Abstract

The rational expectations paradigm restricts the subjective beliefs of investors to align with the objective distribution. We relax this constraint and analyze how investors optimally choose their subjective beliefs about the information contained in their private signals and in prices. We show that investors systematically choose to deviate from rational expectations. In any symmetric equilibrium, investors optimally exhibit overconfidence in their private information but under-react to the information in prices. However, when aggregate risk aversion is sufficiently low, such symmetric equilibria do not exist. Instead, there exists an asymmetric equilibrium in which investors endogenously separate into (i) fundamental investors, who also ignore the information in prices, and (ii) “technical" traders, who overweight the information in prices when forming beliefs. Relative to the corresponding rational expectations equilibrium, these equilibria feature higher (i) return volatility, (ii) price informativeness, (iii) trading volume and (iv) return predictability. Finally, we show that such deviations by informed investors improve the welfare of liquidity traders under the objective distribution.

JEL Classification: D8, G1
Keywords: difference of opinions, optimal expectations, overconfidence.
1 Introduction

The traditional approach to modeling beliefs in economics and finance is to assume rational expectations i.e., an agent’s subjective beliefs are forced to coincide with the objective distribution. However, a growing body of empirical evidence establishes that people do not behave this way.\textsuperscript{1} For example, a robust finding in both the economics and psychology literatures is that people tend to be overconfident about their own views, but dismissive of others. Moreover, models that incorporate differences of opinions or behavioral biases (e.g., overconfidence, dismissiveness, cursedness) have been useful in understanding how such deviations affect financial markets. However, such models provide little guidance in understanding when such deviations arise because agents are constrained to exhibit such behavior by assumption.

If investors could choose their subjective beliefs, would they choose to exhibit rational expectations? Or would behavioral biases like overconfidence and dismissiveness arise endogenously? If so, under what conditions? To explore the answers to these questions, we develop a model in which a continuum of symmetric, privately informed investors trade a single risky asset against noise traders in a competitive market (a la Hellwig (1980)). However, instead of assuming that investors exhibit rational expectations or a specific behavioral bias, we allow investors to optimally choose how to interpret their private signals and the information in prices.

An investor’s interpretation of these signals determines her subjective beliefs about fundamentals and the potential speculative gains from trade which, in turn, determine her anticipated utility, i.e., the expected utility she receives from optimally trading the risky asset given her subjective beliefs. Each investor optimally chooses her subjective beliefs to maximize her anticipated utility net of the costs of deviating from rational expectations.\textsuperscript{2} We consider two types of costs: (i) direct costs of deviation through an explicit cost function, and (ii) an implicit cost due to the distortions they introduce in behavior, as in the “optimal expectations” framework of Brunnermeier and Parker (2005).

We find that investors optimally choose to deviate from rational expectations in systematic ways. In any symmetric equilibrium, we show that investors endogenously choose to be overconfident about their private signals, but under-react to the information in prices (consistent with models of differences of opinions or cursedness). The key to understanding this result is that, in a speculative setting, each investor prefers to have information that is not shared by others. On the one hand, a more precise signal implies that the investor faces

\textsuperscript{1}See Bénabou and Tirole (2016) and Barberis (2018) for recent surveys of the literatures in economics and finance, respectively.

\textsuperscript{2}One can view the benchmark rational expectations approach as a special case of our model in which the cost of deviations in beliefs is infinite.
less uncertainty about the trading opportunity and can trade more aggressively. This implies that interpreting a signal as being more precise (than it actually is) increases an investor’s anticipated utility — we refer to this as the information channel. This effect applies to both private signals and the information in the price, which leads an investor to overweight both signals. On the other hand, a more precise price tracks fundamentals more closely, which reduces the gains from speculating against other investors. As a result, believing that the price is more informative reduces anticipated utility — we term this the speculative channel. We show that the information channel leads investors to overweight their private information while the offsetting effects of both channels leads investors to underweight the information in prices.

We also show that allowing investors to choose their beliefs leads not only to different implications for individual behavior but also changes the very nature of the observed equilibrium. When investors choose their beliefs under optimal expectations, we find that a unique, symmetric equilibrium exists when investor risk-aversion is sufficiently large. However, when risk aversion is sufficiently low, we also show that a symmetric equilibrium cannot exist. Instead, there may exist asymmetric equilibria in which a majority of investors objectively interpret their private signals and ignore the price, but a minority over-weight both their private signals and the information in the price. Such asymmetric equilibria help us understand why ex-ante symmetric investors can endogenously choose to separate into fundamental (or value) investors and technical traders. Moreover, such endogenous investor heterogeneity does not commonly arise in standard models, because investors are simply assumed to exhibit rational expectations or specific behavioral biases. As such, our analysis may help shed light on why some systematic deviations are more prevalent in certain settings, but not in others.

We find that the implications of allowing for belief choice on observable return-volume characteristics appear to be generally consistent with the empirical evidence. For instance, we show that across both the symmetric and asymmetric equilibria, our model generates higher return predictability, higher trading volume, and higher price informativeness than the rational expectations benchmark. Intuitively, these results are driven by the fact that investors (in the aggregate) under-react to the price and over-react to their private information. We also show that return volatility is higher than in the rational expectations equilibrium when price informativeness is sufficiently high, but lower otherwise.

Finally, we characterize investor welfare in our setting and compare it to the rational expectations benchmark. Note that the expected utility of investors depends on the reference

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3In the limit, if the price is perfectly informative, there are no gains from trading.
beliefs.\footnote{Given that they choose to deviate from rational expectations, the investors’ anticipated utility net of costs must be higher than under the rational expectations equilibrium (according to their subjective beliefs). However, from the perspective of a social planner who is restricted to objective beliefs, investors’ expected utility is lower when they deviate from rational expectations because their demand for the risky asset is suboptimal.} We take a conservative stance on welfare by (i) evaluating expected utility under objective beliefs and (ii) focusing on the disutility that investors incur due to their suboptimal choices. Using this measure, informed investors are always worse off than in a rational expectations equilibrium. However, we find that the noise traders are always better off under optimal expectations because they incur a lower price impact on their trades relative to the rational expectations equilibrium. Surprisingly, we find that overall welfare can even be higher under optimal expectations. We show that this depends on the aggregate risk aversion: welfare is lower under the rational expectations equilibrium when risk aversion is sufficiently high, but higher otherwise.

Our analysis suggests that focusing on the rational expectations benchmark is restrictive: when given a choice, investors optimally deviate from holding such objective beliefs. Moreover, our results suggest that allowing for such deviations may be helpful in understanding observable patterns in returns and trading volume. Our analysis also highlights a potential advantage of allowing for belief choice over the standard approach in the behavioral economics literature which assumes the existence of such deviations (e.g., over-confidence and differences of opinions) and then studies their impact in isolation. The equilibria which arise in our setting suggest that, in a richer framework, such deviations can arise together and interact with each other in novel ways, as in our symmetric equilibria. As importantly, even in settings where investors are ex-ante identical, the analyzed asymmetric equilibria highlights that allowing for belief choice can lead to endogenous heterogeneity in investor type in equilibrium where subsets of investors exhibit different systematic deviations in the same economy.

We conclude this section by reviewing the related literature. Section 2 introduces the model setup and financial market equilibrium, given investor beliefs. Section 3 studies the tradeoffs associated with deviating from rational expectations. Section 4 studies a setting in which the investor chooses beliefs only about her private signal while in Section 5, we solve the more general setting and explore the models’ implications. Section 6 analyzes the impact on welfare implications and Section 7 concludes. Proofs and extensions can be found in the Appendix.
Related literature

The investor behavior we propose has an important precedent in Brunnermeier and Parker (2005)’s influential work on optimal expectations. Brunnermeier and Parker introduce a model of utility-serving biases in beliefs in which agents care about expected future utility flows: they are happier (sadder) if they overestimate (underestimate) the probabilities of states of the world in which their investments pay off. The investors in our framework behave similarly: they trade off a higher anticipated utility against the utility loss of deviating from the objective distribution. Brunnermeier et al. (2007) apply this framework to understand portfolio under-diversification and a preference for skewness in a representative agent setting. In contrast, we extend this behavior to settings where investors form beliefs not only about exogenous variables (fundamentals, signals) but also endogenous objects (equilibrium prices).

Bénabou and Tirole (2016) survey the broader literature on belief choice. They emphasize that subjective beliefs often fill both psychological and functional needs for individuals. Our paper is closely related to the former channel, in which people optimally manipulate their beliefs in order to improve their perception of the future, trading off such benefits against the costly mistakes they induce. In this class of models, beliefs directly enter into the agent’s utility. Following closely related reasoning, Caplin and Leahy (2001) assume that today’s utility (happiness) depends in some way on beliefs regarding future outcomes.

Our paper contributes to the growing literature that studies the effects of deviations from rational expectations in financial markets, and is most closely related to two strands. The first strand focuses on differences of opinion, whereby investors “agree to disagree” about the joint distribution of payoffs and signals and therefore, incorrectly condition on the information in prices (e.g., Banerjee et al. (2009), Banerjee (2011) and the references in these papers). The second strand of the literature focuses on the impact of overconfidence in financial markets, whereby investors have incorrect valuations and believe in them too strongly (see Daniel and Hirshleifer (2015) for a recent survey). These papers highlight how differences of opinion and overconfidence can help explain a number of stylized facts about financial markets that are difficult to reconcile in the rational expectations framework, including excess trading volume and return predictability. In contrast to the existing literature, however, we do not assume that investors exhibit differences of opinions or over-confidence. Instead, investors are allowed to choose their beliefs, and importantly, exhibiting rational expectations is within their choice set. We show that, given the choice, investors optimally choose to deviate from rational expectations, and we characterize conditions under which they endogenously choose to exhibit both overconfidence and differences of opinions.

Other approaches that lead investors to discount the information in prices include models that feature cursedness (e.g., Vayanos et al. (2018)) and costly learning from prices (e.g., Vives and Yang (2017)).
2 Model setup

Asset payoffs. There are two securities. The gross return on the risk-free security is normalized to one. The terminal payoff (fundamental value) of the risky security is $F$, which is normally distributed with mean $m$ and prior precision $\tau$, i.e.,

$$F \sim \mathcal{N}(m, \frac{1}{\tau}). \quad (1)$$

We denote the market-determined price of the risky security by $P$, and the aggregate supply of the risky asset by $z$, where

$$z \sim \mathcal{N}\left(0, \frac{1}{\tau_z}\right). \quad (2)$$

Information. There is a continuum of investors, indexed by $i \in [0, 1]$. Before trading, each investor is endowed with a private signal $s_i$, where

$$s_i = F + \varepsilon_i \quad \varepsilon_i \sim N\left(0, \frac{1}{\tau_\varepsilon}\right) \quad (3)$$

and $\varepsilon_i$ is independent and identically distributed across investors so that $\int \varepsilon_i \, di = 0$. Moreover, investors can update their beliefs about $F$ by conditioning on the information in the price $P$.

Beliefs and preferences. Each investor $i$ is endowed with initial wealth $W_0$ and zero shares of the risky security, and exhibits CARA utility with coefficient of absolute risk aversion $\gamma$ over terminal wealth $W_i$:

$$W_i = W_0 + x_i (F - P), \quad (4)$$

where $x_i$ denotes her demand for the risky security. We allow each investor to interpret the quality of the information in both her private signals and the price subjectively. Specifically, we assume that investor $i$ believes that the noise in her private signal is given by\(^6\)

$$\varepsilon_i \sim_i \mathcal{N}\left(0, \frac{1}{\delta_{e,i} \tau_\varepsilon}\right), \quad (5)$$

\(^6\)Here $\sim_i$ denotes “distributed as, according to investor $i$’s beliefs.”
and believes the distribution of the asset’s aggregate supply, which as we show below is proportional to the noise in the signal investors can extract from the price, is given by

$$z_i \sim N \left(0, \frac{1}{\delta_{z,i} \tau_z} \right).$$

(6)

In what follows, we will denote the expectation and variance of random variable $X$ under investor $i$’s beliefs about the information environment, i.e., $\delta_{e,i}$ and $\delta_{z,i}$ by $\mathbb{E}_i [X]$ and $\text{var}_i [X]$, respectively. As is standard, we will denote the expectation and variance of $X$ under the true (objective) distribution by $\mathbb{E} [X]$ and $\text{var} [X]$, respectively.

The parameters $\delta_{e,i}, \delta_{z,i} \in [0, \infty]$ reflect the degree to which investor $i$ over- or under-estimates the precision of the private signal and aggregate noise, respectively. When $\delta_{e,i} = \delta_{z,i} = 1$, investor $i$’s beliefs satisfy rational expectations: her beliefs coincide with the objective distribution of the underlying shocks. On the other hand, when $\delta_{e,i} > 1$, investor $i$ is overconfident about her private signal: she believes her private signal is more informative than it objectively is and she overweights it in forming her beliefs. The opposite is true when $\delta_{e,i}$ is less than one. Similarly, when $\delta_{z,i} > 1$ ($\delta_{z,i} < 1$), investor $i$ believes the price to be more informative (less informative, respectively) about fundamentals. We assume that such deviations from the objective distribution impose a utility cost, denoted by $C(\delta_{e,i}, \delta_{z,i})$.

Given her choice of subjective beliefs, each investor optimally chooses her position in the risky security. Thus, optimally chosen subjective beliefs maximize her anticipated utility, net of cost $C(\cdot)$. Formally, denote investor $i$’s optimal demand, given her beliefs, by:

$$x^*_i(\delta_{e,i}, \delta_{z,i}) = \arg \max_{x_i} \mathbb{E}_i \left[- \exp \{-\gamma x_i (F - P) - \gamma W_0\} \mid s_i, P \right].$$

(7)

and denote investor $i$’s anticipated utility by

$$AU_i(\delta_{e,i}, \delta_{z,i}) \equiv \mathbb{E}_i \left[- \exp \{-\gamma x^*_i (F - P) - \gamma W_0\} \mid s_i, P \right].$$

(8)

Then, investor $i$ optimally chooses subjective beliefs $\delta_{e,i}$ and $\delta_{z,i}$ to maximize:

$$\max_{\delta_{e,i}, \delta_{z,i}} AU_i(\delta_{e,i}, \delta_{z,i}) - C(\delta_{e,i}, \delta_{z,i}).$$

(9)

We assume that the cost function $C(\cdot)$ penalizes deviations from the objective distribution (i.e., from $\delta_{e,i} = \delta_{z,i} = 1$) and is well-behaved as defined below.

**Definition 1.** A cost function $C(\delta_{e,i}, \delta_{z,i})$ is well-behaved if $C(1, 1) = \frac{\partial C}{\partial \delta_{e,i}} (1, 1) = \frac{\partial C}{\partial \delta_{z,i}} (1, 1) = 0$, and $C$ is strictly convex (i.e., its global minimum is at $(1, 1)$).
While many of our results apply to general cost functions, our main analysis focuses on a closely related setting: the “optimal expectations” framework introduced in Brunnermeier and Parker (2005). In this setting, the cost each investor incurs by distorting her subjective beliefs is the reduction in expected utility (under the objective distribution) when her position in the risky asset, \( x^*_i (\delta_{e,i}, \delta_{z,i}) \), is determined by the chosen subjective distribution. As is well-established, any deviation from the rational expectations benchmark \((\delta_{e,i} = \delta_{z,i} = 1)\) is objectively inefficient: the investor is over- or under-weighting the information she receives.

**Definition 2.** Investor \( i \) exhibits **optimal expectations** if her beliefs maximize:

\[
\max_{\delta_{e,i}, \delta_{z,i}} AU_i (\delta_{e,i}, \delta_{z,i}) + \mathbb{E} \left[ -\gamma \exp \left\{ -\gamma x^*_i (\delta_{e,i}, \delta_{z,i}) \times (F - P) - \gamma W_0 \right\} \right].
\]

As we show in the appendix, the cost of forming optimal expectations \((C_{\text{obj}} (\delta_{e,i}, \delta_{z,i}))\) is well-behaved, where we define

\[
C_{\text{obj}} (\delta_{e,i}, \delta_{z,i}) = \mathbb{E} \left[ -\gamma \exp \left\{ -\gamma x^*_i (\delta_{e,i}, \delta_{z,i}) \times (F - P) - \gamma W_0 \right\} \right] - \mathbb{E} \left[ -\gamma \exp \left\{ -\gamma x^*_i (1, 1) \times (F - P) - \gamma W_0 \right\} \right].
\]

### 2.1 Financial market equilibrium

We first solve for the financial market equilibrium, taking investors’ chosen subjective beliefs as given. We consider equilibria in which the price \( P \) is a linear combination of fundamentals \( F \) and noise trading \( z \), and conjecture that observing the price is equivalent to observing a signal of the form:

\[
s_p = F + \beta z.
\]

The variance of this signal is \( \tau_p^{-1} = \beta^2 / \tau_z \), and \( \beta \) is a constant determined in equilibrium. Given investor \( i \)’s subjective beliefs \( \delta_{e,i} \) and \( \delta_{z,i} \), and conditional on her observed signals, \( s_i \) and \( s_p \), investor \( i \)’s posterior subjective beliefs are given by:

\[
F|s_i, s_p \sim_i \mathcal{N} \left( \mu_i, \frac{1}{\omega_i} \right), \quad \text{where}
\]

\[
\mu_i \equiv \mathbb{E}_i [F|s_i, s_p] = m + A_i (s_i - m) + B_i (s_p - m),
\]

\[
\omega_i \equiv \frac{1}{\text{var}_i [F|s_i, s_p]} = \frac{\tau}{1 - A_i - B_i}; \quad \text{and}
\]

\[
A_i \equiv \frac{\delta_{e,i} \tau_e}{\tau + \delta_{e,i} \tau_e + \delta_{z,i} \tau_p}, \quad \text{and} \quad B_i \equiv \frac{\delta_{z,i} \tau_p}{\tau + \delta_{e,i} \tau_e + \delta_{z,i} \tau_p}.
\]
The optimal demand for investor $i$, given her subjective beliefs, is given by

$$x_i^* = \frac{E_i [F | s_i, P] - P}{\gamma \text{var}_i [F | s_i, P]} = \frac{\omega_i}{\gamma} (\mu_i - P).$$

Equilibrium prices are determined by market clearing:

$$\int_i x_i^* di = z,$$  \hspace{1cm} (18)

which implies:

$$P = \frac{\int_i \omega_i \{m + A_i (F - m) + B_i (s_p - m)\} di}{\int_i \omega_i di} - \frac{\gamma}{\int_i \omega_i d\bar{z}}.$$  \hspace{1cm} (19)

This verifies our conjecture for functional form of the price and we can write

$$\beta = \frac{-\gamma}{\int_i \omega_i A_i di} = -\frac{\gamma}{\tau_e \int_i \delta_{e,i} di}.$$  \hspace{1cm} (20)

The above results are summarized in the following lemma.

**Lemma 1.** Given investor $i$’s subjective beliefs $\delta_{e,i}$ and $\delta_{z,i}$, there always exists a unique, linear, financial market equilibrium in which

$$P = m + \Lambda (s_p - m),$$

where

$$\Lambda = \frac{\int_i \delta_{e,i} \tau_e + \delta_{z,i} \tau_p di}{\int_i \tau + \delta_{e,i} \tau_e + \delta_{z,i} \tau_p di},$$

$$s_p = F + \beta z,$$

$$\tau_p = \tau_z / \beta^2,$$

and

$$\beta = -\frac{\gamma}{\tau_e \int_i \delta_{e,i} di}.$$  \hspace{1cm} (21)

When subjective belief choices are symmetric (i.e., $\delta_{e,i} = \delta_e$ and $\delta_{z,i} = \delta_z$ for all $i$), then the price is given by:

$$P = m + \Lambda (s_p - m),$$

where

$$s_p = \left(F - \frac{\gamma}{\tau_e \delta_e} \bar{z}\right),$$

$$\Lambda = \frac{\delta_e \tau_e + \delta_z \tau_p}{\tau + \delta_e \tau_e + \delta_z \tau_p},$$

and

$$\tau_p = \frac{\tau_z \tau_e^2 \delta_e^2}{\gamma^2}.  \hspace{1cm} (22)$$

As the above lemma makes apparent, the choice of investor beliefs affect equilibrium prices through two channels. First, an increase in the perceived precision of private signals (higher $\delta_{e,i}$) increases the signal to noise ratio of the signal $s_p$ (since $|\beta|$ is decreasing in $\delta_{e,i}$). Investors trade more aggressively on their private information which is then reflected in the objective quality of the information in the price. Second, an increase in the perceived precision of either private signals (i.e., higher $\delta_{e,i}$) or price information (i.e., higher $\delta_{z,i}$) increases the sensitivity of the price to fundamentals ($F$) through $\Lambda$. These channels interact to affect a number of empirically observable features of the financial market equilibrium, which we characterize next.
2.2 Return and volume characteristics

Given the financial market equilibrium in Lemma 1, we characterize how the underlying parameters of the model, in combination with investors’ choice of beliefs, affect observable return and volume characteristics and the degree to which prices reflect information. Since the risk-free security is the numeraire, the (net) return on it is zero. Consequently, the (dollar) return on the risky security is given by

\[ R \equiv F - P. \] (23)

Furthermore, because the risky security is in zero net supply, the unconditional expected return is zero i.e., \( \mathbb{E}[R] = 0 \). However, conditional on the price, the expected return can be expressed as:

\[ \mathbb{E}[F - P|P] = m + \theta (P - m), \text{ where } \theta \equiv \frac{\text{cov}(F - P, P)}{\text{var}(P)}. \] (24)

Here \( \theta \) reflects the degree to which the returns are predictable and, as such, we refer to it as the return predictability coefficient. The unconditional variance in returns is given by

\[ \sigma^2_R = \text{var}(F - P), \] (25)

while the conditional variance in returns is characterized by

\[ \text{var}(F - P|P) = \tau_p^{-1}. \] (26)

Note that the conditional variance in returns is inversely related to one measure of price informativeness, as \( \tau_p \) reflects how precise the price signal is about fundamentals \( F \). Finally, since investors start without an endowment of the risky security, trading volume in our economy can be characterized as

\[ V \equiv \int_i |x_i^*| \, di. \] (27)

Given investor beliefs, the following result describes how these return-volume characteristics depend on the underlying parameters.

**Lemma 2.** Consider the unique financial market equilibrium described in Lemma 1. Then,

- (ii) the unconditional variance in returns is \( \sigma^2_R = \frac{(1-\Lambda)^2}{\tau} + \frac{\Lambda^2 \beta^2}{\tau_2} \),
- (iii) the return predictability coefficient is \( \theta = \frac{1}{\Lambda} \left( \frac{1/\tau}{\beta^2/\tau_2 + 1/\tau} - \Lambda \right) \),
- (iv) price informativeness is \( \tau_p = \tau_2 / \beta^2 \), and
expected trading volume is

\[ 
\mathbb{E} [\mathcal{V}] = \int \frac{\omega_i}{\gamma} |\mu_i - P| \, di = \int \frac{\omega_i}{\gamma} \sqrt{\frac{2}{\pi}} \left( \frac{A_i^2}{\tau_e} + \frac{(B_i - \Lambda)^2 \beta^2}{\tau_z} + \frac{(A_i + B_i - \Lambda)^2}{\tau} \right) \, di, 
\]

(28)

where \( \omega_i, A_i, B_i, \beta \) and \( \Lambda \) are defined in (15)-(16) and Lemma 1.

To provide intuition for the dependence of these equilibrium characteristics on the underlying parameters, we make use of the signal to noise ratio (or Kalman gain) for the price signal \( s_p \), which can be written as

\[ 
\kappa \equiv \frac{\text{var}(F)}{\text{var}(s_p)} = \frac{1/\tau}{\beta^2/\tau_z + 1/\tau} = \frac{\tau_p}{\tau + \tau_p}. 
\]

(29)

First, note that an increase in the price sensitivity \( \Lambda \) has two offsetting effects on return volatility. On the one hand, when the price is more sensitive to \( s_p \), it reflects fundamentals more closely, and this decreases volatility (through the \( (1-\Lambda)^2/\tau \) term). On the other hand, a higher \( \Lambda \) also implies that the price is more sensitive to shocks to the asset supply, which increases volatility (through the \( \Lambda^2 \beta^2/\tau_z \) term). The first effect dominates when the price sensitivity \( \Lambda \) is lower than the signal to noise ratio \( \kappa \) (i.e., \( \Lambda < \kappa \)), while the latter effect dominates when price sensitivity is higher.

Second, note that the return predictability coefficient is positive when the signal to noise ratio is higher than the price sensitivity (i.e., \( \theta > 0 \)). In this case, prices exhibit drift — a higher price today predicts higher future returns. On the other hand, when the signal to noise ratio is lower than \( \Lambda \), prices exhibit reversals. Comparing the expression of \( \kappa \) above to \( \Lambda \) in (21), prices cannot exhibit drift unless investors under-react to price information (i.e., \( \delta_e,i < 1 \)). Conversely, when investors correctly interpret the precision of price information, the prices exhibit reversals (i.e., \( \delta_e,i = 1 \) implies \( \Lambda > \kappa \)).\footnote{This is because, as we show below, in equilibrium investors never choose to set \( \delta_{e,i} = 0 \).} In particular, prices exhibit reversals when investors exhibit rational expectations. Such reversals arise because the aggregate supply of the asset is subject to transitory shocks and investors are risk-averse. Moreover, we note that holding fixed the signal to noise ratio \( \kappa \), an increase in price sensitivity \( \Lambda \) decreases return predictability: even though an increase in \( \Lambda \) increases the covariance of \( F \) and \( P \), this effect is swamped by the increase in the variance of the price.

Third, price informativeness, \( \tau_p \), naturally decreases in the magnitude of \( \beta \) — when investors have less private information, the price is more sensitive to aggregate supply shocks. Finally, note that volume reflects the cross-sectional variation across investor valuations (i.e., \( \mu_i \)), scaled by their posterior variance (i.e., \( \omega_i^{-1} \)). The variation is driven by three channels:
(i) the weight each investor’s beliefs place on the noise in their private signals (i.e., the $\frac{1}{\tau_e}$ term), (ii) the relative weight on the noise in prices (i.e., the $\frac{\beta^2}{\tau_z}$ term) and (iii) the relative weight on the true fundamental value (i.e., the $\frac{1}{\tau_f}$ term). Note that the last term is absent in symmetric equilibria, since $A_i + B_i = \Lambda$ in this case. However, in asymmetric equilibria, this final term reflects the variation in valuations due to asymmetric reaction to private signals and the information in prices (see Section 5.2).

3 Anticipated utility and subjective beliefs

With the financial market equilibrium established, we can now characterize the optimal subjective beliefs of an investor. Importantly, we assume that each investor takes as given the subjective belief distortion chosen by other investors: she does not assume that other investors hold rational expectations.

Given the optimal demand for the risky asset (17), anticipated utility is given by

$$AU_i (\delta_{e,i}, \delta_{z,i}) = \mathbb{E}_i \left[ -\exp \left\{ -\frac{\left( \mathbb{E}_i [F|s_i, P] - P \right)^2}{2 \text{var}_i [F|s_i, P]} \right\} \right].$$

Moreover, given the characterization of the equilibrium price in Lemma 1, investor $i$’s beliefs about the conditional return are given by:

$$\mathbb{E}_i [\mathbb{E}_i [F|s_i, P] - P] = m - m = 0,$$  

$$\text{var}_i [\mathbb{E}_i [F|s_i, P] - P] = \text{var}_i [F - P] - \text{var}_i [F|s_i, P],$$

where the first equality follows from the law of iterated expectations and the second equality follows from the law of total variance.\(^8\) With this in mind, the above expectation reduces to

$$AU_i (\delta_{e,i}, \delta_{z,i}) = -\sqrt{\frac{\text{var}_i [F|s_i, P]}{\text{var}_i [F - P]}}.$$  

From this, we derive the following result.

**Lemma 3.** Anticipated utility is always (weakly) increasing in $\delta_{e,i}$; it is strictly increasing as long as $\delta_{z,i} > 0$. Anticipated utility is non-monotonic in $\delta_{z,i}$: there exists some $\tilde{\delta}$ such that

\(^8\)The law of total variance implies

$$\text{var}_i [F - P] = \mathbb{E}_i [\text{var}_i [F - P|s_i, P]] + \text{var}_i [\mathbb{E}_i [F - P|s_i, P]],$$

which in turn, implies the above expression.
for all $\delta_{z,i} < \bar{\delta}$ anticipated utility is decreasing in $\delta_{z,i}$ while for all $\delta_{z,i} > \bar{\delta}$ it is increasing.

To gain some intuition, we note that anticipated utility is simply a monotonic transformation of

$$\frac{\text{var}_i [F - P]}{\text{var}_i [F|s_i, P]} = \text{var}_i \left( \frac{\mathbb{E}_i [F - P|s_i, P]}{\sqrt{\text{var}_i [F - P|s_i, P]}} \right) \equiv \text{var}_i (SR_i)$$

(35)

where

$$SR_i \equiv \frac{\mathbb{E}_i [F - P|s_i, P]}{\sqrt{\text{var}_i [F - P|s_i, P]}}$$

(36)

is investor $i$’s conditional Sharpe ratio, given her beliefs. When the variance of the conditional Sharpe ratio is higher, the investor expects to observe both (i) more profitable trading opportunities and (ii) the opportunity to trade more aggressively. Of course, she also faces an increased likelihood of facing the opposite scenario, but the benefit on the upside always outweighs the reduction in expected profits on the downside.\(^9\) As a result, anticipated utility is higher when the variance in the conditional Sharpe ratio is higher.

Intuitively, reducing the perceived uncertainty (i.e., $\text{var}_i [F - P|s_i, P]$) about the trading opportunity is valuable - if the investor has better information about the value of the asset this increases her utility. Increasing the perceived precision of the private signal (i.e., increasing $\delta_{e,i}$) has this effect and so anticipated utility increases when the investor inflates her perception of the quality of the private signal.

On the other hand, increasing the perceived precision of the price signal (i.e., increasing $\delta_{z,i}$) has two off-setting effects. First, the **information effect** of learning from prices reduces the conditional variance $\text{var}_i [F - P|s_i, P]$: the investor has better information about the asset’s value which increases anticipated utility. This information effect reduces the volatility of the perceived return on the risky security, a benefit in and of itself, but it also allows the investor to scale up her trading position. Second, the **speculative effect** of believing prices are more informative decreases the perceived variance of the conditional expected return (i.e., $\text{var}_i (\mathbb{E}_i [F|s_i, P] - P)$), which lowers anticipated utility. Intuitively, when the price is more informative, it tracks fundamentals more closely and, as a result, the trading opportunity is less profitable. The overall effect of changing the perceived precision of the price signal depends on the relative magnitude of these two effects. As we show in the proof of Lemma 3, the latter effect dominates when $\delta_{z,i}$ is low, while the former effect dominates when $\delta_{z,i}$ is sufficiently high, which is what drives the non-monotonicity in $\delta_{z,i}$.

This is the key distinction between learning from private signals and learning from price information and it drives our equilibrium results below. Learning from either source is

\(^9\)This arises because the trading opportunity and the investor’s position act as complements - effectively, the utility is convex in the trading opportunity (as captured by the conditional Sharpe ratio), and so an increase in the perceived variance is beneficial.
informative about fundamentals which naturally increases utility. However, learning from prices also reduces the investor’s perception of the potential trading opportunity. We explore how this distinction leads to differences in investors’ subjective interpretation of private signals and the information in the price in the next two sections.

4 Benchmark: Belief choice about private signals

We begin with a benchmark in which investors are forced to have objective beliefs about the price signal (i.e., we assume $\delta_{z,i} = 1$ for all $i$) but can choose their beliefs about the precision of their private signals. Unsurprisingly, given the intuition laid out above, we find that all investors choose to exhibit over-confidence about their private information in equilibrium.

Proposition 1. Suppose investors have objective beliefs about the price signal i.e., $\delta_{z,i} = 1$ for all $i$, and the cost function $C(\delta_{e,i}, \delta_{z,i})$ is well-behaved. Then, there exists a unique symmetric equilibrium in which all agents are over-confident about their private signal.

With objective beliefs about the informativeness of the price, Lemma 3 implies that an investor’s anticipated utility strictly increases in the perceived precision of her private signal. Since the cost of setting $\delta_{e,i} = 1$ is zero (i.e., $C(1,1) = 0$) and the marginal cost of increasing $\delta_{e,i}$ at $\delta_{e,i} = \delta_{z,i} = 1$ is also zero, investors prefer to optimally choose $\delta_{e,i} > 1$ i.e., they optimally choose to be over-confident about her private signal.

Proposition 2. Suppose investors have objective beliefs about the price signal i.e., $\delta_{z,i} = 1$ for all $i$, and exhibit optimal expectations, i.e., they solve (10). Then, there exists a unique equilibrium in which the optimal choice of $\delta_{e,i} = \delta_e$ satisfies:

$$
\frac{(\tau + \tau_p + \tau_e \delta_e (2 - \delta_e))^{\frac{3}{2}}}{(\tau + \tau_p + \tau_e \delta_e)^{\frac{3}{2}}} = 2 (\delta_e - 1).
$$

(37)

Moreover, the equilibrium overconfidence parameter, $\delta_e$, (i) increases with $\tau$ and $\tau_z$, (ii) decreases with risk-aversion $\gamma$, and (iii) is U-shaped in $\tau_e$.

Consistent with the intuition laid out above, equation (37) shows that in a symmetric equilibrium, $\delta_{e,i} > 1$ for all agents. What drives the degree of overconfidence? As prior uncertainty falls ($\uparrow \tau$) and as the quality of the information in prices rises ($\uparrow \tau_z, \downarrow \gamma$), both the benefit and cost of being overconfident falls: overconfidence is less distortive of the investor’s perceived information advantage. Interestingly, as overconfidence grows, the

\[\text{10The LHS of (37) is always positive, which indicates that } \delta_e > 1.\]
cost falls more quickly, and so when investors have access to better outside information, overconfidence is higher. While similar logic applies with respect to the quality of the investor’s private signal, increasing overconfidence directly distorts how this information is utilized. A result, for low values of $\tau_e$ the benefit of increased overconfidence falls more quickly, which introduces a non-monotonicity in $\delta_e$ as $\tau_e$ increases.

**Corollary 1.** Relative to the rational expectations equilibrium (i.e., when $\delta_{z,i} = \delta_{e,i} = 1$ for all $i$), the equilibrium characterized in Proposition 1 features: (i) lower return volatility, (ii) a less negative predictability coefficient, (iii) higher price informativeness, and (iv) higher expected volume.

These implications follow naturally from the expressions in 2 and noting the fact that because $\delta_e > 1$, the price sensitivity $\Lambda$ in this equilibrium is higher than in the rational expectations equilibrium and both are higher than the signal to noise ratio i.e.,

$$\Lambda = \frac{\delta_e \tau_e + \tau_p}{\tau + \delta_e \tau_e + \tau_p} > \Lambda_{RE} = \frac{\tau_e + \tau_p}{\tau + \tau_e + \tau_p} > \kappa. \quad (38)$$

Our result on return predictability is consistent with Daniel et al. (1998), who argue that investor overconfidence results in price reversals. Moreover, overconfidence induces investors to trade more aggressively based on their signals. This results in more informative prices, which is consistent with empirical evidence in Hirshleifer et al. (1994); Kyle and Wang (1997); Odean (1998); Hirshleifer and Luo (2001). Finally, consistent with the large literature on overconfidence, our model suggests that such behavior by investors can help explain the relatively high trading volume that has been extensively documented empirically.

5 Belief choices and equilibrium characterization

We now turn to the more general setting in which investors can optimally choose their beliefs about both the quality of their private signal as well as the information contained in prices. We begin by characterizing the characteristics of any feasible symmetric equilibrium.

**Proposition 3.** Suppose the cost function $C(\delta_{e,i}, \delta_{z,i})$ is well-behaved. In any symmetric equilibrium, all investors are (weakly) over-confident about their private signal (i.e., $\delta_{e,i} \geq 1$ for all $i$) but choose to under-react to the information in prices (i.e., $\delta_{z,i} < 1$ for all $i$).

Thus, in any symmetric equilibrium, investors always choose to over-confident about their private information (as above) but under-react to the information in prices. This

---

11 This can be seen by evaluating the numerator of equation (37).
is a robust outcome in our setting. Consider the choices $\delta_{e,i}$ and $\delta_{z,i}$ of investor $i$ in a symmetric equilibrium where all other investors choose $\delta_e$ and $\delta_z$, respectively. Recall that for a well-behaved cost function, deviations away from rational expectations (i.e., $\delta_{e,i} = 1$ and $\delta_{z,i} = 1$) are penalized i.e., the cost function is decreasing below one and increasing above one. Since anticipated utility is always (weakly) increasing in $\delta_{e,i}$, this immediately implies any symmetric equilibrium features (weak) over-confidence about private information (i.e., $\delta_{e,i} \geq 1$). Intuitively, increasing the perceived precision of private information always increases anticipated utility, and so it is natural that investors choose to be over-confident about their private signals.

However, as Lemma 3 establishes, anticipated utility is U-shaped in $\delta_{z,i}$. Moreover, as we show in the proof of Proposition 3, when other investors choose $\delta_z$, anticipated utility is decreasing in $\delta_{z,i}$ at $\delta_{z,i} = \delta_z$. This implies the equilibrium choice of $\delta_{z,i}$ cannot be higher than one, since if this were the case, investor $i$ could increase her anticipated utility and decreases her costs by reducing $\delta_{z,i}$, an unambiguously better outcome. Intuitively, in a symmetric equilibrium, investor $i$ has an incentive to decrease the perceived precision of price information relative to the choice of others because by doing so, she improves her ability to speculate on her private information by decreasing the correlation between her conditional valuation ($\mu_i$) and those of other investors (i.e., $\int_j \mu_j dj$), which is reflected in the equilibrium price.

Given the above characterization for arbitrary cost functions, the next subsections characterize conditions for the existence of symmetric equilibria in an optimal expectations setting.

5.1 Symmetric equilibrium and under-reaction to prices

We begin with a sufficient condition for the existence and uniqueness of symmetric equilibria.

**Proposition 4.** Suppose investors exhibit optimal expectations i.e., their beliefs satisfy (10). There exists a $\bar{\gamma}$ such that for all $\gamma \geq \bar{\gamma}$, there exists a unique, symmetric equilibrium in which all investors ignore the information in prices (i.e., $\delta_{z,i} = 0$ for all $i$), and correctly interpret their private signals (i.e., $\delta_{e,i} = 1$ for all $i$).

The plot in Figure 1, panel (a), provides a numerical illustration. The figure plots the marginal anticipated utility (solid) and the marginal cost function (dashed) for an investor $i$ as a function of her choice $\delta_{z,i}$. Recall that deviations away from $\delta_{z,i}$ are costly — as a result, the marginal cost for $\delta_{z,i} < 1$ is negative. Moreover, note that Lemma 3 implies that the marginal anticipated utility is negative below a threshold $\bar{\delta}$ (which is a little above 1.5 in the plot). Finally, note that while the marginal anticipated utility when $\delta_{z,i} = 0$ is $-\infty$, the marginal cost in an optimal expectations framework is always negative but finite. At any
Figure 1: Marginal Anticipated Utility vs. Marginal Cost for Optimal Expectations

The figure shows marginal anticipated utility (solid black line) and marginal cost function (dashed orange line) as a function of $\delta_{zi}$. The marginal cost function is under the assumption that investors exhibit optimal expectations (i.e., their beliefs satisfy (10)). Other parameters are: $\tau = \tau_e = \tau_z = \delta_e = \delta_{e,i} = 1$, $\delta_z = 0.5$.

alternative symmetric equilibrium the marginal benefit and marginal cost must intersect.\(^{12}\)

A sufficiently high $\gamma$ ensures that (i) the marginal anticipated utility curve intersects zero at a point to the right of $\delta_{z,i} = 1$, and (ii) the marginal cost curve is sufficiently flat between $\delta_{z,i} = 0$ and $\delta_{z,i} = 1$. This in turn ensures that the two curves never intersect, and the symmetric equilibrium is a corner solution at $\delta_{z,i} = 0$.

Intuitively, when $\gamma$ is high, price informativeness $\tau_p$ is relatively low. In this case, the speculative effect of learning from prices dominates the information effect, and investors prefer to under-weight the information in prices. When $\gamma$ is sufficiently high, the price is sufficiently uninformative, and investors optimally choose to ignore the information in prices. Since the marginal anticipated utility does not change with $\delta_{e,i}$ when $\delta_{z,i} = 0$, investors optimally choose to correctly interpret their private information (i.e., $\delta_{e,i} = 1$).

Given the above equilibrium, the following result compares return-volume characteristics to the rational expectations benchmark.

**Corollary 2.** Relative to the rational expectations equilibrium (i.e., when $\delta_{z,i} = \delta_{e,i} = 1$ for all $i$), the equilibrium characterized in Proposition 4 features: (i) higher predictability coefficient, (ii) equal price informativeness, and (iii) equal expected volume. Return volatility is higher than in the rational expectations equilibrium iff price informativeness is sufficiently

\(^{12}\)Specifically, any other potential maximum lies at every second intersection of the two curves.
high (i.e., \( \tau_p \geq \sqrt{\frac{\tau^2 + 8\tau \tau_e + 8\tau_e^2}{2}} - \frac{\tau}{2} \)).

Since \( \delta_e = 1 \) in the symmetric equilibrium, the signal to noise ratio \( \kappa \) is the same as in the rational expectations equilibrium (since \( \beta = -\gamma/\tau_e \)). However, the price sensitivity in the symmetric equilibrium, \( \Lambda_{SE} = \frac{\tau_e}{\tau + \tau_e} \), is lower than that in the rational expectations equilibrium (i.e., \( \Lambda_{RE} = \frac{\tau_e + \tau_p}{\tau + \tau_e + \tau_p} \)). Following the discussion of Lemma 2, this immediately implies the results on the predictability coefficient and price informativeness. The volume remains the same across the two symmetric equilibria since the investors weight their private signals correctly (i.e., \( \delta_{e,i} = 1 \)) in either case. This implies that the cross-sectional variation in valuations, scaled by their posterior variance, remains unaffected by whether or not they condition on prices.\(^{13}\) Finally, we characterize conditions under which return volatility is higher under the optimal expectations equilibrium. Recall that return volatility is decreasing in \( \Lambda \) when the signal to noise ratio \( \kappa = \frac{\tau_e}{\tau + \tau_p} \) is sufficiently high (relative to \( \Lambda \)). The condition on \( \tau_p \) above ensures that the decrease in \( \Lambda \) (from \( \Lambda_{RE} \) to \( \Lambda_{SE} \)) leads to an increase in volatility.

5.2 Asymmetric equilibrium and specialization of beliefs

The next result establishes sufficient conditions to rule out the existence of a symmetric equilibrium.

**Proposition 5.** Suppose investors exhibit optimal expectations i.e., their beliefs satisfy (10). There exists a \( \gamma \) such that for all \( \gamma < \gamma \), there cannot exist a symmetric equilibrium.

Again, consider the numerical example plotted in Figure 1, panel (b). When \( \gamma \) is sufficiently low, the marginal anticipated utility curve crosses zero below \( \delta_{z,i} = 1 \). This implies that there is always a (local) maximum, corresponding to the second intersection of the solid and dotted lines. Investor \( i \) might prefer to deviate from the corner (\( \delta_{z,i} = 0 \)) to this interior maximum, if her expected anticipated utility, net of cost, is higher.

Intuitively, this can occur when \( \gamma \) is sufficiently low because price informativeness is sufficiently high (investors trade more aggressively on their information). Moreover, in any symmetric equilibrium, investors under-react to the information in prices. Together, these imply that an individual investor may have an incentive to deviate and condition more aggressively on the information in prices — in such a case, the speculative effect is dominated by the information effect. But such profitable deviations rule out a symmetric equilibrium.

The plots in Figure 2 provide a numerical example. The panels show investor \( i \)'s anticipated utility, net of costs, as a function of \( \delta_{z,i} \), given the behavior of others. In panel

\(^{13}\)This is apparent in the limit when there is no private information i.e., \( \tau_e = 0 \). In this case, volume is zero in any symmetric equilibrium, irrespective of whether investors condition on prices or not.
(a), all other investors choose \( \delta_z = 0 \). In this case, investor \( i \) has an incentive to deviate by over-weighting the information in prices (i.e., by setting \( \delta_{zi} \approx 1.3 \)). Even though the price is objectively very informative (large information effect), because other investors are ignoring it (\( \delta_z = 0 \)), the speculative effect of over-weighting the price is relatively small. In panel (b), we consider an alternative symmetric equilibrium in which all other investors choose \( \delta_z > 0 \). Now, the speculative effect dominates and investor \( i \) strictly prefers to ignore the information in prices. In both cases, a symmetric equilibrium is ruled out because an individual investor has an incentive to deviate from the equilibrium behavior.

Given the non-existence of symmetric equilibria, we numerically explore the existence of asymmetric equilibria in which investors mix between two sets of beliefs. We assume a fraction \( \lambda \) optimally chooses \( \delta_{e,i} = 1 \) and \( \delta_{z,i} = 0 \), while the remaining fraction \( 1 - \lambda \) optimally chooses \( \delta_{e,i} = \delta_e \) and \( \delta_{z,i} = \delta_z \). The following result characterizes the existence of such an equilibrium.

**Lemma 4.** An asymmetric equilibrium is characterized by the triple \((\lambda, \delta_e, \delta_z)\) which solve a system of three equations (specified in the Appendix). The equilibrium price is given by:

\[
P = m + \Lambda_{AE} \left( s_p - m \right),
\]

where \( s_p = F - \frac{\gamma}{\delta_e \tau_e} z \).

\[
\Lambda_{AE} \equiv \frac{\delta_e \tau_e + \delta_z \tau_{p,AE}}{\tau + \delta_e \tau_e + \delta_z \tau_{p,AE}}, \quad \tau_{p,AE} = \frac{\tau_z \tau_e^2 \delta_e^2}{\gamma^2},
\]

(39)

\( \bar{\delta}_e = (\lambda + (1 - \lambda) \delta_e) \) and \( \bar{\delta}_z = (1 - \lambda) \delta_z \). Moreover, \( \delta_e, \delta_z \leq 3/2 \), and \( \Lambda_{AE} < \Lambda_{RE} = \frac{\tau_e + \tau_p}{\tau + \tau_e + \tau_p} \) under some conditions reported in the appendix.

Panel (c) of Figure 2 illustrates an instance of the asymmetric equilibrium. In this case, each investor is indifferent between two (sets of) beliefs. In equilibrium, one set of investors (a fraction \( \lambda = 0.95 \)) ignore the information in prices while the remaining fraction \( 1 - \lambda = 0.05 \) over-weight the information in prices.

**Corollary 3.** Relative to the rational expectations equilibrium (i.e., when \( \delta_{z,i} = \delta_{e,i} = 1 \) for all \( i \)), the equilibrium characterized in Proposition 5 features: (i) higher predictability coefficient, (ii) higher price informativeness, and (iii) higher expected volume.

The asymmetric equilibrium has three main implications. First, the return predictability coefficient is higher than in a rational expectations equilibrium and can even be positive. In the traditional noisy-rational expectations setting with exogenous, transient noise trading (e.g., Hellwig (1980)), returns exhibit reversals. Intuitively, an aggregate demand (supply) shock temporarily pushes the current price up (down, respectively), but since the shock is not persistent, prices revert in the future. In our model, because some investors underweight
Figure 2: Anticipated utility net of costs versus $\delta_{z,i}$
The figure plots the anticipated utility net of costs for investor $i$ as a function of her choice $\delta_{z,i}$. Other parameters are: $\tau = \tau_e = \tau_z = \delta_e = \delta_{e,i} = 1$, $\gamma = 0.3$

Figure 3 provides a numerical illustration of these results. Specifically, the figure plots price sensitivity, predictability, volume and volatility for the rational expectations (dashed) and optimal expectations equilibria (solid) as a function of risk aversion $\gamma$. Moreover, the kink in the solid lines corresponds to the value of $\gamma$ at which the optimal expectations equilibrium switches from the asymmetric equilibrium (low $\gamma$) to the symmetric equilibrium (higher $\gamma$). Consistent with the predictions of Corollaries 2 and 3, predictability and volume are (weakly) higher under optimal expectations than under rational expectations. Moreover, volatility is higher under optimal expectations when risk aversion is sufficiently low (i.e., $\tau_p$ is sufficiently high), but is higher otherwise. Finally, in this parameter region, the price
Figure 3: Comparison of return and volume characteristics
The figure plots price sensitivity, return predictability, trading volume and return volatility (variance) as a function of risk aversion for optimal expectations (solid line) and rational expectations (dotted line). Other parameters are set to $\tau = \tau_e = \tau_z = 1$.

sensitivity $\Lambda$ is always lower for the optimal expectations equilibrium.

6 Welfare implications

In this section, we explore the welfare implications of allowing investors to choose their beliefs optimally. We begin by noting that welfare for the informed investors depends on the reference beliefs used. From the perspective of the investors' subjective beliefs, expected utility is higher when they deviate from rational expectations. However, from the perspective of a social planner who is restricted to hold objective beliefs, expected utility for investors is lower when they deviate from rational expectations - their demand for the risky asset is suboptimal given their information sets.

We can also consider the effect of deviations from rational expectations for liquidity or noise traders. Recall that the aggregate supply, $z$, is noisy. Suppose this reflects the sale
of the risky asset by a liquidity trader, who has CARA utility with risk aversion $\gamma$ and is endowed with initial wealth $W_0$. Then, her expected utility is given by

$$U_z \equiv \mathbb{E} \left[ -\gamma \exp \left\{ -\gamma (-x) \times (F - P) - \gamma W_0 \right\} \right].$$

(40)

The following result compares the expected utility of liquidity traders in the constrained and unconstrained equilibria.

**Proposition 6.** Liquidity traders have higher expected utility in the symmetric equilibrium than in the rational expectations equilibrium. In any asymmetric equilibrium in which $\Lambda_{AE} < \Lambda_{RE}$, liquidity traders have higher expected utility in the asymmetric equilibrium than in the rational expectations equilibrium.

The result suggests, perhaps surprisingly, that liquidity traders are better off when informed investors choose their beliefs. To see why, it is illustrative to consider the (conditional) expected utility of the liquidity trader if she had mean-variance preferences.\(^{14}\) Selling $z$ units gives her utility $u(z)$, where

$$u(z) = -z \mathbb{E} [F - P | z] - \frac{1}{2} \gamma z^2 \text{var} (F - P | z)$$

(42)

$$= \beta \Lambda z^2 - \frac{1}{2} \gamma z^2 \left( 1 - \Lambda \right)^2 \frac{1}{\tau},$$

(43)

where $\beta = -\frac{\gamma}{\tau \Lambda} < 0$. A liquidity trader’s utility is driven by two components. The first component ($\beta \Lambda z^2$) reflects her disutility from price impact — for instance, a larger sale (higher $z$) pushes prices downward, which reduces her proceeds. The second term ($- \frac{1}{2} \gamma z^2 \left( 1 - \Lambda \right)^2 \frac{1}{\tau}$) reflects a standard risk-aversion channel — when prices are less informative about fundamentals, the liquidity trader faces more uncertainty about her payoff, which reduces utility.

As discussed earlier, price sensitivity, $\Lambda$, is higher when investors exhibit rational expectations: $\Lambda_{RE} > \Lambda_{SE}, \Lambda_{AE}$. This has offsetting effects on the liquidity trader’s utility. On the one hand, a higher $\Lambda$ implies that the price is more sensitive to her trade and so utility falls through the price impact channel. On the other hand, a higher $\Lambda$ implies prices track fundamentals more closely which reduces the risk in the liquidity trader’s payoff. In the proof of Proposition 6, we show that in our setting the price impact effect always dominates the risk-aversion effect. As a result, liquidity traders are always better off when investors choose to deviate from rational expectations.

\(^{14}\)Note that by the Law of Iterated Expectations, we have:

$$U_z \propto \mathbb{E} \left[ \mathbb{E} \left[ -\gamma \exp \left\{ -\gamma (-x) \times (F - P) \right\} | z \right] \right] = \mathbb{E} \left[ -\exp \left\{ -\gamma u(z) \right\} \right],$$

(41)

so considering mean-variance preferences is qualitatively without loss of generality.
The overall effect on welfare (accounting for both liquidity and informed investors) depends on the relative magnitude of the effects. Figure 4 plots the difference in expected utility between the optimal expectations equilibrium and the rational expectations equilibrium for each group separately, and for both groups as a whole. In line with the above result, the liquidity traders are always better off under optimal expectations — the dotted line is always above zero. Similarly, and as expected, under the objective distribution, the informed investors are worse off under the optimal expectations equilibrium — the dashed line is always below zero. Note that this is a conservative measure of expected utility as it only accounts for the costs of deviating from rational expectations and does not include any of the gains from anticipated utility.

Aggregate welfare appears to be higher in the rational expectations equilibrium when risk aversion is low, but higher under optimal expectations when risk aversion is high. Note that an increase in $\gamma$ amplifies the wedge in expected utility (since these are scaled by $\gamma$). Moreover, when risk aversion is sufficiently high, price informativeness is low and investors trade less aggressively. These two effects imply that the relative cost of distorted beliefs in the optimal expectations equilibrium is lower (the dashed line is eventually increasing in $\gamma$). Together these effects lead to the relative ranking of welfare across the two equilibria.
7 Conclusions

In this paper, we analyze how investors rationally distort their beliefs about their informational environment in the context of an otherwise standard competitive trading environment. We show that in any symmetric equilibrium, investors are always (i) weakly overconfident in the quality of their private signal (so that their perceived private information advantage is preserved or amplified), and (ii) discount the quality of the information in prices (so that their perceived trading opportunity is maximized). We then demonstrate conditions under which a pure differences of opinion equilibrium can arise, i.e., where agents use their private signals objectively but do not update on the information in prices. Finally, we analyze a type of asymmetric equilibrium in which ex-ante symmetric investors choose to separate into two types: fundamental investors who focus on their private signals, and technical traders who overweight the information in prices. Our analysis of these equilibria yields several insights that can help better explain patterns in both asset prices and trading volume. Moreover, we show that such systematic deviations in subjective beliefs can lead to aggregate welfare improvements, relative to a rational expectations benchmark.
References


8 Appendix - Proofs

8.1 Proof of Lemma 3

Lemma 1 implies that the price is of the form: \( P = m + \Lambda (s_p - m) \). This implies anticipated utility is given by

\[
AU(\delta_{e,i}, \delta_{z,i}) = -\frac{1}{\tau + \delta_{e,i} \tau_e + \delta_{z,i} \tau_p} \sqrt{(1 - \Lambda)^2 \frac{1}{\tau} + \Lambda^2 \frac{1}{\delta_{z,i} \tau_p}}. \tag{44}
\]

Note that given other investors’ choices, investor \( i \)'s marginal anticipated utility is

\[
\frac{\partial}{\partial \delta_{e,i}} AU = \frac{\tau_e}{2 (\tau + \delta_{e,i} \tau_e + \delta_{z,i} \tau_p)} \times \sqrt{\frac{1}{(1 - \Lambda)^2 \frac{1}{\tau} + \Lambda^2 \frac{1}{\delta_{z,i} \tau_p}}} \geq 0 \tag{45}
\]

\[
\frac{\partial}{\partial \delta_{z,i}} AU = \frac{(1 - \Lambda)^2 \delta_{z,i} \tau_p^2 - \Lambda^2 \tau (\delta_{e,i} \tau_e + \tau)}{2 \delta_{z,i} (\Lambda^2 \tau + (1 - \Lambda)^2 \delta_{z,i} \tau_p) (\delta_{e,i} \tau_e + \delta_{z,i} \tau_p + \tau)} \times \sqrt{\frac{1}{(1 - \Lambda)^2 \frac{1}{\tau} + \Lambda^2 \frac{1}{\delta_{z,i} \tau_p}}} \tag{46}
\]

This implies anticipated utility is always increasing in \( \delta_{e,i} \), and increasing in \( \delta_{z,i} \) when

\[
\frac{\delta_{z,i}^2}{\delta_{e,i} \tau_e + \tau} > \frac{\Lambda^2 \tau}{(1 - \Lambda)^2 \tau_p^2}, \tag{47}
\]

i.e., it is initially decreasing and then increasing in \( \delta_{z,i} \). Moreover, note that

\[
\lim_{\delta_{z,i} \to 0} \frac{\partial}{\partial \delta_{z,i}} AU = -\infty, \quad \lim_{\delta_{z,i} \to 0} \frac{\partial}{\partial \delta_{z,i}} AU = 0 \tag{48}
\]

and \( \frac{\partial}{\partial \delta_{z,i}} AU \) equals zero at:

\[
\delta_{z,i}^* = \frac{1}{\tau_p} \left( \frac{\Lambda}{1 - \Lambda} \right) \sqrt{\tau (\delta_{e,i} \tau_e + \tau)} \tag{49}
\]

\( \square \)

8.2 Proof of Proposition 1

The objective of investor \( i \) given by

\[
\max_{\delta_{e_i}} AU_i(\delta_{e_i}) - C(\delta_{e_i}) .
\]
Lemma 3 implies that anticipated utility increases with overconfidence parameter $\delta_{ei}$. So, investor tries to balance the benefit of increasing $\delta_{ei}$ with the cost of increasing $\delta_{ei}$. The FOC with respect to $\delta_{ei}$ is

$$
\frac{\tau_e}{2 \left( \frac{(1-\Lambda)^2}{\tau_0} + \frac{\Lambda^2}{\tau_p} \right)^{\frac{1}{2}} (\tau_e \delta_{ei} + \tau_p + \tau_0)^{\frac{3}{2}}} = \frac{\partial C}{\partial \delta_{ei}}
$$

(50)

and the second order condition is

$$
- \frac{3 \tau_e^2}{4 \left( \frac{(1-\Lambda)^2}{\tau_0} + \frac{\Lambda^2}{\tau_p} \right)^{\frac{1}{2}} (\tau_e \delta_{ei} + \tau_p + \tau_0)^{\frac{5}{2}}} - \frac{\partial^2 C}{\partial \delta_{ei}^2} < 0.
$$

Condition (50) implies that the optimal overconfidence parameter is always greater than one i.e., $\delta_e^* \geq 1$. \qed

### 8.3 Proof of Proposition 2

The cost function in the case of optimal expectations is given by

$$
C(\delta_{ei}) = \frac{1}{\sqrt{\left( \frac{(1-\Lambda)^2}{\tau_0} + \frac{\Lambda^2}{\tau_p} \right) (\tau_0 + \tau_p + \tau_e \delta_{ei} (2 - \delta_{ei}))}}
$$

The FOC in the case of optimal expectations is given by

$$
\frac{\tau_e}{2 \left( \frac{(1-\Lambda)^2}{\tau_0} + \frac{\Lambda^2}{\tau_p} \right)^{\frac{1}{2}} (\tau_e \delta_{ei} + \tau_p + \tau_0)^{\frac{3}{2}}} = \frac{\tau_e (\delta_{ei} - 1)}{\left( \frac{(1-\Lambda)^2}{\tau_0} + \frac{\Lambda^2}{\tau_p} \right)^{\frac{1}{2}} \left[ (\tau_0 + \tau_p + \tau_e \delta_{ei} (2 - \delta_{ei})) \right]^{\frac{3}{2}}}
$$

(51)

which simplifies to

$$
\frac{(\tau_0 + \tau_p + \tau_e \delta_{ei} (2 - \delta_{ei}))^{\frac{3}{2}}}{(\tau_p + \tau_0 + \tau_e \delta_{ei})^{\frac{3}{2}}} = 2 (\delta_{ei} - 1)
$$

(52)

which establishes the result. \qed
8.4 Proof of Corollary 1

Taking derivatives of the return-volume characteristics in Lemma (2) with respect to $\delta_e$ gives:

$$\frac{\partial \sigma^2_R}{\partial \delta_e} = -2\gamma^2 \tau_e \left( \gamma^6 + \delta_e \tau_e \tau_z \left( 3\gamma^4 + \gamma^2 \tau_z \left( 3\delta_e \tau_e + \tau \right) + \delta_e^2 \tau_e^2 \tau_z^2 \right) \right) / \tau_z \left( \gamma^2 \tau + \delta_e \tau_e \left( \gamma^2 + \delta_e \tau_e \tau_z \right) \right)^3 < 0 \quad (53)$$

$$\frac{\partial \theta}{\partial \delta_e} = \gamma^4 \tau \tau_e \tau_z \left( \gamma^2 \left( 2\delta_e \tau_e + \tau \right) + 3\delta_e^2 \tau_e^2 \tau_z \right) / \left( \gamma^2 \tau + \delta_e \tau_e \tau_z \right)^2 > 0 \quad (54)$$

$$\frac{\partial \tau_p}{\partial \delta_e} = \frac{2\delta_e \tau_e^2 \tau_z}{\gamma^2} > 0 \quad (55)$$

$$\frac{\partial E[\mathcal{V}]}{\partial \delta_e} = \frac{\sqrt{\frac{2}{\pi}} \delta_e \tau_e}{\gamma \sqrt{\tau^2 + \delta_e^2 \tau_e \tau_z}} > 0 \quad (56)$$

which establishes the result.

8.5 Proof of Proposition 3

Equation (45) shows that marginal anticipated utility is weakly increasing in $\delta_{e,i}$. As long as $\frac{\partial C(1,\delta_e)}{\partial \delta_{e,i}} = 0$ (which holds under any well-behaved cost function), then the first-order condition in a symmetric equilibrium

$$\frac{\tau_e}{2 \left( \frac{(1-\Lambda)^2}{\tau} + \frac{\Lambda^2}{\delta_{e,p}} \right)^{\frac{3}{2}} \left( \tau_e \delta_e + \delta_z \tau_p + \tau \right)^{\frac{3}{2}}} = \frac{\partial C(1,\delta_z)}{\partial \delta_{e,i}} \quad (57)$$

implies that $\delta_e \geq 1$ with $\delta_e > 1$ if $\delta_z \neq 0$. This proves the first half of the proposition.

Lemma 1 implies that in any symmetric equilibrium, we have $\Lambda = \frac{\delta_e \tau_e + \delta_z \tau_p}{\tau + \delta_e \tau_e + \delta_z \tau_p}$. Moreover, note that $\frac{\partial}{\partial \delta_{z,i}} AU = 0$ at

$$\tilde{\delta}_{z,i} = \frac{1}{\tau_p} \left( \frac{\Lambda}{1 - \Lambda} \right) \sqrt{\tau \left( \delta_e \tau_e + \tau \right)} \quad (58)$$

$$= \sqrt{1 + \frac{\tau_e}{\tau} \left( \delta_z + \frac{\tau_p}{\tau_p} \delta_{z,i} \right)} > \delta_z \quad (59)$$

But this implies $\frac{\partial}{\partial \delta_{z,i}} AU (\delta_{z,i} = \delta_z) < 0$ since $\frac{\partial AU}{\partial \delta_{z,i}} < (>)0$ for all $\delta_{z,i} < (>)\tilde{\delta}_{z,i}$. Next, note that if $\delta_{z,i} = \delta_z \geq 1$, then $C''(\delta_{z,i}) > 0$. Taken together, this proves that at any proposed symmetric equilibrium where $\delta_z > 1$, investor $i$ has an incentive to deviate. Thus, the only possible symmetric equilibrium is one in which each investor chooses $\delta_{z,i} < 1$. This proves the second half of the proposition.
8.6 Proof of Propositions 4 and 5

For an investor exhibiting optimal expectations, choosing \((\delta_{ei}, \delta_{zi})\) yields anticipated utility and cost given by:

\[
AU(\delta_{ei}, \delta_{zi}) = -\sqrt{\frac{1}{\tau + \delta_{ei}\tau_e + \delta_{zi}\tau_p}} \frac{1}{(1 - \Lambda)^2 \frac{1}{\tau} + \Lambda^2 \frac{1}{\delta_{zi}\tau_p}}
\]

(60)

\[
C(\delta_{ei}, \delta_{zi}) = \frac{1}{\sqrt{\Lambda^2 (\delta_{zi} - 1)^2 + \left(\frac{(1-\Lambda)^2}{\tau} + \frac{\Lambda^2}{\tau_p}\right) (\tau + \tau e\delta_{ei} (2 - \delta_{ei}) + \tau p\delta_{zi} (2 - \delta_{zi}))}}.
\]

(61)

Let \(\kappa \equiv \left(\frac{\Lambda}{1 - \Lambda}\right)^2 \frac{\tau}{\tau_p}\). Then,

\[
AU(\delta_{ei}, \delta_{zi}) = -\sqrt{\frac{\kappa}{(1 - \Lambda)^2}} \sqrt{\frac{1}{\tau + \delta_{ei}\tau_e + \delta_{zi}\tau_p}} \frac{1}{(1 + \frac{\kappa}{\delta_{zi}}) (\tau + \delta_{ei}\tau_e + \delta_{zi}\tau_p)}
\]

(62)

\[
C(\delta_{ei}, \delta_{zi}) = \sqrt{\frac{\tau}{(1 - \Lambda)^2}} \sqrt{\frac{1}{(1 - \delta_{zi})^2 \kappa\tau_p + (1 + \kappa) (\tau + \tau e\delta_{ei} (2 - \delta_{ei}) + \tau p\delta_{zi} (2 - \delta_{zi}))}}
\]

(63)

Suppose all others are playing \(\bar{\delta}_e, \bar{\delta}_z\). Then, \(\Lambda = \frac{\tau e\bar{\delta}_e + \tau p\bar{\delta}_z}{\tau + \tau e\bar{\delta}_e + \tau p\bar{\delta}_z}\) and so

\[
\kappa = \left(\frac{\Lambda}{1 - \Lambda}\right)^2 \frac{\tau}{\tau_p} = \frac{\gamma^2 (\tau e\bar{\delta}_e + \tau p\bar{\delta}_z)^2}{\tau\tau_e^2\tau_p^2\delta_e^2}.
\]

(64)

Then, \((1, 0)\) is a symmetric equilibrium iff

\[
AU(1, 0) - C(1, 0) > AU(\delta_{ei}, \delta_{zi}) - C(\delta_{ei}, \delta_{zi})
\]

(65)

for all \(\delta_{ei}, \delta_{zi}\), or equivalently,

\[
H \equiv 1 + R - L > 0
\]

(66)
where

\[ R \equiv \frac{AU(\delta_{e,i}, \delta_{z,i})}{C(\delta_{e,i}, \delta_{z,i})} \]  

\[ = \sqrt{\gamma^4 \tau_e \tau_z \left( \tau_e \left( \frac{\tau_e \tau_z \delta_e^2}{\gamma^2} (\delta_e - (\delta_e - 2) \delta_e) + 2 \delta_e \delta_z - (\delta_e - 2) \delta_e \right) + \frac{\tau_e \tau_z \delta_e^2 - \gamma^2 (\delta_e - 2) \delta_e}{\gamma^4 \tau_e} \right)^2 - \tau_e + \gamma^2} \]  

\[ L \equiv \frac{C(0, 1)}{C(\delta_{e,i}, \delta_{z,i})} \]  

\[ = \sqrt{\gamma^4 \tau_e \tau_z \left( \tau_e \left( \frac{\tau_e \tau_z \delta_e^2}{\gamma^2} (\delta_e - (\delta_e - 2) \delta_e) + 2 \delta_e \delta_z - (\delta_e - 2) \delta_e \right) + \frac{\tau_e \tau_z \delta_e^2 - \gamma^2 (\delta_e - 2) \delta_e}{\gamma^4 \tau_e} \right)^2 + \gamma^2} \]  

Note that

\[ \lim_{\gamma \to \infty} R = \sqrt{\frac{(2 - \delta_e) \delta_e \tau_e + \tau}{\delta_e \tau_e + \tau}}, \quad \lim_{\gamma \to \infty} L = \sqrt{\frac{(2 - \delta_e) \delta_e \tau_e + \tau}{\tau_e + \tau}} \]  

\[ \lim_{\gamma \to \infty} H = 1 + \sqrt{\frac{(2 - \delta_e) \delta_e \tau_e + \tau}{\delta_e \tau_e + \tau}} - \sqrt{\frac{(2 - \delta_e) \delta_e \tau_e + \tau}{\tau_e + \tau}} \]  

\[ \geq 1 + \sqrt{\frac{(2 - \delta_e) \delta_e \tau_e + \tau}{\delta_e \tau_e + \tau}} - \sqrt{\frac{\tau_e + \tau}{\tau_e + \tau}} \geq 0 \]  

which implies \((0, 0)\) is an equilibrium for \(\gamma\) sufficiently high.

Next, note that,

\[ \lim_{\gamma \to 0} R = \lim_{\gamma \to 0} L = \sqrt{\frac{\delta_e^2 \delta_e^2 \delta_z^2}{\tau_e^2 \tau_z^2 \delta_e^2}} = 1, \]  

so that

\[ \lim_{\gamma \to 0} H = 1 + R - L > 0 \]  

which implies that for sufficiently low \(\gamma\), an investor prefers to deviate to \((0, 0)\) for any
\( \bar{\delta}_e, \bar{\delta}_z \neq 0 \). Finally, consider an equilibrium in which \( \bar{\delta}_e = 1, \bar{\delta}_z = 0 \). In this case,

\[
\lim_{\gamma \to 0} R = \lim_{\gamma \to 0} \sqrt{2 - \delta_z} \tag{76}
\]

\[
\lim_{\gamma \to 0} L = \lim_{\gamma \to 0} \frac{1}{\gamma} \sqrt{(2 - \delta_z)} = \infty \tag{77}
\]

which suggests that

\[
\lim_{\gamma \to 0} H < 0 \tag{78}
\]

and so \((1, 0)\) cannot be an equilibrium for \(\gamma\) sufficiently low. However,

\[
\lim_{\gamma \to \infty} R = \lim_{\gamma \to \infty} \sqrt{\frac{(2 - \delta_e) \delta_e \tau_e + \tau}{\delta_e \tau_e + \tau}} \quad \lim_{\gamma \to \infty} L = \lim_{\gamma \to \infty} \sqrt{\frac{(2 - \delta_e) \delta_e \tau_e + \tau}{\tau_e + \tau}} \tag{79}
\]

which implies \(\lim_{\gamma \to \infty} H \geq 0\) as before, and so \((1, 0)\) is an equilibrium for \(\gamma\) sufficiently high.

8.7 Proof of Corollary 2

Denote the return characteristics in the rational expectations equilibrium (symmetric equilibrium) by \(RE\) (\(SE\), respectively). Note that

\[
\theta_{RE} - \theta_{OE} = -\frac{\tau \tau_e^2 \tau_z^2}{(\gamma^2 + \tau_e \tau_z) (\gamma^2 \tau + \tau_e^2 \tau_z)} < 0 \tag{80}
\]

\[
\tau_{p,RE} - \tau_{p,OE} = 0 \tag{81}
\]

\[
\mathbb{E} [V_{RE}] - \mathbb{E} [V_{OE}] = \sqrt{\frac{2}{\pi}} \left( \frac{\sqrt{\gamma_z^2 + \tau_e \tau_z}}{\gamma \sqrt{\gamma_e (\gamma_e + \tau_e) + \tau_e^2 \tau_z}} (\gamma^2 (\tau_e + \tau) + \tau_e^2 \tau_z) - \gamma^2 (\tau_e + \tau) \sqrt{\frac{\gamma_z^2 + \tau_e \tau_z}{\gamma (\gamma_e + \tau_e)^2 \tau_z}} \right) \tag{82}
\]

Finally, note that

\[
\sigma^2_{R,RE} - \sigma^2_{R,OE} = \frac{\tau \left( \tau_p^2 + \tau \tau_e - 2 \tau_e (\tau_e + \tau) \right)}{(\tau_e + \tau)^2 (\tau_e + \tau_p + \tau)^2} \tag{83}
\]

which is positive iff \(\tau_p > \frac{1}{2} \sqrt{8 \tau \tau_e + 8 \tau_e^2 + \tau^2 - \tau} \).
8.8 Proof of Lemma 4

Suppose \( \lambda \) fraction of investors chose \((\delta_{e1}, \delta_{c1})\) and \((1 - \lambda)\) investors chose \((\delta_{e2}, \delta_{c2})\). This implies that price is given by

\[
P = m + \Lambda (s_p - m) ,
\]

where \( \Lambda = \frac{(\lambda \delta_{e1} + (1 - \lambda) \delta_{e2}) \tau_e + (\lambda \delta_{c1} + (1 - \lambda) \delta_{c2}) \tau_p}{\tau + (\lambda \delta_{e1} + (1 - \lambda) \delta_{e2}) \tau_e + (\lambda \delta_{c1} + (1 - \lambda) \delta_{c2}) \tau_p}.
\]

Assume that risk aversion is not sufficiently high, this implies that investor’s objective function has a local interior maxima. Investor then evaluates his objective at this interior maxima and chooses the one where his objective is highest. For the mixed equilibrium to sustain, we need \( \delta_{e1} = 0 \) (which implies \( \delta_{e1} = 1 \)) and \( \delta_{e2} = \delta_e^* \geq 1 \) (and let \( \delta_{e2} = \delta_e^* \)). For this mixed equilibrium, investor has to be indifferent between the two points, which implies that the following conditions have to hold:

\[
\frac{\partial AU}{\partial \delta_e} \big|_{(\delta_{e1}, \delta_{c1} = \delta_e^*)} = C'(\delta_e^*)
\]

\[
\frac{\partial AU}{\partial \delta_z} \big|_{(\delta_{e1}, \delta_{c1} = \delta_e^*)} = C'(\delta_e^*)
\]

\[
AU (0, 1) - C (0, 1) = AU (\delta_e^*, \delta_e^*) - C (\delta_e^*, \delta_e^*).
\]

(84)

The first two conditions are the FOCs for local maxima \((\delta_e^*, \delta_e^*)\) and the third condition says that investors are indifferent between the local maxima and the corner solution \((0, 1)\). These three equations will help us solve for 3 unknowns \( \delta_e^* \), \( \delta_e^* \) and \( \lambda \). Suppose \( \bar{\delta}_e = \bar{\delta}_e = (\lambda + (1 - \lambda) \delta_e) \) and \( \bar{\delta}_z = (1 - \lambda) \delta_z \) denote the average action of investors. The FOCs can be rewritten as

\[
R^3 = \frac{2 (\delta_e^* - 1)}{1 - (\delta_e^* \tau_e + \tau) \frac{(\delta_e \tau_e + \delta_e \tau_p)^2}{\tau_p \tau_e \delta_e^* \tau_e}}.
\]

(85)

\[
R^3 = \frac{2 (\delta_e^* - 1)}{1 + \frac{\Lambda^2}{\tau_p \text{var}(F - P)} \left( \frac{1}{\delta_e^* - 1} \right)},
\]

(86)

where

\[
R^2 = \frac{(\delta_e \tau_e + \delta_e \tau_p)^2}{\tau_e^2} + \left( \frac{1}{\tau} + \frac{(\delta_e \tau_e + \delta_e \tau_p)^2}{\tau_p \delta_e^* \tau_e} \right) \left( \tau + \tau_e \delta_e^* (2 - \delta_e^*) + \frac{\tau_p}{\delta_e^* \tau_p} \delta_e^* (2 - \delta_e^*) \right).
\]

Since any deviations from rational expectations generate higher anticipated utility and lower true utility, \( R < 1 \). Using this inequality in equation 85 gives us that \( \delta_e^* < \frac{3}{2} \). Similarly, using
$R < 1$ in equation 86 gives us that $1 < \delta^*_e < \frac{3}{2}$. Moreover $\Lambda_{RE} > \lambda_{AE} \iff \tau_e + \frac{\tau_p}{\delta^*_e} > \delta^*_e \tau_e + \delta^*_z \tau_p$. Let $x$ denote the ratio of these two i.e.,

$$x = \frac{\delta^*_e \tau_e + \delta^*_z \tau_p}{\tau_e + \frac{\tau_p}{\delta^*_e}}.$$ 

We need to prove that $x < 1$. Take the limit as $\gamma \to 0$,

$$\lim_{\gamma \to 0} R^2 = \lim_{\tau_p \to \infty} \frac{(\frac{\tau_p}{\tau^2})^2 x^2 + \left(\frac{1}{\tau} + \left(\frac{\tau_p}{\delta^*_e \tau^2}\right)^2 x^2\right) (\tau + \tau_e \delta^*_e (2 - \delta^*_e)) + \frac{\tau_p}{\tau^2} \delta^*_z (2 - \delta^*_z) \left(\frac{\delta^*_z \tau_p}{\tau} + \left(\frac{\tau_p}{\tau} \right)^2 x^2\right)}{1 + \left(\frac{\tau_p}{\delta^*_e \tau^2}\right)^2 (\tau + \delta^*_e \tau_e + \delta^*_z \tau_p)}.$$ 

There are 2 cases to consider, Case 1: $\tau_p x \to \infty$, Case 2: $\tau_p x \to constant$. In case 2, it is immediate that $x < 1$ for sufficiently large $\tau_p$ i.e., for sufficiently large $\gamma$. Next, I will prove that case 1 is not possible in our setting.

Suppose, for now, case 1 is possible. In this case, $\tau_p x \to \infty$ as $\gamma \to 0$. In this case,

$$\lim_{\gamma \to 0} R^2 = \lim_{\tau_p \to \infty} \frac{(\frac{\tau_p}{\tau^2})^2 x^2 + \left(\frac{1}{\tau} + \left(\frac{\tau_p}{\delta^*_e \tau^2}\right)^2 x^2\right) (\tau + \tau_e \delta^*_e (2 - \delta^*_e)) + \frac{\tau_p}{\tau^2} \delta^*_z (2 - \delta^*_z) \left(\frac{\delta^*_z \tau_p}{\tau} + \left(\frac{\tau_p}{\tau} \right)^2 x^2\right)}{\tau + \delta^*_e \tau_e + \delta^*_z \tau_p} = 1$$

and the indifference condition of the investor becomes

$$\sqrt{\frac{(\frac{\tau_p}{\tau^2})^2 x^2 + \left(\frac{1}{\tau} + \left(\frac{\tau_p}{\delta^*_e \tau^2}\right)^2 x^2\right) (\tau + \tau_e \delta^*_e (2 - \delta^*_e)) + \frac{\tau_p}{\tau^2} \delta^*_z (2 - \delta^*_z) \left(\frac{\tau_p}{\tau} \right)^2 x^2}{\tau^2}} = 2$$

The LHS of above expression is 1 and hence indifference condition cannot be satisfied. This
implies that case 1 is not possible. This implies that $\tau_p x$ tends to a finite constant. This immediately implies that for $\gamma$ sufficiently low, $x < 1$.

\[\blacksquare\]

### 8.9 Proof of Corollary 3

Denote the return characteristics in the rational expectations equilibrium (optimal expectations asymmetric equilibrium) by $RE$ ($AE$, respectively). Let $\bar{\delta}_e = \lambda + (1 - \lambda) \delta_e^*$ and $\bar{\delta}_z = (1 - \lambda) \delta_z^*$ denote the average beliefs about the precision of private signals and price signal. Note that

$$\tau_{p,AE} - \tau_{p,RE} = \frac{\tau_z}{\gamma^2} \left( \delta_e^2 - 1 \right) > 0.$$

$$\theta_{AE} - \theta_{RE} = \frac{\tau_z}{\Lambda_{AE} \left( \beta_{AE}^2 \tau + \tau_z \right)} - \frac{\tau_z}{\Lambda_{RE} \left( \beta_{RE}^2 \tau + \tau_z \right)} = \frac{\tau_z}{\Lambda_{AE} \left( \beta_{AE}^2 \tau + \tau_z \right)} \left( 1 - \frac{\Lambda_{AE}}{\Lambda_{RE}} \frac{\beta_{AE}^2 \tau + \beta_{AE} \delta_e^2 \tau + \beta_{AE} \delta_z^2 \tau}{\lambda} \right).$$

Since $\bar{\delta}_e > 1$, we have $\frac{\beta_{2\tau + \delta_e^2 \tau_z}}{\delta_e^2 \beta_{2\tau} + \delta_e^2 \tau_z} < 1$. Moreover, since $\frac{\Lambda_{AE}}{\Lambda_{RE}} < 1$ by Lemma 4, we have $\theta_{AE} > \theta_{RE}$.

$$\mathbb{E} [\mathcal{V}] = \int \frac{\omega_i}{\gamma} \left[ \frac{2}{\pi} \left( \frac{(A_i + B_i - \Lambda)^2}{\tau} + \frac{A_i^2}{\tau_e^2} + \frac{(B_i - \Lambda)^2}{\tau_p^2} \right) \right] di \quad (87)$$

$$\mathbb{E} [\mathcal{V}_{AE}] = \lambda V_1 + (1 - \lambda) V_2, \quad (88)$$

$$V_1 \equiv \frac{(\tau_e + \tau_p)}{\gamma} \sqrt{\frac{2}{\pi}} \left( \frac{1}{\tau} \left( \frac{\tau_e}{\tau + \tau_e} - \Lambda_{AE} \right)^2 + \frac{1}{\tau_e} \left( \frac{\tau_e}{\tau + \tau_e} \right)^2 + \frac{1}{\tau_{p,AE}^2 \Lambda_{AE}} \right) \quad (89)$$

$$V_2 \equiv \frac{\tau + \delta_e^2 \tau_e + \delta_e^2 \tau_{p,AE}}{\gamma} \sqrt{\frac{2}{\pi}} \left( \frac{1}{\tau} \left( \frac{\delta_e^2 \tau_e + \delta_z^2 \tau_{p,AE}}{\tau + \delta_e^2 \tau_e + \delta_z^2 \tau_{p,AE}} - \Lambda_{AE} \right)^2 + \frac{1}{\tau_e} \left( \frac{\delta_z^2 \tau_e}{\tau + \delta_e^2 \tau_e + \delta_z^2 \tau_{p,AE}} \right)^2 + \frac{1}{\tau_{p,AE}^2 \Lambda_{AE}} \right) \quad (90)$$

35
Let

\[
A(x) = \frac{x\tau_e + (1 - x)\delta^*_e\tau_e}{x(\tau + \tau_e) + (1 - x)(\tau + \delta^*_e\tau_e + \delta^*_pAE)}
\]

(91)

\[
B(x) = \frac{x0 + (1 - x)(\delta^*_p)}{x(\tau + \tau_e) + (1 - x)(\tau + \delta^*_e\tau_e + \delta^*_pAE)}
\]

(92)

\[
\omega(x) = x(\tau_e + \tau) + (1 - x)(\tau + \delta^*_e\tau_e + \delta^*_pAE)
\]

(93)

\[
V(x) = \frac{\omega(x)}{\gamma} \sqrt{\frac{2}{\pi} \left( \frac{1}{\tau_e} A(x) + B(x) - \Lambda \right)^2 + \frac{1}{\tau_e} A(x)^2 + \frac{1}{\tau_p} (B(x) - \Lambda)^2}
\]

(94)

and

\[
\mathbb{E}[\mathcal{V}_{AE}] = \lambda V(1) + (1 - \lambda) V(0)
\]

(95)

Note that

\[
V(\lambda) = \frac{\omega(\lambda)}{\gamma} \sqrt{\frac{2}{\pi} \left( \frac{1}{\tau_e} A(\lambda)^2 + \frac{1}{\tau_{p,RE}} (B(\lambda) - \Lambda)^2 \right)}
\]

(96)

\[
= \frac{1}{\gamma} \sqrt{\frac{2}{\pi} \left( \frac{(\lambda\tau_e + (1 - \lambda)\delta^*_e\tau_e)^2}{\tau_e} + \frac{(1 - \lambda)(\delta^*_e\tau_e - \lambda(\tau_e + \delta^*_p)) - (1 - \lambda)(\delta^*_e\tau_e + \delta^*_p))}{\tau_{p,RE}} \right)}
\]

(97)

\[
= \frac{1}{\gamma} \sqrt{\frac{2}{\pi} \left( \frac{(\lambda\tau_e + (1 - \lambda)\delta^*_e\tau_e)^2}{\tau_e} + \frac{(\lambda\tau_e + (1 - \lambda)\delta^*_e\tau_e)^2}{\tau_{p,RE}} \right)}
\]

(98)

\[
= \frac{1}{\gamma} \sqrt{\frac{2}{\pi} \left( \delta^*_e\tau_e + \frac{\tau^2_e}{\tau_p} \right)} \geq \frac{1}{\gamma} \sqrt{\frac{2}{\pi} \left( \tau_e + \frac{\tau^2}{\tau_p} \right)} = \mathbb{E}[\mathcal{V}_{RE}]
\]

(99)

where \(\delta_e \equiv \lambda + (1 - \lambda)\delta^*_e\). It remains to be shown that:

\[
\lambda V(1) + (1 - \lambda) V(0) \geq V(\lambda)
\]

(100)

Note that

\[
V(x) = \frac{1}{\gamma} \sqrt{\frac{2}{\pi} \left( \frac{1}{\tau_e} (\alpha(x) + \beta(x) - \Lambda\omega(x))^2 + \frac{1}{\tau_e} \alpha(x)^2 + \frac{1}{\tau_p} (\beta(x) - \Lambda\omega(x))^2 \right)}
\]

(101)

where

\[
\alpha(x) = x\tau_e + (1 - x)\delta^*_e\tau_e \equiv a_0 + a_1 x
\]

(102)

\[
\beta(x) = x0 + (1 - x)(\delta^*_p) \equiv b_0 + b_1 x
\]

(103)

\[
\omega(x) = x(\tau_e + \tau) + (1 - x)(\tau + \delta^*_e\tau_e + \delta^*_pAE) \equiv w_0 + w_1 x
\]

(104)

\[
\frac{V_{xx}}{V^3} = 4 \frac{\tau_e + \tau_{p,RE}}{\pi^2 \gamma^4 \tau_e \tau_p} \left(-a_0 b_1 + a_0 \Lambda w_1 + a_1 b_0 - a_1 \Lambda w_0\right)^2 > 0
\]

(105)
which implies $V(x)$ is convex, which implies:

$$
\mathbb{E}[V_{AE}] = \lambda V(1) + (1 - \lambda) V(0) \geq V(\lambda) \geq \mathbb{E}[V_{RE}] \quad (106)
$$

### 8.10 Proof of Proposition 6

The utility of noise traders is

$$
U_z = -E(\gamma\exp\{z(F - P)\}) = -E(\gamma\exp\{z(F - \Lambda(F + \beta z))\}) = -E(\gamma\exp\{zF(1 - \Lambda) - \gamma\Lambda\beta z^2\})
$$

$$
= -E\left(\gamma\exp\left\{\left(\frac{\gamma^2(1 - \Lambda)^2}{2\tau} - \gamma\Lambda\beta\right)z^2\right\}\right)
$$

$$
= -\gamma \frac{1}{\sqrt{1 - 2\frac{1}{\tau_e}\left(\frac{\gamma^2(1 - \Lambda)^2}{2\tau} - \gamma\Lambda\beta\right)}}
$$

$$
= -\gamma \frac{\tau_z}{\sqrt{\tau_z - \frac{\tau^2(1 - \Lambda)^2}{2\tau} + 2\gamma\Lambda\beta}}
$$

where we used the fact that $E(e^{\alpha^2}) = \frac{1}{\sqrt{1 - 2\alpha^2}}$. This implies that utility of noise traders is monotonically decreasing in $\frac{\gamma(1 - \Lambda)^2}{2\tau} - \Lambda\beta$.

Comparing the symmetric equilibrium with rational expectations, since $\beta_{SE} = \beta_{RE} = -\frac{\tau}{\tau_e}$, and $\Lambda_{SE} = \frac{\tau_e}{\tau + \tau_e}$ and $\Lambda_{RE} = \frac{\tau_e + \tau_p}{\tau + \tau_e + \tau_p}$, we have

$$
U_{SE} - U_{RE} > 0 \quad (107)
$$

$$
\Leftrightarrow \left(\frac{\Lambda_{SE}}{\tau_e} - \frac{\Lambda_{RE}}{\tau_e} + \frac{(1 - \Lambda_{SE})^2}{2\tau} - \frac{(1 - \Lambda_{RE})^2}{2\tau}\right) < 0 \quad (108)
$$

$$
\Leftrightarrow \left(\frac{\tau_p}{\tau_e} \left(\left(\frac{\tau_e + 2\tau}{\tau} + 2\tau (\tau_e + \tau)\right)\right)\right) > 0 \quad (109)
$$

which implies $U_{SE} > U_{RE}$ always. The price impact effect dominates the risk aversion effect.

For the asymmetric equilibrium, we have $\beta_{AE} = -\frac{\tau}{\tau_e\delta_e}$, and $\Lambda_{AE} = \frac{\delta_e\tau_e + \delta_z\tau_p}{\tau + \delta_e\tau_e + \delta_z\tau_p}$, so that

$$
U_{AE} - U_{RE} > 0 \quad (110)
$$

$$
\Leftrightarrow \left(\frac{\Lambda_{AE}}{\tau_e\delta_e} - \frac{\Lambda_{RE}}{\tau_e} + \frac{(1 - \Lambda_{AE})^2}{2\tau} - \frac{(1 - \Lambda_{RE})^2}{2\tau}\right) < 0 \quad (111)
$$

$$
\Leftrightarrow \frac{1}{\tau_e} \left(\Lambda_{RE} - \Lambda_{AE}\right) (2 - (\Lambda_{AE} + \Lambda_{RE})) < \frac{1}{\tau_e} \left(\Lambda_{RE} - \frac{\Lambda_{AE}}{\delta_e}\right) \quad (112)
$$

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Note that $\bar{\delta_e} \geq 1$, so it is sufficient to establish:

$$\frac{1}{2\tau} (\Lambda_{RE} - \Lambda_{AE}) (2 - (\Lambda_{AE} + \Lambda_{RE})) < \frac{1}{\tau_e} (\Lambda_{RE} - \Lambda_{AE})$$

(113)

Suppose $\Lambda_{RE} > \Lambda_{AE}$. Then, the above is equivalent to:

$$2 - (\Lambda_{AE} + \Lambda_{RE}) < \frac{2\tau}{\tau_e}$$

(114)

$$\Leftrightarrow \left( \frac{1}{\tau + \tau_e + \tau_p} + \frac{1}{\tau + \delta_e \tau_e + \delta_z \tau_p} \right) \leq \frac{2}{\tau_e}$$

(115)

which is always true. So, if $\Lambda_{RE} > \Lambda_{AE}$, we have $U_{AE} > U_{RE}$. 

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