Detecting phase synchronisation in time series data sets

Colm Daly
Dylan Morrissey
David Arrowsmith
Wolfram Just

0 Content

1 Motivation
2 What is a phase?
3 Phase synchronisation
4 Analysis of movement data
5 Epilogue
1 Motivation

Movement data (time series)

- Non stationary
- “Chaotic” oscillation
- Classification
2 What is a phase?

Idea: Rewrite the time signal $x(t)$ as $x(t) = A(t) \cos(\Theta(t))$ where $A(t)$ is called the amplitude and $\Theta(t)$ the phase. But how (e.g. lack of uniqueness etc.)?

Harmonic signal: $x(t) = a \cos(\omega t) \ (a, \omega > 0)$

Observe: $x(t) = \text{Re}(z(t))$ where

$z(t) = a e^{i\omega t} = a \cos(\omega t) + ia \sin(\omega t)$

and $\Theta(t) = \text{arg}(z(t))$. 

Beat: \( x(t) = a \cos(\omega_1 t) + b \cos(\omega_2 t) \) \((a > b > 0)\).

How to express this as \( x(t) = A(t) \cos(\Theta(t)) \) in a meaningful way?

Use \( x(t) = \text{Re}(z(t)) \) with

\[
\begin{align*}
  z(t) &= A(t) \exp(i\Theta(t)) = a \exp(i\omega_1 t) + b \exp(i\omega_2 t) \\
  A(t) &= \left( a^2 + 2ab \cos(\Delta \omega t) + b^2 \right)^{1/2} \\
  \Theta(t) &= \omega_1 t + \arctan(b \sin(\Delta \omega t)/(a + b \cos(\Delta \omega t)))
\end{align*}
\]
Recall:

\[ x(t) = a \cos(\omega t) = a/2 \exp(i\omega t) + a/2 \exp(-i\omega t) \]

\[ \rightarrow z(t) = a/2 \exp(i\omega t) \text{ (positive frequency component !)} \]

General case:

\[ x(t) \rightarrow \hat{x}(\omega) = \int_{-\infty}^{\infty} \exp(-i\omega t)x(t)dt \]

\[ \rightarrow \hat{z}(\omega) = \begin{cases} \hat{x}(\omega) & \text{if } \omega > 0, \\ 0 & \text{if } \omega \leq 0. \end{cases} \]

\[ \rightarrow z(t) = \int_{-\infty}^{\infty} \exp(i\omega t)\hat{z}(\omega)d\omega/(2\pi) \]

\[ \rightarrow \Theta(t) = \arg(z(t)) \text{ phase} \]

\[ A(t) = 2|z(t)| \text{ instantaneous amplitude} \]
Properties:

\[ z(t) = x(t) + i/\pi(\mathcal{H}x)(t) = x(t) + i/\pi \int_{-\infty}^{\infty} x(s)/(t-s)ds \]

- Amplitude continuity: a small change in the value of the signal \( x(t) \) should induce a correspondingly small change in the instantaneous amplitude \( A(t) \).

- Phase independence of scale: multiplying the real signal \( x(t) \) by a real positive constant \( c \) should have no effect on the instantaneous phase and should multiply the instantaneous amplitude by the same constant.

- Harmonic correspondence: the instantaneous amplitude and phase of a pure sinusoid \( a \cos(\omega t + \phi) \) should be given, respectively, by \( A(t) = a, \Theta(t) = \omega t + \phi \).
3 Phase synchronisation

Coupled chaotic oscillators

\[
\begin{align*}
\dot{x}_1 &= -1.015 y_1 - z_1 + C(x_2 - x_1) \\
\dot{y}_1 &= 1.015 x_1 + 0.15 y_1 \\
\dot{z}_1 &= 0.2 + z_1(x_1 - 10) \\
\dot{x}_2 &= -0.985 y_2 - z_2 + C(x_1 - x_2) \\
\dot{y}_2 &= 0.985 x_2 + 0.15 y_2 \\
\dot{z}_2 &= 0.2 + z_2(x_2 - 10)
\end{align*}
\]

small coupling
\[C = 0.015\]

intermediate coupling
\[C = 0.027\]

moderate coupling
\[C = 0.035\]
Phase representation

small coupling
$C = 0.015$
desynchronised
\[ |\Theta_1(t) - \Theta_2(t)| \sim t \]

intermediate coupling
$C = 0.027$
phase slips

moderate coupling
$C = 0.035$
phase synchronisation
\[ |\Theta_1(t) - \Theta_2(t)| < \text{const.} \]
4 Analysis of movement data

time series

phases and phase difference
time series

phases and phase difference

hip, coronal, ipsilateral

hip, coronal, ipsilateral/contralateral

hip, coronal, contralateral

hip, coronal, contralateral - ipsilateral
5 Epilogue

Structure of the data set

- **Position**: Ankle, Foot, Hip, Knee, Pelvis
- **Coordinate**: Coronal, Sagittal, Horizontal
- **Orientation**: Ipsilateral, Contralateral
Normalisation

![Graph 1](image1)

![Graph 2](image2)