t+1} n$$

As before, firms distribute at $t+1$ to each household born at t dividends

$$d_{t+1} = F_K\Big(\frac{k^t}{n} + \frac{1}{n}\sum_{i=1}^{+\infty}(\frac{\delta}{n})^i k^{t-i}, 1\Big)\sum_{i=1}^{+\infty}[1-(1-\phi)^i](\frac{\delta}{n})^i k^{t-i} \tag{76}$$

and factor prices are

$$\begin{aligned} r_{t+1} &= F_K(K_{t+1}, N_{t+1}) \\ w_t &= F_L(K_t, N_t) \end{aligned} \tag{77}$$

The market clearing condition can again be obtained adding up the budget constraints of the agents alive at any given period t, of which there are n young agents *per* old one, *i.e.* adding up

$$c_0^t + k^t + s^t + m^t = w_t \tag{78}$$

and

$$\frac{c_1^{t-1}}{n} = (r_t+\tau)\Big[\frac{k^{t-1}}{n} + \frac{1}{n}\sum_{i=1}^{+\infty}(1-\phi)^i(\frac{\delta}{n})^i k^{t-1-i}\Big] + \frac{d_t}{n} + \sigma s^t + \frac{p_{t-1}}{p_t}\frac{m^{t-1}}{n} \tag{79}$$

which after taking into account the feasibility condition amounts to

$$m^t = \tau\Big[\frac{k^{t-1}}{n} + \frac{1}{n}\sum_{i=1}^{+\infty}(1-\phi)^i(\frac{\delta}{n})^i k^{t-1-i}\Big] + (\sigma-1)s^t + \frac{p_{t-1}}{p_t}\frac{m^{t-1}}{n} \tag{80}$$

holding at any given t.

A competitive equilibrium, under such a (not necessarily balanced yet) policy, is therefore characterised by the following conditions.

Proposition 9. *In the overlapping generations economy in Section 2, in which a fraction ϕ of private capital falls into the public domain each period, a competitive equilibrium under a policy that (i) subsidises at a rate τ the returns to capital and (ii) taxes at a rate $\sigma-1$ the net position between saving in firms and capital ownership and borrowing against future dividends and resale value, is characterised by a consumption profile c_0^t, c_1^t, a loan to firms k^t, a net position s^t between firms and capital ownership and borrowing, a real balance m^t, and distributed profits*

d_{t+1}, for each agent born in each period t, as well as prices p_t, for all t, such that

$$\frac{1}{\beta}\frac{u'(c_0^t)}{u'(c_1^t)} = \frac{p_t}{p_{t+1}} = F_K(\frac{1}{\delta}\sum_{i=1}^{+\infty}(\frac{\delta}{n})^i k^{t+1-i}, 1) + \tau$$

$$c_0^t + k^t + s^t + m^t = F_L(\frac{1}{\delta}\sum_{i=1}^{+\infty}(\frac{\delta}{n})^i k^{t-i}, 1)$$

$$\frac{c_1^t}{n} = \left[F_K(\frac{1}{\delta}\sum_{i=1}^{+\infty}(\frac{\delta}{n})^i k^{t+1-i}, 1) + \tau\right]\left[\frac{k^t}{n} + \frac{1}{n}\sum_{i=1}^{+\infty}(1-\phi)^i(\frac{\delta}{n})^i k^{t-i}\right] + \frac{d_{t+1}}{n} + \sigma s^{t+1} + \frac{p_t}{p_{t+1}}\frac{m^t}{n}$$

$$\left[F_K(\frac{1}{\delta}\sum_{i=1}^{+\infty}(\frac{\delta}{n})^i k^{t+1-i}, 1) + \tau\right]\left[s^t - \sum_{i=1}^{+\infty}(1-\phi)^i(\frac{\delta}{n})^i k^{t-i}\right] = d_{t+1} + \sigma s^{t+1} n$$

$$d_{t+1} = F_K(\frac{k^t}{n} + \frac{1}{n}\sum_{i=1}^{+\infty}(\frac{\delta}{n})^i k^{t-i}, 1)\sum_{i=1}^{+\infty}[1-(1-\phi)^i](\frac{\delta}{n})^i k^{t-i}$$

$$m^t = \tau\left[\frac{k^{t-1}}{n} + \frac{1}{n}\sum_{i=1}^{+\infty}(1-\phi)^i(\frac{\delta}{n})^i k^{t-1-i}\right] + (\sigma-1)s^t + \frac{p_{t-1}}{p_t}\frac{m^{t-1}}{n} \tag{81}$$

As opposed to what happens under laissez-faire, at the equilibrium that decentralises under a *period-by-period balanced* policy of this kind the planner's steady state s^t and m^t stay constant as well, as Proposition 10 next establishes. This policy finances a subsidy to the return to capital —at the exact depreciation rate— through a tax on households' savings on firm and depreciated capital ownership if $s^t > 0$, or on household debt issued against it if $s^t < 0$.

Proposition 10. *In the overlapping generations economy in Section 2, in which a fraction ϕ of private capital falls into the public domain each period, the planner's steady state is decentralised as a competitive equilibrium steady state by a period-by-period balanced policy that (i) subsidises the returns to capital by the depreciationfactor, i.e.*

$$\tau = \delta \tag{82}$$

and (ii) taxes the net position s between savings in firms and depreciated capital ownership and debt against future profits and resale value by means of increasing (respectively, decreasing) it by a factor $\sigma > 1$ (resp. < 1) if $s < 0$, (resp. > 0), i.e.

in case of net borrowing (resp. saving), at a rate

$$\sigma - 1 = -\frac{\delta}{s}\frac{k^*}{n-(1-\phi)\delta} \tag{83}$$

depending on the net position s of households, which is moreover bounded above by an $\bar{s}$ equal to the planner's steady state labor income next of first period consumption and loans to firms.

Proof. Should there be values for c_0, c_1, k, s, m, d, and p_t/p_{t+1} such that

$$\begin{aligned}
\frac{1}{\beta}\frac{u'(c_0)}{u'(c_1)} &= \frac{p_t}{p_{t+1}} = F_K(\frac{k}{n-\delta},1)+\delta \\
c_0+k+s+m &= F_L(\frac{k}{n-\delta},1) \\
\frac{c_1}{n} &= \left[F_K(\frac{k}{n-\delta},1)+\delta\right]\left[\frac{k}{n}+\frac{1}{n}\frac{(1-\phi)\delta}{n-(1-\phi)\delta}k\right]+\frac{d}{n}+\sigma s+\frac{p_t}{p_{t+1}}\frac{m}{n} \\
\left[F_K(\frac{k}{n-\delta},1)+\delta\right]&\left[s-\frac{(1-\phi)\delta}{n-(1-\phi)\delta}k\right] = d+\sigma s n \\
d &= F_K(\frac{k}{n-\delta},1)\left[\frac{\delta}{n-\delta}-\frac{(1-\phi)\delta}{n-(1-\phi)\delta}\right]k \\
m &= \delta\left[\frac{k}{n}+\frac{1}{n}\frac{(1-\phi)\delta}{n-(1-\phi)\delta}k\right]+(\sigma-1)s+\frac{p_{t-1}}{p_t}\frac{m}{n}
\end{aligned} \tag{84}$$

for a given σ, then —according to (81)— they would characterise a competitive equilibrium steady state under the policy of subsidising returns to savings at a rate δ and taxing positive (respectively, negative) net savings in firms and capital ownership minus borrowing against future dividends and resale by a factor $\sigma < 1$ (resp. $\sigma > 1$). But, is there a σ for which such values exist, and are they those of the planner's steady state characterised by (26)? The answer to this question is yes, as established next.

Indeed, note that the system (84) —including σ as endogenous variable and augmented to include the additional equation balancing taxes and subsidies

$$\delta\left[\frac{k}{n}+\frac{1}{n}\frac{(1-\phi)\delta}{n-(1-\phi)\delta}k\right]+(\sigma-1)s = 0 \tag{85}$$

— pins down any balanced policy implementing the planner's steady state. Indeed, according to (85) the last equation in (84) implies $p_t/p_{t+1} = n$ so that the first line becomes

$$\frac{1}{\beta}\frac{u'(c_0)}{u'(c_1)} = n = F_K(\frac{k}{n-\delta}, 1) + \delta \tag{86}$$

which is the first line of the planner's system in (26). Moreover, the equations in the second, third, fifth, and sixth lines of (84) —along with the balanced policy condition (85)— imply the feasibility of the allocation, so that the solution to (84) is the unique solution c_0^*, c_1^*, k^* to (26).

Thus the equation (85) pins down the necessary σ to be

$$\sigma = 1 - \frac{\delta}{s}\frac{k^*}{n-(1-\phi)\delta} \tag{87}$$

so that, since σ depends on s, there is a 1-dimensional continuum of (τ, σ) policies decentralising the planner's steady state.[33] Indeed, the planner's k^* in the solution to (26) pins down d through the fifth line of (84). As for s,m, and σ, their values should be determined by the remaining equations —namely the first and second period budget constraints and the no-arbitrage condition, i.e. second, third and fourth lines in (84)— but these equations happen not to be independent,[34] and as a consequence the household's net *financial* worth $s+m$ decentralising the planner's steady state is determined by the first period budget constraint

$$s + m = F_L(\frac{k^*}{n-\delta}, 1) - c_0^* - k^* \tag{88}$$

[33]Since necessarily $\tau = \delta$, the indeterminacy of the policy is on σ, which will depend —as shown next— on the net position s chosen by households at the market steady state decentralising the planners.

[34]After some simple algebra, they can be rewritten as

$$(\sigma - 1)s = A$$
$$(\sigma - C)s = B$$

with

$$A = c_0 + \frac{c_1}{n} + k - F_L(\frac{k}{n-\delta}, 1) - \left[F_K(\frac{k}{n-\delta}, 1) + \delta\right]\left[1 + \frac{(1-\phi)\delta}{n-(1-\phi)\delta}\right]\frac{k}{n} - \frac{d}{n}$$
$$B = -\left[F_K(\frac{k}{n-\delta}, 1) + \delta\right]\frac{(1-\phi)\delta}{n-(1-\phi)\delta}\frac{k}{n} - \frac{d}{n}$$
$$C = \frac{1}{n}\left[F_K(\frac{k}{n-\delta}, 1) + \delta\right]$$

and it is straightforward to check that at the planner's steady state $A = B$ (on top of being negative) and $C = 1$, from which it follows that one of the budget constraints or the no-arbitrage condition is implied by the two other equations.

and its distribution in assets s and m is linked to σ through the no-arbitrage condition, written as

$$(\sigma-1)s=-\left[F_K(\frac{k^*}{n-\delta},1)+\delta\right]\frac{(1-\phi)\delta}{n-(1-\phi)\delta}\frac{k^*}{n}-\frac{d}{n} \tag{89}$$

It follows from the last condition that

$$(\sigma-1)s<0 \tag{90}$$

so that, in principle,

(1) either $s<0$ and $\sigma>1$, i.e. σ is a tax on debt —households repay more than they borrowed in excess of their savings in firm and depreciated capital ownership, the excess payment being used to finance the subsidy on the rental rate of capital

(2) or $s>0$ and $0<\sigma<1$, i.e. σ is a tax on savings —households get less than they saved in excess of their borrowing against firm and depreciated capital ownership, the missing savings being used to finance the subsidy on the rental rate of capital

Note, nonetheless, that the net position of the household decentralising the planner's steady state, although indeterminate, is bounded above. Indeed, since from the first period budget constraint it holds that

$$0\leq m=F_L(\frac{k^*}{n-\delta},1)-c_0^*-k^*-s \tag{91}$$

it follows that s is bounded above

$$s\leq\bar{s}=F_L(\frac{k^*}{n-\delta},1)-c_0^*-k^* \tag{92}$$

—i.e. savings in firm and depreciated capital ownership cannot be too high.[35] □

[35]Incidentally, similarly from the second period budget constraint

$$0\leq m=\frac{c_1^*}{n}-\left[F_K(\frac{k^*}{n-\delta},1)+\delta\right]\left[\frac{k^*}{n}+\frac{1}{n}\frac{(1-\phi)\delta}{n-(1-\phi)\delta}k^*\right]-\frac{d}{n}-\sigma s \tag{93}$$

and the distributed dividends in (84), it follows that

$$\sigma s\leq\frac{c_1^*}{n}-F_K(\frac{k^*}{n-\delta},1)\frac{k^*}{n-\delta}-\delta\frac{k^*}{n-(1-\phi)\delta} \tag{94}$$

too.

In order to have quantitative assessment —in the overlapping generations setup too— of the policy needed to decentralise the planner's steady state, it is straightforward to solve the system (26) and evaluate the solution with the parameters assumed $\beta = .98$, $\delta = .85$, $n = 1.007$ and α slightly below 1/3 (see footnote 30) to find the planner's steady state to be

$$\begin{aligned} k^* &= \left(\frac{(n-\delta)^\alpha}{\alpha}\right)^{\frac{1}{\alpha-1}} = 0.4757 \\ c_0^* &= \frac{1}{1+\beta}\left[\left(\frac{k^*}{n-\delta}\right)^\alpha - k^*\right] = 0.4878 \\ c_1^* &= \frac{n\beta}{1+\beta}\left[\left(\frac{k^*}{n-\delta}\right)^\alpha - k^*\right] = 0.4814 \end{aligned} \tag{95}$$

so that

$$\bar{s} = (1-\alpha)\frac{k^*}{n-\delta} - c_0^* - k^* = 1.0666 \tag{96}$$

and at the planner's steady state output is

$$c_0^* + \frac{c_1^*}{n} + k^* + \bar{s} = 2.5082 \tag{97}$$

that is to say, the positive net position (i.e. savings) in instruments other than loans to firms —i.e. in money and firm and depreciated capital ownership net of borrowing against future distributed dividends and resale value— that decentralises the planner's and best steady state, under this policy and given the gradual slide of capital into the public domain, is 42.52% of output ($\bar{s}$ over output in (97)), while the overall savings rate ($k^* + \bar{s}$ over output) is 61.49% (!)

Note, however, households cannot hold, to support the planner's steady state, too much of the excess savings $\bar{s}$ above just in firm and depreciated capital ownership in excess of the borrowing against future distributed profits and resale value, since that would imply a negative σ which would render the households' budget set unbounded —indeed, if $s = \bar{s}$ and $m = 0$, then (from (89) and the dividends in (84), after simplifications) σ would satisfy

$$(\sigma - 1)s = -\left[\frac{(1-\phi)\delta}{n-(1-\phi)\delta} + \alpha\left(\frac{k^*}{n-\delta}\right)^\alpha\right]\frac{\delta}{n} \in (-2.9506, -3.6440) \tag{98}$$

i.e.

$$\sigma - 1 \in (-2.7665, -3.4165) \tag{99}$$

for the empirically relevant range for the share of capital falling into the public domain each period, $0.06 \leq \phi \leq 0.11$.[36] As a matter of fact, for the log utility and Cobb-Douglas production function and the parameters considered, from (95) and in the relevant empirical range for $\phi \in (0.06, 0.11)$, it holds

$$\sigma s < \frac{c_1^*}{n} - F_K(\frac{k^*}{n-\delta}, 1)\frac{k^*}{n-\delta} - \delta\frac{k^*}{n-(1-\phi)\delta} \in (-1.6117, -1.9415) \qquad (100)$$

so that, actually, $s < 0$ must hold if $\sigma > 0$ is to hold (and, thus, the budget set remains compact).

As for the value of σ needed, for negative net positions $s < 0$ —i.e for borrowing in excess of firm and depreciated capital ownership and resale value— and big enough savings in monetary real balances, the tax rate $\sigma - 1$ on excess borrowing becomes positive and, from (98), smaller the bigger are s (in absolute value) and m. Thus, although there is a continuum of policies of this type decentralising the planner's steady state, large monetary real balances —offset by large indebtedness against future dividends and resale value— lower tax rate on debt needed. In other words —in, at least, this log utility/Cobb-Douglass production setup and the parameters considered— the bigger the amounts of both money and indebtedness, the lower the tax rate on debt needed to implement the planner's steady state, so that while the levels of money and indebtedness have no allocational consequence, they matter nonetheless to pin down the decentralising policy and, in particular, they can make the needed tax as small as wished.

5. Conclusion

The models presented in this paper aim at pointing to a source of inefficiency that seems to have been overlooked until now. Namely, that the impossibility of maintaining property rights on the capital generated by *all* past savings distorts households consumption/saving decision away from the optimal one, with potentially very significant consequences for the long-run steady state level of consumption. Although the models are deliberately stripped of any details other than those essential to make the point, for that very same reason the mechanisms shown to be at play will, with all likelihood, stay in any other model with the additional elements necessary to make it more apt to match empirical evidence.

[36]See footnote 17 and the discussion that prompted it.

The simple estimates of the deviations of the market from the planner's levels of consumption and capital accumulation, are intended only to give an idea of the order of magnitude of the inefficiency and the potential gains from addressing it. While the specific figures obtained for these estimates may, therefore, be taken with a pinch of salt, they unequivocally point to these gains to be significant. Thus, empirically supported, detailed models that incorporate the mechanisms uncovered here should be able to deliver quantified policy recommendations able to undo the inefficiencies pointed at in this paper, and to steer the market outcome towards the optimal one that a planner would choose.

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