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Schelling's spatial proximity model of segregation revisited ☆

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Abstract

Schelling [Schelling, T.C., 1969. Models of Segregation. American Economic Review, Papers and Proceedings, 59, 488-493, Schelling, T.C., 1971a. Dynamic Models of Segregation. Journal of Mathematical Sociology, 1 (2), 143–186, Schelling, T.C., 1971b. On the Ecology of Micromotives. The Public Interest, 25, 61–98, Schelling, T.C., 1978. Micromotives and Macrobehavior. New York: Norton.] presented a microeconomic model showing how an integrated city could unravel to a rather segregated city, notwithstanding relatively mild assumptions concerning the individual agents' preferences, i.e., no agent preferring the resulting segregation. We examine the robustness of Schelling's model, focusing in particular on its driving force: the individual preferences. We show that even if all individual agents have a *strict preference for perfect integration*, best-response dynamics may lead to segregation. This raises some doubts on the ability of public policies to generate integration through the promotion of openness and tolerance with respect to diversity. We also argue that the one-dimensional and two-dimensional versions of Schelling's spatial proximity model are in fact two qualitatively very different models of segregation.

JEL classification: C72; C73; D62
Keywords: Neighborhood segregation; Myopic Nash Equilibria; Best-response dynamics; Markov chain; Limit-behavior

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1. Introduction

Schelling (1969, 1971a,b, 1978) presented a microeconomic model of neighborhood segregation that Schelling (1971a) called a "*spatial proximity model*" (p. 149), as it specifies a spatial setup in which the individual agents care only about the composition of their own local neighborhood. More specifically, distinguishing two types of agents, every agent is assumed to be equally happy with any configuration of his neighborhood except that he does not tolerate more than a certain fraction of it, e.g. 50 or 60%, to be populated by agents of the other type. Unsatisfied agents get the chance (in some arbitrary order) to move to a satisfactory position, until nobody wants to move anymore. Lo and behold, an unraveling process starts from a more or less integrated city into a rather segregated one.

Schelling's neighborhood segregation model has become one of the most widely cited and acclaimed models in economics.[1] There are several reasons for this. First, the emergence of segregation seems intellectually intriguing. Micromotives at the local level give rise to macrobehavior at the aggregate (global) level, but this emerging macrobehavior does not simply correspond to the underlying micromotives, i.e., segregation occurs although no individual agent strictly prefers this. Moreover, it appears to be one of the very first models of complexity and self-organization in economics. Another reason for the fame of Schelling's model is educational. It is unusually simple. Combined with its intellectual appeal, this makes it a convenient means to illustrate the idea of unintended consequences resulting from the interaction between individuals. What is more, the model is a 'do-it-yourself' model of self-organization, as it can be easily verified by anyone with a pen and paper. The other main reason for the interest in Schelling's model is related to the fact that segregation has become one of the most important socio-political and public economic issues. It has been so for some time in the USA, and has increasingly become one also in many Western-European countries.[2,3]

Notwithstanding the general recognition given to Schelling's model nowadays, it leaves some question marks concerning the assumptions made with respect to the individual agents' preferences. That is, although these assumptions are relatively mild in the sense that no agent prefers segregation, it is also true that no agent is against it. In other words, while agents in Schelling's model are content to live together in a ratio of 50–50, they are equally content to live in a completely segregated city, as long as they can live in a ghetto of like agents. Since these preferences are the driving force in Schelling's model of segregation, it is worth investigating whether this is essential. Therefore, we provide a formal as well as numerical analysis of the properties of Schelling's model of segregation.

The emphasis of our analysis is on myopic best-response dynamics. But we also provide a complete myopic equilibrium characterization. This characterization makes clear that the best-response dynamics in the model are not an equilibrium selection issue. While in the two-dimensional setup, the equilibria are over-represented in the set of best-response outcomes, in the

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[1] See, e.g., Akerlof (1997), Arrow (1998), Rosser (1999), Binmore (1992), Blume (1997), Brock and Durlauf (2001), Clark (1991), Dixit and Nalebuff (1991), Glaeser and Scheinkman (2002), Ioannides and Seslen (2002), Krugman (1996), Lindbeck et al. (1999), Manski (2000), Skyrms and Pemantle (2000), or Young (1998).

[2] See, e.g., The Economist (2001).

[3] Notice that segregation occurs not only in a racial context. It can also be found between followers of different religions, between men and women in an office canteen, between tourists and locals at a city square, between faculty and students in a seminar room, between different nationalities at a conference dinner, between workers with different skills in different firms, or between different species occupying their own territory.

one-dimensional model with a strict preference for integration, the best-response dynamics do not select an equilibrium at all. Instead, the completely segregated outcome prevails, which is the very opposite of any equilibrium, in the sense that the latter are all perfectly integrated. This suggests that one should be careful not to focus exclusively on equilibria when studying social dynamic processes.

Focusing on the driving force behind the dynamic behavior of Schelling's model, the individual preferences, we keep all other details as general and simple as possible. The main insight from our analysis can be summarized as follows. Schelling's model of segregation is robust. Schelling's mild assumptions on the individual preferences for integration can be made considerably more extreme. Even strict preferences for perfect integration by all individual agents may lead to neighborhood segregation. Moreover, for the one-dimensional setup, we prove that such preferences are sufficient for complete segregation. Thus, the paper sharpens Schelling's insight: altering agents' preferences might do nothing to reverse the undesirable outcome. In particular, stronger (in the sense made clear below) preferences for integration need not result in more integration.

The remainder of this paper is structured as follows. Section 2 explains why segregation is a key public economic policy issue. In Section 3 we briefly recapitulate Schelling's (1969, 1971a,b, 1978) spatial proximity model of segregation and outline the features of the model that we are going to analyze in detail. Section 4 presents some analytical tools and benchmark allocations. Schelling's two-dimensional model is analyzed in Section 5, while the one-dimensional, linear model is considered in Section 6, and Section 7 concludes.

2. Segregation as public economic policy issue

The spatial proximity model relates the segregation issue directly to a number of traditional issues in the public economics literature. The dynamics of the spatial proximity model, i.e., the entire unraveling process from integration to segregation, is driven by externalities, leading to a failure of the Coase theorem. These externalities arise as the private location decision of an individual alters the composition of neighborhoods of other people. The individual characteristics that enter neighbors' preferences are not priced in the housing market. For instance, the ask price of a house is independent of how valuable a neighbor a prospective buyer would be. But the neighbors might care for the characteristics of who eventually buys the house. The spatial proximity model focuses on the importance of these externalities.

The spatial proximity model of segregation is closely related to the Tiebout model. According to Tiebout (1956), people can sort themselves into communities according to their public goods preferences. Various problems with this model have been pointed out in the literature, "*such as the restricted number of communities, the multidimensional nature of public goods, limitations to mobility, and economies of scale in public goods provision*" (see Alesina et al., 1999, p. 1246–47, and the references therein). The spatial proximity model qualifies the optimistic conclusion of the Tiebout model. Once a society is segregated, this restricts the mobility of the individuals, as entire areas may become 'no-go' zones for certain groups. This inhibits Tiebout sorting by preferences for the public good.

Alesina et al. (1999) argue that in cities where ethnic groups are polarized, and where politicians have ethnic constituencies, the share of spending that goes to public goods is low. More specifically, they document how ethnic fragmentation may influence local public goods, as reflected in the composition of spending, the aggregate total of spending, and the budget balance. This concerns in particular core public goods such as education, roads and urban transportation systems, sewerage, and trash pickup. This is because people might value public goods more to the extent that they benefit their own group and discount the benefits for other groups.

The relationships between segregation, school finance and school choice have been studied by Nechyba (2003). Residential and school segregation seem to go hand-in-hand, and school segregation may lead in turn to underachievement and disadvantages in the labor market. The effect of segregation on educational outcomes has been explored by Cutler and Glaeser (1997).

Baughman (2004) studies the effect of segregation on the effectiveness of campaigns to expand public health insurance for children. She argues that residential segregation affects child health insurance coverage through its effect on community-level outreach and information networks.

Low supply of public goods such as education, transportation or health care, may in turn lead to vicious cycles, with segregated groups falling further behind. As Alesina et al. note, if public schools provision is low because of ethnic conflict, *"the relative skill levels of minorities in ghettos does not improve and their poverty level increases, making problems of central cities' unemployment and decay even worse-and ethnic conflicts even more acute"* (p. 1247). Similarly, Wilson (1996) notes that poor public transportation systems from inner city ghettos to the location of job opportunities increase the costs of finding and keeping jobs for inner city minorities.

Brender (2005) argues that segregation may affect local tax collection as a share of the total amount charged by law, thus worsening the situation for the disadvantaged minority through a low supply of public services.

Besides these more traditional public economics issues mentioned above, a number of related public policy issues have emerged in the last few years. Segregation enhances a lack of shared language, cultural values and norms. This makes social coordination more difficult. Some have argued that segregation puts the whole idea of a peaceful society with its constitutional and civic liberties at risk (see, e.g., Scheffer, 2000). One of the remarkable aspects in the war on terror has been that a number of terror suspects turned out to be citizens that had been born and raised in the countries they were fighting. Kepel (2004) even claims that the war on terror will be decided in the suburbs of Europe, where the key question will be whether the Muslim communities there will successfully integrate or not.

Substantial public spending is related to the aforementioned issues. In addition, a further sign of the increased recognition that integration is a key public policy problem can be seen in the amounts explicitly earmarked for 'integration' in the governments' budgets. For example, in The Netherlands this increased from just €9 million in 1970 to €1.1 billion in 2003 (see Commissie Blok, 2004).

As segregation has increasingly been recognized as one of the most important public policy issues in countries such as the UK, the Netherlands, France, and Germany, various countries have started evaluating and questioning the effectiveness of decades of integration policies (see Baldwin and Rozenberg, 2004; Commissie Blok, 2004). The prevalent form of integration policy in countries such as the UK and the Netherlands has been one promoting multiculturalism: respecting diversity and allowing integration. This policy has failed in that integration simply did not happen.[4] The spatial proximity model provides a possible answer why this could have been expected.

[4] Although the formal conclusion of Commissie Blok (2004) was positive about the Dutch integration policies, all political parties were unanimous in their fierce criticism during the parliamentary debate following its presentation.

Before we go into the details of the spatial proximity model, let us discuss some possible integration policies. A first possibility is to internalize the externality that drives the unraveling of integration, i.e., the externality stemming from the moving agent's neglect of the wellbeing of his past and future neighbors. This is, for example, partly achieved by the so-called Manhattan co-ops,[5] where newcomers are screened by their prospective neighbors, and board approval is required for apartment purchases (see New York Law Journal, 1997).

The first-best outcome could be difficult to achieve and maintain even for a benevolent planner. Integrated allocations are almost by definition characterized by very particular residential patterns. It might be difficult to work out an integrated pattern in which nobody wants to move, while it might be even more difficult to find such integrated patterns that are robust to small disturbances. Furthermore, centralized imposition of integration may often entail a more substantial policy intervention than merely enforcing coordination. In particular, in many countries, the law does not give the right to impose and enforce integration, as shown, e.g., by the Missouri v. Jenkins Supreme Court case in the US in 1995. More recently, attempts by the Dutch government to increase educational integration failed due to the constitutional right of (state-funded) religious schools to reject pupils with different beliefs.

Given the difficulties experienced with other policy measures aimed at integration, it is not surprising that the most frequently adopted policy has been to shape the individual citizen's preferences, promoting openness and tolerance with respect to diversity. The spatial proximity model focuses on the ramifications of such a policy. The stark conclusion is that a wide class of preferences for integration results in extreme segregation. Any integration policy must be based on a good understanding of the mechanisms underlying this result. The analysis suggests that education of preferences is not sufficient to achieve integration. It should be complemented by the coordinating role of a government designed to avoid convergence towards extremely segregated outcomes.

3. Schelling's spatial proximity model

3.1. Recapitulation of Schelling (1969, 1971a,b, 1978)

There are two basic variants of Schelling's spatial proximity model. The first version, presented in Schelling (1969), is a one-dimensional model. Besides this linear model, Schelling (1971a) presents a two-dimensional version as well, which is also the version appearing in Schelling (1971b, 1978).

Schelling (1969, 1971a) considers a number of individual agents, distinguishing two types of individuals (O and X), distributed along a line, i.e., in one dimension (1D).[6] An agent's position is defined relative to his neighbors only, and there are no absolute positions. A given individual's neighborhood is defined as the four nearest neighbors on either side of him.[7] Agents towards the end of the line will have less than eight neighbors. Each individual is

[5] Most apartment blocks in Manhattan — about 80%— are co-ops.

[6] The number of individuals can be finite, but Schelling (1971a) also refers to the possibility of an infinitely continuing line or a line closing in a ring (p. 152).

[7] Notice that the spatial proximity model differs from the so-called 'bounded neighborhood' or 'tipping' model of segregation (see Schelling, 1969, 1971a, 1972, 1978)) in that each individual has his own, locally defined neighborhood.

concerned only with the number of like and unlike neighbors.[8] More specifically, each agent wants at most 50% unlike neighbors; otherwise agents are indifferent. The starting configuration is created by distributing equal numbers of agents of each type in random order. The dynamics, then, are an iterative process of agents choosing myopic best-responses. At each stage all agents that are not satisfied are put in some arbitrary order. When an agent's turn comes, he moves to the nearest satisfactory position. Since all positions are relative only, he simply inserts himself between two agents (or at either end of the line). Similarly, his own departure does not lead to an empty position.[9] This process continues until no agent wants to move anymore. The typical outcome is a highly segregated state, although nobody actually prefers segregation to integration.

Schelling (1971a,b, 1978) considers a regular lattice with bounds, such as a checkerboard. There are again two types of agents, who can each occupy one cell of the board. But now there are also some free cells left. The neighborhood of an individual agent is the so-called Moore neighborhood. For an agent in the interior of the board this consists of the eight cells directly surrounding his own location, with fewer neighbors for agents at the boundary. Absolute rather than relative positions characterize this two dimensional (2D) setup, and agents can only move to empty positions.[10,11]

The preferences considered in Schelling (1971a) are the same as for the one-dimensional model (each agent accepts up to 50% of unlike neighbors), whereas Schelling (1971b, 1978) also considers the possibility that agents accept up to 2/3 of unlike neighbors. The starting state is typically highly integrated.[12] The best-response dynamics, then, work as follows. All unsatisfied agents are put on a list in some arbitrary order. When an agent's turn comes, he moves to the nearest available satisfactory position. At the next stage a new list is compiled, and so on. This process continues until no agent wants to move anymore. Again, the typical outcome is a highly segregated state.

3.2. Schelling's model revisited

Many details of Schelling (1969, 1971a,b, 1978) can be varied, but our focus is on the driving force behind the dynamic process, the individual preferences, while other details are kept as simple and general as possible. We consider utility functions that imply a *strict preference for perfect integration*. That is, all individual agents with such preferences strictly dislike living in a segregated neighborhood, even if they are a part of the majority. Our analysis is invariant to any positive monotonic transformation of these utility functions.

Let the utility function of an individual agent be u. Denote the percentage of his neighbors consisting of the other type as x $(0 \leq x \leq 100)$, and the maximum tolerable percentage of unlike

[8] Notice that local preferences are exemplified in the so-called NIMBY (Not In My BackYard) attitude. They may be explained by the fact that, as Quinn puts it, "(t)hat's where you mow the lawn, you shovel the snow, and the kids play together" (The Economist, 2003, p. 50).

[9] Alternatively, one could interpret the 1D model as one with a continuous action space, and the agents taking up a negligible amount of space themselves, such that there is always space between any two agents available.

[10] The reason is that transferring the moving technique used in 1D to 2D leads to some complications. It is not clear in which dimension one should create space or close empty spaces on a lattice. Further, when creating or closing space in one direction, *all* other agents on that row (column) would see their neighborhood altered, as would all agents on the adjacent rows (columns).

[11] See also Sakoda (1971), which is based on Sakoda (1949), for a very similar model of endogenous interactions.

[12] Schelling (1971a) starts with a random initial distribution of agents, while Schelling (1971b, 1978) creates the starting configuration by reshuffling a perfectly integrated board.

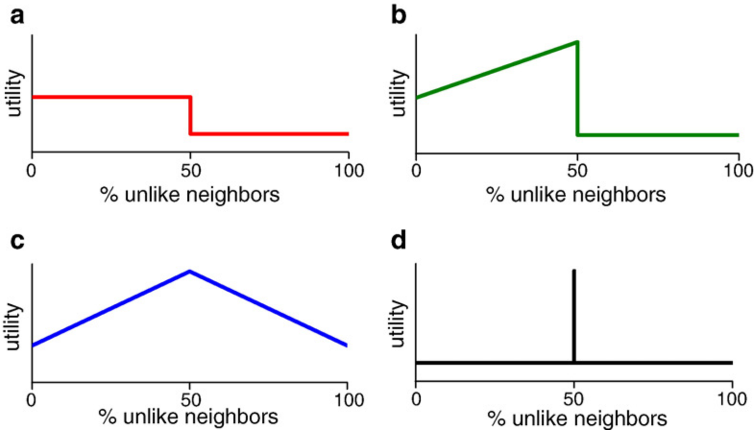

Fig. 1. a. flat utility. b. p50 utility. c. p100 utility. d. spiked utility.

neighbors as c. Then, the class of individual preferences considered can be represented as follows:

$$u(x) = \begin{cases} a + d(50-|x-50|) & \text{for } x<50 \text{ and } 50<x\leq c \\ a + d(50-|x-50|) + b & \text{for } x = 50 \\ 0 & \text{for } x>c \text{ and if } x \text{ not defined (i.e., no neighbors),} \end{cases}$$[13]

with $a>0$, $50\leq c\leq 100$, $d\geq 0$, and $b\geq 0$.

The first utility function that we consider is based on Schelling (1969, 1971a,b, 1978), and is the weakest one with respect to preferences for integration, setting $b=d=0$ and $c=50$ (see Fig. 1a). This utility function consists of two entirely flat pieces with a discrete drop in utility at a cut-off point of 50%. Thus, an agent is indifferent between a neighborhood without any unlike agents and any neighborhood with up to 50% unlike neighbors, and perfect integration is no better than complete segregation as long as an agent can live among his own type.

An essential change we introduce next concerns the rising part up to a peak at a 50–50 neighborhood ($d>0$), while retaining the cut-off point and the flat part beyond it, i.e., $c=50$ (see the p50 utility function in Fig. 1b). While such an agent still has aversion to being in a minority, he now has a strict preference for perfect integration, preferring this to any majority of like agents.

We also consider a peaked utility function ($d>0$) with the cut-off point removed ($c=100$). This is represented by the perfectly symmetric single-peaked utility function p100 in Fig. 1c. Such an agent has no bias in favor of like agents at all. He strictly prefers living in a perfectly integrated neighborhood, but any neighborhood with x% like or x% unlike neighbors is equally good.

[13] For what the two-dimensional model concerns, Schelling (1971a,b, 1978) distinguishes preferences expressed in absolute terms (number of like or unlike agents within a neighborhood) or relative terms (ratio of like to unlike neighbors within a neighborhood), but in neither case specifies preferences over completely empty neighborhoods. We assume that empty neighborhoods are the least preferred, on which none of our results depend.

Finally, we consider a spiked utility function that emphasizes the strict preference for perfect integration by rendering an individual indifferent with respect to any other configuration, setting $a=d=0$ and $b>0$ (see Fig. 1d). Here agents are driven exclusively by their obsession to live in a perfectly integrated neighborhood.

Given the preferences, the behavioral assumption made by Schelling (1969, 1971a,b, 1978) is that of myopic best-responses (BR). Schelling, however, also assumes inertia. That is, satisfied agents will always stay put, whereas it is not clear what happens with non-satisfied agents who cannot find a satisfactory position. What is more, Schelling assumes that players move to the nearest satisfactory position. Both assumptions can be justified by some moving technology. The cost of moving would need to be strictly increasing with distance, but even for the largest possible distance it would need to be smaller than any possible strictly positive difference in utility between two locations.

We abstract from this implicit assumption of moving costs to focus on preferences. That is, a player chooses a best-response with probability one, but in case of indifference between more than one optimal position (possibly including his current position), he chooses equiprobably among them.

We also simplify the order of the moves. In Schelling (1969, 1971a,b, 1978) all unsatisfied agents simultaneously put their name on a list, which is, then, processed sequentially in some arbitrary order, after which a new list is drawn, etc.[14] We simply select at each stage one agent uniform randomly, and ask him to choose a best-response.[15] In Pancs and Vriend (2003) we also analyze the possibility that all agents simultaneously choose a best-response to the current state.[16]

Next, we specify the spatial setting. In all setups analyzed, we consider neighborhoods defined in terms of the eight nearest surrounding positions (for agents in the interior). In 2D we focus on the standard board specification, while in the 1D setup we concentrate on a ring. The reason to focus on a board instead of a torus is that 2D tori do not seem to appear frequently in reality. The usual justification for considering a torus is that it is an approximation of an indefinitely extending two-dimensional space. It is unclear, though, that this is a meaningful approximation, especially when the underlying lattice is relatively small. No results depend crucially on this choice.

As to the one-dimensional setup, we focus on a ring for two reasons. First, a ring, unlike a 2D torus, is relatively natural (e.g., in the form of a ring road in a city, chairs around a table, the shoreline of an island or lake, or the 24 h around the clock). Second, the positions near the boundary of a finite line have a huge impact on the existence of equilibria, which is an artifact caused by the shrinking size of a neighborhood for agents close to the edges. Schelling (1971a) explains that for $k+k$ neighborhoods with odd k, alternating equilibria disappear. However, this is also the case for even k. Furthermore, with the peaked and spiked utility functions that we consider *all* (pure strategy) equilibria disappear for *any* k.[17]

[14] Notice that Schelling's specification does not seem very natural from a game-theoretic perspective: a currently satisfied agent might want to put his name on the list anyway, as he might no longer be satisfied at the moment his turn would come.

[15] Hence, although we do not consider any noise per se, there are three sources of randomness in our model: the initial allocations, the order of moves, and the way indifferences are resolved.

[16] In case of conflicting choices in the 2D setup (two or more agents simultaneously choosing the same location), we randomly allow one of these to be realized. We, however, do not consider the possibility of simultaneously choices in the 1D setup, because of the conceptual difficulties arising from conflicting choices.

[17] Any *ad hoc* cures, e.g., modifying the utility function for agents near the edges, would rob the model of one of its major advantages: simplicity.

4. Analytical tools and benchmark allocations

Given the model as specified in Section 3, we need to characterize the outcomes of the BR dynamics. To do so, in this subsection, we present two benchmark allocations, and define the measures summarizing the degree of segregation of these allocations.

4.1. Benchmark allocations

In order to characterize the set of all possible steady-states of BR dynamics, we first introduce the concept of a Myopic Nash Equilibrium (MNE), describing all those configurations in which no agent can find a better location given the locations currently chosen by the other players. This equilibrium is myopic since the dynamic structure of the game is disregarded.[18] Notice that the set of MNE would be of special interest to a social planner who could affect the initial allocation of agents.

Definition 1. Let $z_i \in Z_i$ be the location of player i, such that $z_i \neq z_j$ for $i \neq j$, and let $v_i(z)$ be the payoff of agent i from z. Then $z^* = (z_i^*, z_{-i}^*)$ is a Myopic Nash Equilibrium (MNE) iff, for all $z_i \in Z_i$ such that $z_i \neq z_j^*$ for all i and j, $v_i(z_i^*, z_{-i}^*) \geq v_i(z_i, z_{-i}^*)$. A MNE is strict if these inequalities are all strict, and a MNE is weak otherwise.

The second benchmark is the set of all possible allocations, which in case of the 2D version we approximate by a sample of random allocations. This benchmark allows to tell in which sense the MNE or the BR outcomes are out of the ordinary.

Definition 2. In a random allocation the probability that any given location is occupied by a particular type equals the ratio of the number of agents of this type to the number of possible locations.

A random allocation is likely to be rather integrated, with many agents having neighbors of both types. A segregated allocation describes the opposite case, when few agents have neighbors of the opposite type. The concept of a completely segregated allocation is specific to the segregation measure used. Complete segregation occurs if the segregation measure records the lowest possible level of integration. When describing an individual neighborhood, the concepts of perfect integration and complete segregation have the following meanings.

Definition 3. A neighborhood is perfectly integrated if it contains equal numbers of agents of each type.

Definition 4. A neighborhood is completely segregated if it contains agents of only one type.

When analyzing the myopic best-response dynamics, the following two terms are used: a period and a run.

Definition 5. A period is an instance when an agent is offered an opportunity to move.

Definition 6. A run corresponds to an independently executed best-response sequence. In a run of n periods, n agents are chosen sequentially to make a location decision. Each choice is an identical and independent draw from the set of agents, with each agent facing the same probability of being chosen.

[18] In other words, the equilibrium is in locations rather than dynamic strategies.

4.2. Segregation measures

We use the following measures of segregation.

4.2.1. Clusters

This measure counts the number of clusters that can be distinguished. Two agents belong to the same cluster if they are of the same type and they are, either directly or indirectly, linked laterally. Two agents are directly linked if they neighbor each other either horizontally or vertically. Moreover, if agent i belongs to the same cluster as agent j, and agent j to the same cluster as agent k, then agents i and k belong to the same cluster as well. An indirect lateral link goes through an uncontended zone of empty cells ('blanks'). Two blanks belong to the same zone if they are laterally linked, applying the same transitive relationship as above. Such a zone is contended if its neighbors, horizontally or vertically, are agents of different types, otherwise it is uncontended.[19] The one-dimensional version of the cluster measure is a straightforward simplification, as there are no blanks. A cluster measure, which is equivalent to the average cluster size, was used in Schelling (1969, 1971a,b, 1978), without being formally defined. The cluster measure does not take into account how large individual clusters are, or how integrated (or not) agents within a cluster are.

4.2.2. Switch rate

Take the position of a given agent, and make one full turn to observe all his neighbors. Let m_i be the number of agent i's neighbors if it exceeds one, while it is zero otherwise. Let l_i be the number of switches, defined as the number of times that the type of a neighbor changes as we complete the turn, ignoring blanks. The switch rate, then, is $\Sigma_i l_i / \Sigma_i m_i$. The switch rate, unlike the cluster measure, cares about patterns. It measures how integrated neighborhoods are, as seen by the individual agents.[20]

4.2.3. Distance

Let r_i be the minimal number of cells which need to be traveled by agent i (either laterally or diagonally) to reach an unlike agent, and t_i be the minimal number of cells to reach a like agent. The distance measure is: $(1/N)\Sigma_i(r_i/t_i)$, where N denotes the total number of agents.

4.2.4. Mix deviation

For a given agent i, let p_i be the absolute deviation from a 50–50 neighborhood: $p_i = |0.50 - g_i/(g_i+f_i)|$, where g_i is the number of like agents in agent i's neighborhood (excluding the agent himself), f_i is the number of unlike agents, and $p_i = 0$ for agents with empty neighborhoods. The mix deviation measure, then, is: $(1/N)\Sigma_i p_i$.

4.2.5. Share

For a given agent i, let g_i and f_i be again the number of like and unlike agents respectively in his neighborhood. The share measure, then, is: $\Sigma_i g_i / (\Sigma_i g_i + \Sigma_i f_i)$, where agents with empty neighborhoods are ignored. The measure is based on Schelling (1969, 1971a). A difference with the mix deviation is that it computes a weighted average of individual shares.

[19] The extension of the measure to diagonal links is not straightforward, as one would need to define the concept of contended diagonal links, which can give rise to counterintuitive results.

[20] Notice that agents with one or no neighbors are ignored, as no switches are possible for such neighborhoods. In the 1D setup the switch rate is equivalent to the clusters measure.

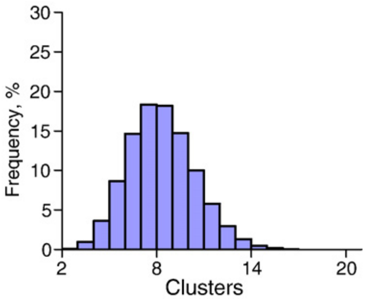

Fig. 2. Random allocations, 5 × 5 board.

4.2.6. Ghetto rate

This measures the number of agents that lives in a neighborhood without any unlike neighbor. This measure, due to Schelling (1969), is a somewhat crude one as it treats having one unlike neighbor the same as having eight unlike neighbors.

These measures will be correlated to some extent, but they will each stress slightly different aspects of segregation. The emphasis is on segregation, rather than on some utility based measure, because this gives us an exogenous criterion to assess the consequences of different individual utility functions.[21]

5. Analysis of two-dimensional setup

5.1. 5 × 5 board

We start analyzing the model with a 5 × 5 board, with ten agents of each type, and five empty locations.[22] Fig. 2 shows the distribution of clusters for a million random allocations. The distribution is rather symmetric, with the 20 agents forming on average 7.8 clusters.

Table 1 shows the number of existing MNE for the flat and two peaked (p50 and p100) utility functions.[23] The search for MNE is exhaustive, i.e., we try all the possible configurations on the board. The number of MNE is of a similar order of magnitude for the flat and the p100 utility functions, whereas it is much lower for the p50 function.[24] Also, all MNE for the flat and p50 utility functions are non-strict, whereas almost 10% is strict with the p100 function.

[21] This approach would correspond to a government envisaging a desired outcome, such as multiculturalism or integration, and then implementing policies to induce particular individual attitudes. Typically, the literature on segregation uses measures designed on the basis of pre-defined and distinguishable neighborhoods (e.g., Census tracts). These measures can be classified along five dimensions: evenness (dissimilarity), exposure (isolation), concentration, centralization, and clustering (see, e.g., Cutler and Glaeser, 1997; Cutler et al., 1999; Frankel and Volij, 2004; Massey and Denton, 1988; White, 1983, 1986). While there are no pre-defined neighborhoods in the Schelling model, our mix deviation and share measures can be seen as measures of evenness, our ghetto and switch rate measure exposure, and our cluster and distance measures concern clustering.

[22] The reason to start with a 5 × 5 board is its tractability. In the analysis of the 2D we always allocate 40% of locations to each type, leaving 20% empty.

[23] Since for the 2D setup the findings for the spiked utility function are very similar to the p100 function, we omit them throughout. These results are available from the authors upon request.

[24] When counting allocations, those obtained by swapping Os and Xs are not distinguished. Mirrored or rotated allocations are, however, distinguished because, for example, the number of distinct rotations depends on the degree of symmetry of a particular allocation.

Table 1
Number of existing MNE, 5 × 5 board

	MNE		
	Flat utility	p50	p100
Non-strict MNE	430,110	2880	351,472
Strict MNE	0	0	36,482
Total MNE	430,110	2880	387,954

Fig. 3a to c show the frequency distribution of the cluster measure for the set of MNE for the flat and peaked utility functions. The MNE with the flat utility function are concentrated in the lower half of the range found for random allocations, with an average of 4.3 clusters per MNE.

Notice that the set of MNE for the p50 utility function is a subset of the MNE for the flat utility function.[25] The question, then, is which MNE of Fig. 3a will survive with the p50 function. Given the strict preference for perfect integration, one might conjecture that the subset of MNE will be more integrated. However, as Fig. 3b shows, this turns out to be incorrect. The average of the 2880 MNE with the p50 utility function has 3.6 clusters, with a majority (64%) characterized by complete segregation. The distribution of the set of MNE for the p100 utility function looks similar to that for the flat utility function, with 4.0 clusters on average.

While there is little reason to expect MNE to be a focal point of the best-response dynamics, it is instructive to see by how much and in which direction the best-response dynamics outcomes deviate from the set of MNE. The best-response dynamics are started from random allocations. The initial allocations do not matter, unless it is a strict MNE with p100 utility, in which case no agent would wish to move. Notice that the number of strict MNE is negligible relative to the total number of possible allocations.

Fig. 4a to c show the outcomes of best-response (BR) dynamics, without inertia, for each of the three utility functions. Each diagram depicts the distribution of clusters for 1000 runs after 100,000 periods. In Fig. 4a, for the flat utility function, 91% of the runs end in complete segregation, with 2.1 clusters on average. As Fig. 4b shows, with strict preferences for perfect integration, there is even more segregation. In 98% of the runs we observe complete segregation after 100,000 periods. The average is 2.0 clusters. The results for the p100 utility function are less stark. In Fig. 4c, the distribution of outcomes of the BR dynamics is not very different from the set of MNE. On average there are 5.0 clusters. Although complete segregation seems to be avoided, this still implies more segregation than with random allocations, where we observed on average 7.8 clusters.

To demonstrate that our results are not sensitive to the choice of the segregation measure, Table 2 summarizes the results for all segregation measures we defined. For each of the measures there is hardly any difference between the outcomes of the BR dynamics with the flat and p50 function. While there are fewer clusters with the p50 than with the flat function, the other measures suggest there is slightly more segregation with the flat than with the p50 utility function.[26]

[25] If it is not possible to find a better location for an agent with the p50 utility function, then it is also impossible with the flat utility function. But the opposite is not true. Suppose an agent lives in a large-majority neighborhood while a perfectly integrated location is available. With a flat utility function this could be part of a MNE, whereas an agent with the p50 utility function would deviate to that empty position.

[26] The reason to emphasize the clusters measure is that it seems to capture the notion of segregation best at the intuitive level.

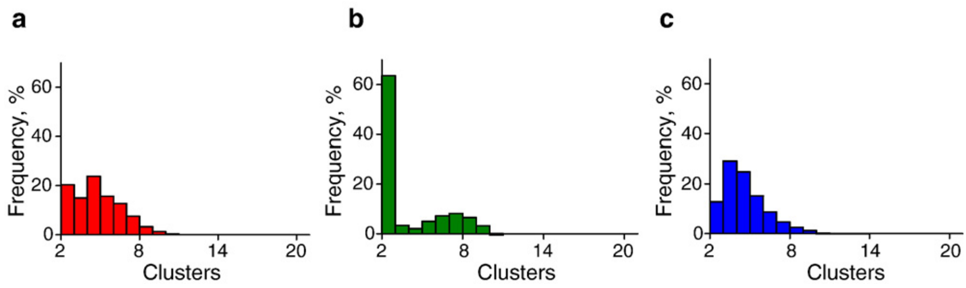

Fig. 3. a. MNE, flat utility, 5 × 5 board. b. MNE, p50 utility, 5 × 5 board. c. MNE, p100 utility, 5 × 5 board.

Table 3 shows the substantial number of MNE reached by the BR dynamics. Although the number of existing MNE is very small with the p50 utility function, BR dynamics lead to a MNE in 37% of cases. This is of the same order of magnitude as the number of MNE reached with the flat utility function (40%). With the p100 function we essentially always end up in a MNE, with most cases being a strict MNE. This shows that even non-strict MNE act as attractors.

The numbers in Table 3 can be compared to the number of MNE one would expect in a sample of random allocations. Accounting for X/O symmetry, there are $\frac{1}{2}\binom{25}{10}\binom{15}{10} = 4,908,043,140$ possible allocations. This means that, given the number of MNE shown in Table 1, for the flat and p50 utility functions, where all MNE are non-strict, a random sample of 1000 allocations most likely would contain no MNE. For the flat utility function the expected number of MNE in 1000 random allocations is $1000\frac{430,110}{4,908,043,140} \approx 0.09$ and for the p50 function it is even much lower. For the p100 function matters are slightly different, because some of the MNE are strict. Assuming that each of the 100,000 periods is a random draw, out of 1000 runs one would expect 524 strict MNE and no weak ones: $1000\left(1-\left(1-\frac{36,482}{4,908,043,140}\right)^{100,000}\right) \approx 524$. The BR dynamics, however, give 854 strict and 145 non-strict MNE. The distinction between the strict and weak MNE matters, because once a strict MNE is encountered, BR process stops. All weak MNE are transient, since no inertia is assumed. This establishes that MNE are over-represented as the outcome of BR dynamics.

While looking at the 100,000th period as evidence of the limiting behavior is reasonable, an analysis of the time series of the average cluster measure plus the 5th and 95th percentile for 10,000 runs (see Pancs and Vriend, 2003) suggests that we could illustrate this by observing a considerably shorter spell of BR dynamics. In each case the degree of segregation stabilizes already within the first few hundred periods. With both the flat and the p50 utility functions average segregation is not only

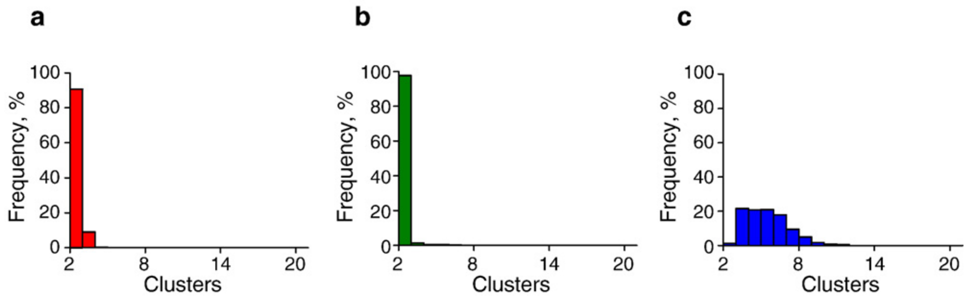

Fig. 4. a. BR, 100,000 periods, flat utility, 5 × 5 board. b. BR, 100,000 periods, p50 utility, 5 × 5 board. c. BR, 100,000 periods, p100 utility, 5 × 5 board.

Table 2
Final distributions, 5 × 5 boards

		Random	MNE			BR dynamics (100,000 periods)		
			Flat	p50	p100	Flat	p50	p100
	Num. Obs.	1,000,000	430,110	2880	387,954	1000	1000	1000
Clusters	Average	7.82	4.30	3.61	4.14	2.10	2.04	4.99
	5%	5	2	2	2	2	2	3
	95%	11	7	8	7	3	2	8
Switch	Average	0.53	0.35	0.31	0.43	0.21	0.23	0.51
	5%	0.40	0.19	0.19	0.28	0.16	0.19	0.42
	95%	0.65	0.52	0.52	0.60	0.27	0.28	0.62
Distance	Average	1.02	1.25	1.31	1.10	1.58	1.50	1.00
	5%	0.93	1.00	1.00	0.98	1.40	1.00	1.00
	95%	1.15	1.65	1.60	1.33	1.80	1.65	1.05
Mix. dev.	Average	0.18	0.23	0.23	0.18	0.34	0.29	0.14
	5%	0.12	0.13	0.12	0.10	0.28	0.26	0.09
	95%	0.25	0.36	0.29	0.26	0.40	0.32	0.18
Share	Average	0.47	0.67	0.66	0.55	0.80	0.73	0.50
	5%	0.38	0.56	0.50	0.43	0.73	0.70	0.42
	95%	0.59	0.84	0.75	0.69	0.88	0.78	0.57
Ghetto	Average	1.06	4.89	5.76	2.07	10.49	8.76	0.08
	5%	0	1	1	0	8	6	0
	95%	4	12	10	6	13	10	1

complete but also rather quick. With the perfectly symmetric p100 utility function segregation is substantial, but not extreme. Much of this segregation occurs relatively early on. In fact, initially the graph looks similar to those for the flat and p50 utility functions. Eventually, the 95th percentile is at 8 clusters, near the average of 7.8 clusters for random allocations.

5.2. Extensions and summary of findings in 2D setup

In Pancs and Vriend (2003) we confirmed that the insights from a 5 × 5 lattice generalize to larger lattices, analyzing the BR dynamics over 50 million periods for a setup with 4000 agents of each type on a 100 × 100 board. Starting with just over 2000 clusters, integration initially declines rapidly for each utility function. With the p100 function the number of clusters reaches about 1700, whereas with both the flat and p50 utility functions very strong segregation obtains almost immediately, although there is some difference between the flat and the p50 utility functions concerning the exact pattern of the final allocation.

Table 3
Number of MNE reached, 5 × 5 board

	BR dynamics (100,000 periods)		
	Flat	p50	p100
Observations	1000	1000	1000
Non-strict MNE reached	404	370	145
Strict MNE reached	n.a.	n.a.	854
Total MNE reached	404	370	999

As explained in Section 3, we decided to focus our analysis of the 2D version of Schelling's model on a board, with the players moving sequentially, and without inertia. In Pancs and Vriend (2003), we establish the robustness of the obtained results by introducing inertia, simultaneous moves, and a torus. For each of these three cases, the following two questions are addressed. First, was the original assumption made by Schelling (1969, 1971a,b, 1978) on inertia, a preference for nearby positions, and the specific way in which the order of moves was determined an essential element of his model of segregation? Second, does the introduction of this variation concerning the nature of the order of moves (sequential or simultaneous), the structure of the lattice (a board or a torus) or the presence of inertia change our findings concerning the relative effect of the utility functions with a strict preference for integration relative to Schelling's flat utility? In each of the three cases the answer to both questions is negative (see Pancs and Vriend, 2003 for an extensive analysis).

Our analysis of the 2D setup shows that Schelling's results are not only robust to a class of alternative specifications, but they can also be strengthened enormously. The simple model characterized by sequential moves (in a random order) in the absence of inertia and without a preference for nearby positions exhibits rapid segregation, even with strict preferences for perfect integration.

While the strict preference for perfect integration (as with the p50 function) leads to approximately the same amount of segregation as the flat utility function, segregation for the p100 function is not as stark as for the flat and p50 utility functions. The essential difference is the asymmetry of the latter two. With Schelling's flat utility function there are two separate effects of this asymmetry. First, in case of indifference between a range of satisfactory positions, agents, on average, would choose a relatively segregated option. This implies a 'random drift' away from integration. Second, in the case of facing a choice between a small minority location and a large majority location (i.e., positions with either x% unlike or x% like, where x is greater than the cut-off point), agents favor the latter. With the p50 function, the flat part favoring the drift to segregation has been removed, while the cut-off point is retained, whereas with the p100 function the cut-off point has been removed as well. Since the p100 function does not induce substantial segregation, this further helps to pin down the essential element explaining segregation in the 2D model. It is the asymmetry related to the cut-off point, i.e., the fact that an agent favors his own ghetto over an unlike ghetto, that is the crucial element in the 2D setup.

This finding is consistent with Zhang (2004), who extends the analysis of Young (1998) for the 1D setup to two dimensions. Both Zhang (2004) and Young (1998) consider a modification of Schelling's one-dimensional model, with agents swapping locations, compensating payments between moving players, and in the presence of noise (mistakes). They argue that complete segregation is the only viable long-run outcome of the best-response dynamics if the agents' preferences are biased in favor of their own type.

The concept of a MNE plays an important role in the analysis, as the BR dynamics often arrive at a strict MNE or spend considerable time in weak MNE. Best-responses eliminate attractive locations,[27] thus reducing incentives to deviate, and increasing chances to encounter a MNE. Further, MNE tend to be clustered together, as it is easy to obtain one MNE from another by moving indifferent agents. Consequently, notwithstanding their tiny share in the total number of possible allocations, MNE states tend to be quite persistent. In as far as MNE are reached, the BR dynamics

[27] This begs the question whether the chosen population density (40% of each type of agents) is critical to the behavior of the system, as it influences the availability of choice locations. As the sensitivity analysis presented in Pancs and Vriend (2003) shows, this is not the case.

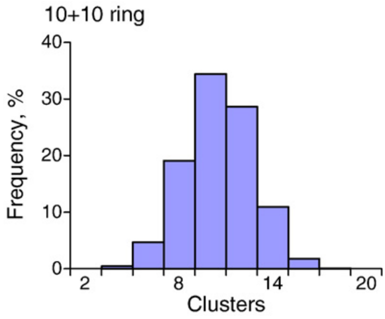

Fig. 5. All possible allocations, 10 + 10 ring.

appear to favor the most segregated among them. But, the attraction of MNE notwithstanding, it would be wrong to conclude that BR dynamics would necessarily lead to an MNE.

6. Analysis of one-dimensional setup

We now turn to an analysis of the 1D version of Schelling's spatial proximity model. In this section we present a number of sharp theoretical results on segregation in 1D. In addition, a numerical analysis serves to highlight implications of alternative specifications of the model, and to demonstrate that asymptotic theoretical results are attained in finite time.

As we saw above, in the 2D setup a considerable role was played by the MNE. Since the existence of MNE cannot be ensured on a line, in most of what follows the 1D space is assumed to be a ring.

6.1. 10 + 10 ring

For a start, and comparison with the 2D model, we look at a ring with ten agents of each type, and a neighborhood formed by eight neighbors (four to the left, and four to the right). Fig. 5 shows the distribution of the cluster measure for all 92,378 possible allocations.[28] On average the 20 agents are located in 10.5 clusters.

The other benchmark we use is the set of MNE. There are 28 non-strict MNE for the flat utility function, and 18 non-strict MNE for the two peaked (p50 and p100) and spiked utility functions. These are exhaustively constructed. The MNE are identical for the two peaked (p50 and p100) and spiked utility functions, forming a subset of the set of MNE for the flat utility function. There are no strict MNE. That the number of MNE is much lower than in the 2D setup is, in part, because there are fewer possible allocations (since there are no empty spaces), and, in part, because the restrictions on equilibrium allocations are much more stringent. The formal analysis of these observations is deferred until Section 6.2.

Fig. 6a and b show the distribution of the cluster measure for the MNE for each of the utility functions considered. MNE with the flat utility range from complete segregation (2 clusters) to perfect integration (20 clusters). This time, however, the intuition that the subset of MNE with the peaked (p50) utility function is characterized by more integration than the set of MNE with the

[28] Again, we neglect equivalent allocations obtained by swapping Os and Xs, so that $\frac{1}{2}\binom{20}{10} = 92,378$.

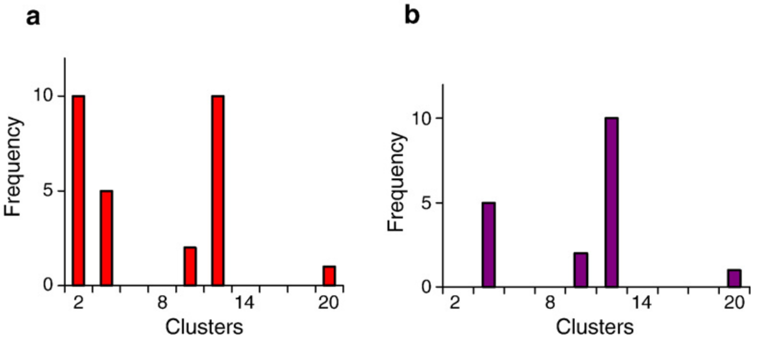

Fig. 6. a. MNE, flat utility, 10+10 ring. b. MNE, peaked or spiked utility, 10+10 ring.

flat utility function is correct. All completely segregated MNE of the flat utility function disappear with the peaked and spiked utility functions.

Fig. 7a and b show the outcomes for 1000 observations of the BR sequences run for 100,000 periods. For each of the utility functions considered, the BR dynamics invariably lead to complete segregation. As will be shown in Section 6.2 this is true for any initial allocation. For the flat utility function one could interpret this as the dynamics always selecting a segregated MNE. However, with any of the peaked or spiked utility functions there is no MNE corresponding to this outcome.[29] Hence, Schelling's (1971a) observation that *"(w)e could have surmised that our rules of movement would lead to equilibria"* (p. 151) is correct for the flat utility function, but it is not generally true for these rules of movement. In particular, the conjecture is not true for the class of peaked or spiked utility functions that we consider.

Since complete segregation is not a MNE with the peaked or spiked utility functions, it cannot be a steady state. Nevertheless, the pattern of complete segregation is stable. With complete segregation, and any of the peaked or spiked utility functions, only the agents at the border of their own ghetto will enjoy the 'bliss' level of utility. Each time an agent can make a move, he will locate himself exactly at such a boundary. Hence, almost all the time the agent whose turn has come moves to a better location, but the configuration of the two ghettos as such is stable. These ghettos only move around on the ring.

Table 4 summarizes the characterization of all possible allocations as well as those of the MNE and the BR outcomes using various segregation measures. MNE for the peaked and spiked utility functions are perfectly integrated according to the mix deviation, share and ghetto measures,[30] while according to the cluster and distance measures they are as integrated as the average random allocation. In each of the MNE with the peaked or spiked utility functions, all agents live in a 50–50 neighborhood, reaching maximum utility. This implies that the adjustment dynamics are an important concern from a social welfare point of view. In sharp contrast to the perfectly satisfactory MNE, BR dynamics always lead to complete segregation, where only four agents reach maximum utility.

[29] As argued below, the state of complete segregation is almost the extreme opposite of any of the MNE. It is not true, however, that complete segregation is a state with the lowest possible utility. For example, with the spiked utility function, alternating clusters of size $(k-1)$ would make all agents unsatisfied.

[30] Although this cannot be read from the table, this applies in fact to each MNE for the peaked and spiked utility functions.

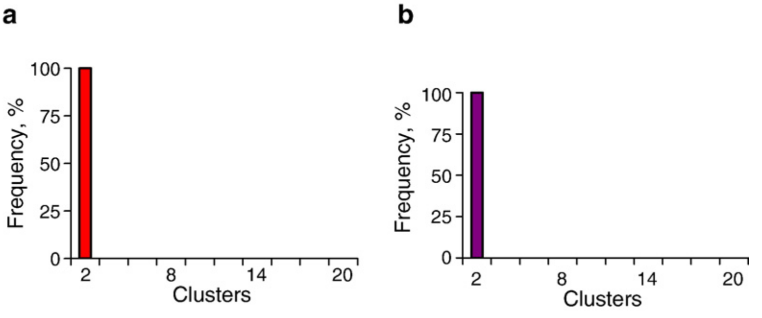

Fig. 7. a. BR, 100,000 periods, flat utility, 10+10 ring. b. BR, 100,000 periods, peaked or spiked utility, 10+10 ring.

In Pancs and Vriend (2003) we also analyzed how these final distributions tend to be approached over time. Starting from identical initial configurations for each of the utility functions, the flat utility function leads to complete segregation within 100 periods in 95% of the 10,000 runs, whereas even with the peaked (p50 and p100) and spiked utility functions it takes just over 300 periods to reach complete segregation.

6.2. Formal analysis

This section demonstrates that the emergence of complete segregation with BR dynamics for each of the utility functions considered holds for any ring size, and any neighborhood size. It is assumed throughout that the number of Xs equals the number of Os.

Proposition A1 shows that in the long run complete segregation is the only possible outcome of best-response dynamics with the spiked utility function. Corollary 1 explains that the same applies to the flat and peaked (p50 and p100) utility functions.

Table 4
Final distributions, 10+10 ring

		All	MNE		BR dynamics (100,000 periods)	
			Flat	Peaked or spiked	Flat	Peaked or spiked
	Num. obs.	92,378	28	18	1000	1000
Clusters	Average	10.53	7.14	10.00	2.00	2.00
	5%	6	2	4	2	2
	95%	14	12	12	2	2
Distance	Average	1.10	1.82	1.16	3.00	3.00
	5%	0.82	0.50	0.50	3.00	3.00
	95%	1.51	3.00	1.80	3.00	3.00
Mix. dev.	Average	0.11	0.09	0.00	0.25	0.25
	5%	0.05	0.00	0.00	0.25	0.25
	95%	0.18	0.25	0.00	0.25	0.25
Share	Average	0.47	0.59	0.50	0.75	0.75
	5%	0.43	0.50	0.50	0.75	0.75
	95%	0.55	0.75	0.50	0.75	0.75
Ghetto	Average	0.00	1.43	0.00	4.00	4.00
	5%	0	0	0	4	4
	95%	0	4	0	4	4

Proposition A1. *If a neighborhood on a circle of size 2m is defined as k neighbors to the left and k neighbors to the right, then, if m is the number of each type, $m>k$, and the utility function is spiked, then the process of best-responses has a unique recurrent class consisting of all completely segregated states.*

Discussion. The complete proof can be found in Appendix A (available on the Journal of Public Economics website). Here is a sketch. We need to show that it is possible to reach complete segregation from each initial configuration with positive probability, while all completely segregated states constitute a single recurrent class. On a completely segregated ring, there are two borders between ghettos. These borders are the only locations that offer positive utility. Hence, any agent can only move to such a border, which does not affect the integrity of the ghettos. Thus, it is sufficient to offer an algorithm showing just one possible path leading from each allocation to complete segregation. In the proof this is done in two steps. A 'seed' is defined as a segment of a ring formed by k agents of one type followed by k agents of the other type. If an allocation contains a seed, then construction of complete segregation by means of positive probability moves is trivial, as this seed can always grow by adding agents at the border inside the seed. If no seed is present initially, we show that it will eventually emerge whatever the initial allocation. Hence, the limit outcome of the BR dynamics is complete segregation.

Corollary 1. *Proposition A1 also holds for the flat and peaked (p50 and p100) utility functions and for any utility function implying a strict preference for perfect integration.*

Proof. Any best-response move according to the spiked utility function would also be a best-response according to the other utility functions. This claim, in turn, relies on the fact that there always exists a location offering the bliss level of utility. That it is indeed so is established in the course of the proof of Proposition A1 (see Appendix A, available on the Journal of Public Economics website). In addition, neither the flat nor any of the peaked utility functions allow an escape from complete segregation. □

It can be shown that Proposition A1 holds even if the number of Xs does not equal the number of Os. Then however, Corollary 1 does not apply.

We now turn to a characterization of the set of MNE for any ring size. For the flow of exposition the number of a proposition is indicated in parentheses after a claim. Its formal statement and a proof are in Appendix B (available on the Journal of Public Economics website).

On a ring there exist at least two positions where an agent can insert itself and enjoy perfect integration (Lemma B1). This implies that for an allocation to be a MNE it is necessary and sufficient that all agents enjoy the highest possible utility (Proposition B1), for otherwise someone would be willing to migrate to one of the perfectly integrated locations. Thus MNE are the only Pareto efficient outcomes. This is so for any of our utility specifications (including the flat utility function), because they all assign the highest possible utility to living in a perfectly integrated neighborhood. Thus the welfare implication drawn for the special case above on the basis of a numerical analysis is a general feature of the 1D model. For all utility functions considered other than the flat one, BR dynamics lead to a Pareto inferior outcome. Pareto efficiency of MNE immediately explains why, with a strict preference for integration, according to the mix, share and ghetto measures, equilibria were characterized as perfectly integrated.

The absence of strict equilibria is also a generic feature (Proposition B2), and it follows from the multiplicity of bliss locations. This claim can also be inferred from the fact that complete segregation is the only feasible long run outcome of BR dynamics. Otherwise the BR dynamics could have become stuck at one of the strict MNE, which would have destroyed the result.

In the 2D case, the set of MNE was biased towards segregation. The opposite is true in the 1D case: in any MNE, no cluster can exceed $k+1$ agents in size with the peaked or spiked utility functions, where k is the size of a neighborhood in either direction (Proposition B3). For a bigger cluster size, agents in the middle of the cluster would enjoy less than the bliss level of utility. But then, this cluster would not be a part of a MNE allocation, because an equilibrium implies that all agents enjoy the highest possible utility. Thus the BR outcome has nothing to do with MNE but for the flat function, when complete segregation happens to be an equilibrium.

The sets of MNE for the spiked and two peaked utility functions coincide (Proposition B4). These are a subset of MNE with the flat utility function (Proposition B7). Extra equilibria in the case of the flat function come from the absence of the upper bound on an admissible cluster size, so that even complete segregation becomes an equilibrium. Logic similar to that of Proposition B4 shows that BR moves for all utility specifications with a strict preference for integration are identical.

The following properties allow to construct the set of MNE for the spiked (and hence peaked) utility function. If the neighborhood size parameter k is odd, then, in and only in a MNE, agents at all locations i and $i+k+1$ are of opposite types (Proposition B5). If k is even, then either agents i and $i+k+1$ are of opposite types or agents i and $i+k$ are of the same type and $a_i+a_{i+1}+\ldots+a_{i+k-2}+a_{i+k-1}=0$. There $a_j=1$ if an agent in the jth position is X and $a_j=-1$ otherwise (Proposition B6). The above properties of MNE imply that many rings, depending on their length, will not have the full set of potential MNE given k. The perfectly integrated alternating MNE are robust to the length of a ring.[31]

If $k=4$, as it is throughout this section, then all equilibria described in Proposition B6 can be summarized by the following five patterns: XXXXXOOOOO, XXXOXOOOXO, XXOX XOOXOO, XXOO and XO. These different patterns cannot be combined; only shifting and concatenation of single patterns are allowed. Accounting for X/O symmetry, the above patterns give rise to 5, 5, 5, 2 and 1 MNE respectively. These 18 equilibria are the only ones possible for the spiked and peaked utility functions, whatever the size of a ring, whereas the number of MNE increases with the size of a ring for the flat utility function.[32]

6.3. Extensions and summary of findings in one-dimensional setup

The formal analysis in Section 6.2 allows us to characterize the set of MNE, and we also know that complete segregation is the only possible long-run outcome. Proposition A1 does not, however, say how soon this happens given the size of a ring, nor does it imply anything about the relative speed of convergence for the various utility specifications. Therefore, in Pancs and Vriend (2003) we also examined the dynamics on a ring with 100 agents of each type ($m=100$) and a neighborhood defined by eight neighbors ($k=4$) for 50 million periods. The random initial allocation, the same for each utility specification, has 110 clusters. After 50 million periods with the peaked or spiked utility function, a level of six clusters is reached, whereas complete segregation occurs within 25 thousand periods with the flat utility function. Moreover, while the flat utility function leads to a steady decline in the number of clusters, the

[31] For instance, if we took an 11+11 ring instead of a 10+10 one, there would be only one MNE, characterized by perfect integration, with 22 clusters, which would make the BR outcome of complete segregation even more striking.

[32] For $m=100$ agents of each type and $k=4$, there are only 18 MNE for the peaked or spiked utility functions, but 60,575,676,973,999,910,976,213 for the flat utility function. The latter comprises $1.34\times10^{-34}\%$ of all possible allocations. The share declines rapidly in m: for $m=10$, 20, 30, 40 and 50 the corresponding shares are 0.03, $6\cdot10^{-6}$, $1.5\cdot10^{-9}$, $4\cdot10^{-13}$ and $10^{-16}\%$.

degree of segregation is more erratic for the peaked and spiked preferences, and in particular we sometimes see sudden outbursts of strong integration that disappear as quickly as they emerge. Thus, the predicted limit result is obtained with the flat utility function, while the peaked and spiked utility functions lead to remarkably extreme segregation in finite time. Strict preferences for integration promote a consistent drive towards segregation, and when segregation is reached, it is permanent.

In Pancs and Vriend (2003) we also consider BR dynamics with inertia, and BR dynamics on a line (instead of a ring) to answer the same two questions as in Section 5. Are inertia and the line essential for Schelling's (1969, 1971a) results? Would inertia or a line change our findings concerning the peaked and spiked utility functions? Again, we find that the answer to each of these questions is negative. The main change that comes with a line is the non-existence of MNE for the peaked and spiked utility functions (see Pancs and Vriend, 2003), whereas the only MNE with the flat utility function consists of complete segregation.

In 1D, the assumptions on the preferences can be extremely mild. In particular, the asymmetry of the utility function (favoring a majority over a minority neighborhood) plays no role. A sufficient (but not necessary) condition on the utility function to get complete *segregation* is that it implies a strict preference for perfect *integration*. Given that, the utility function may have multiple local peaks, and it may even describe a preference for living in any minority neighborhood rather than in any majority neighborhood.[33] Particular specifications of the preferences, however, matter when it comes to the speed of convergence to the limit result. The flat utility function provides the strongest impetus towards quick segregation compared to the family of peaked functions. But, in contrast to the 2D model, in its 1D counterpart, myopic equilibria do not act as attractors in the course of the BR dynamics.

7. Conclusion

This paper establishes robustness of Schelling's spatial proximity models of segregation. This robustness allowed us to abstract from superfluous details on the order of moves, and assume away inertia and a bias in favor of nearby positions, to gain a better understanding of the essence of the spatial proximity model. As it turns out, mild proximity preferences plus externalities of moving agents are sufficient to generate segregation, without needing any further proximity concerns such as, for example, those concerning some information or moving technology. What is more, focusing on the preferences, we found that they can be made much more extreme in favor of integration, with the model still explaining segregation.

Our equilibrium analysis suggests that we can think of segregation in the current model as a coordination issue. We observe a multiplicity of equilibria and, as in the 2D model, some are more segregated than others. The coordination problem, then, consists of not only coordinating on an equilibrium, but doing so on one of the best equilibria. In the 1D model with the spiked or peaked utility functions, we see that all MNE are Pareto optimal. Hence, the problem there is one of coordinating on one of those equilibria. Our analysis shows that one should not expect myopic BR dynamics to solve these coordination problems. It may appear that had the agents been less myopic in the 1D model, coordination on one of the integrated equilibria would have been more likely. However, it is not clear that the coordination problem is related to the fact that our equilibria are myopic, in the sense that they are defined in terms of locations

[33] An example would be the horizontal mirror image of the peaked p50 utility function. Such features seem relevant from a biological perspective, where species might want to avoid living with too many like competitors.

rather than strategies. Solving for the Nash equilibria of the truly dynamic game (i.e., in strategies rather than locations) is beyond the scope of this paper, but one would expect in a dynamic game multiplicity of equilibria to be even more of an issue, and most likely some of those equilibria will be more integrated than others.[34] Hence, one would expect coordination to be even more of a problem when considering dynamic strategies.

What is it about the preferences that leads to the striking outcome of segregation? The answer is different in the 2D and 1D setups. In the 2D model, the dynamics are characterized by a quick disappearance of ideal locations, and agents having to move to less satisfactory ones. Related to this, the key element of the 2D model driving the segregation is the asymmetry in the utility function, i.e., the fact that agents favor a large-majority status over a small-minority status. In the 1D model, however, perfectly integrated locations remain available for choice indefinitely, with newly arriving agents merely pushing incumbents away from such locations. As a result, segregation occurs even if the individuals strictly prefer perfect integration with no bias whatsoever in favor of the agent's own type. In the 1D case complete segregation is the unique long-term outcome. It is the arrangement of the space and the definition of moves that are responsible for the difference between 1D and 2D models.[35] Consequently, the two spatial proximity models proposed by Schelling offer two very different explanations for segregation.

What are the welfare and policy implications of these insights? First, we find that the welfare effect of educating people to have preferences for integration might be adverse. With strict preferences for integration, the segregated outcome will be unsatisfying for the majority of people. This is an adverse welfare effect, as the mere presence of unsatisfied people implies by itself a welfare issue, in particular when more efficiently coordinated equilibria do exist. In contrast, with the flat utility function, as used in Schelling's original setup, this welfare issue does not arise, as people do not mind being segregated as long as they live in the right ghetto. Hence, *without* a strict preference for integration there are far fewer relatively unsatisfied people, even though most may be segregated.

Second, suppose a benevolent planner has some freedom in affecting agents' preferences and aims to maximize integration. What preferences would he like his citizens to have? Our analysis, using measures of integration that are independent of the utility functions assumed, raises some doubts (within the limitations of the model analyzed here) as to what the education of preferences for integration could achieve.[36]

Next, presuming that there is a social welfare case for integration (independent of the specification of the individual preferences), could a migration subsidy or tax system prevent segregation and implement integration? Although we do not explicitly analyze this issue, our analysis suggests that a system consisting of rewards for integrating moves or taxation of segregating moves might not work if it merely emulates the incentive structure represented by the various utility functions analyzed in our paper.

[34] For example, in the 1D model, suppose that we allow for dynamic strategies, and that all other agents choose the dynamic strategy consisting of behaving myopically just as in our BR dynamics. What would be a best-response (in dynamic strategies)? Without providing a full analysis, it would seem that one cannot do better than adhering to the same myopic strategy. In other words, when we would consider dynamic strategies, even in the 1D model there seems to be a Nash equilibrium that is completely segregated.

[35] To check that it is not the dimensionality as such that is the essential difference between the 1D and 2D models, we considered the following one-dimensional version of the 2D model. Take a 1×25 board with ten agents of each type and five empty locations, and connect the first and the last cell of the board. BR dynamics in this variant lead to segregation only for the flat and the p50 utility function, just as for the other 2D models analyzed.

[36] As one referee observed, not all preferences would lead to outcomes that are contrary to the underlying tastes of the agents. Indeed, we have verified that preferences for segregation do not facilitate integration. See also Sakoda (1971).

Another difficulty with integration is that it implies almost by definition a rather special pattern. This makes it hard not only to achieve integration (either in a decentralized or a centralized way), but also renders it unstable with respect to minor disturbances. Hence, one could entertain assimilation as a viable alternative to integration (see, e.g., Baldwin and Rozenberg, 2004).

Finally, in the spatial proximity model, the two groups differ in just one characteristic. Instead suppose there are many more characteristics, that each characteristic has an equal weight in the utility function of individuals, and that these characteristics are not too correlated with each other. Then we conjecture that one should not expect segregation. A possible integration policy would then stress the multidimensionality of multiculturalism instead of reducing it to one factor such as ethnicity or religion (as happens, e.g., with most forms and questionnaires due to Equal Opportunity legislation).

Appendix A. Supplementary data

Supplementary data associated with this article can be found, in the online version, at doi:10.1016/j.jpubeco.2006.03.008.

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Schelling's Spatial Proximity Model of Segregation Revisited. Appendices

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Appendix A. Proof of complete segregation on a ring

Utility is one if the number of O-neighbors equals the number of X-neighbors and zero otherwise. Best response is assumed without inertia. Only dynamics on a circle is considered here. For these proofs it is assumed that the number of Xs equals the number of Os equals *m*.

To prove that complete segregation is the limit outcome of the best response dynamics it should be possible to reach complete segregation from each initial configuration with positive probability. This requires the set of all completely segregated outcomes to form a unique recurrent class while all other states be transient.

On a completely segregated ring, there are two borders between ghettos. These borders are the only locations that offer positive utility. Hence, any agent can only move to such a border, which does not affect the integrity of the ghettos. Thus, only transience of the remaining states needs to be proved. To prove this, it is sufficient to offer an algorithm showing just one possible path leading from each allocation to complete segregation.

In the proofs below this is done in two steps. A 'seed' is discovered or constructed through positive probability moves (PPMs) and then complete segregation is built from this seed. PPMs can not only be used to insert an agent into a position with a utility of one, but also to drop out an agent from a position with a utility of zero, provided the existence of an appropriate destination is assured. PPMs with source and destination within a segment can also be made.

The following notation will be used:

· $2k$ is the size of a neighborhood: k neighbors to the left and to the right.
· $2m$ is the number of agents: m of each type.
· $\{X\}_n$ is a segment consisting of n Xs.
· Square brackets […] are used to highlight a seed.
· $\{X,O\}_{l,n}$ is any segment consisting of l Xs and n Os in any arbitrary order.[1]

LEMMA A1. Any segment of the type $\{X,O\}_{k-l,l}X\{X,O\}_{l,k-l}$, where $2k$ is the size of a neighborhood and $0 < l < k$, can be transformed into $\{O\}_l\{X\}_{k+1}\{O\}_{k-l}$ by means of positive probability moves.

PROOF. The two braces of the given segment contain k Xs and k Os. The agent in the central position neighbors them all and thus enjoys the highest utility. It is possible to move successively every X from each brace into the central position and such move will be a positive probability move. For instance,

$$\{X,O\}_{k-l,l}X\{X,O\}_{l,k-l} \rightarrow \{X,O\}_{k-l-1,l}\{X\}_2\{X,O\}_{l,k-l}.$$

The central position will always enjoy the highest utility, because whenever X is taken from the left brace the former central X shifts to the left and enters the brace from the right end. The removed X is

[1] For instance, $\{X,O\}_{2,3}$ could be XOOOX, as well as OOXOX or OXXOO.

always put in the central position. Similarly, when X is taken from the right brace the former central X shifts to the right and enters the right brace, while the removed X takes its place. □

LEMMA A2. On any circle of size 2*m*, where *m* is the number of each type, and $m \geq k$, where *k* is the size of *k*+*k* neighborhood, there exist at least two perfectly integrated positions between two neighboring agents.

PROOF. Define σ_i as the number of Xs, which a segment of length 2*k* contains, where 2*k* is the size of a neighborhood and *i* is the index of a position between two consecutive agents in the middle of the segment. It follows that $\sigma_i \in \{0,1, ..., 2k\}$ and $i \in \{1,2,...,2m\}$. Position *i* provides positive utility if and only if $\sigma_i = k$. If we go through all such possible segments once, then each agent will be covered exactly 2*k* times – not more because it is given that $m \geq k$. Therefore

$$\sum_{i=1}^{2m} \sigma_i = 2km . \qquad (*)$$

The mapping σ has a property for all *i* that

$$|\Delta\sigma_i| = |\sigma_i - \sigma_{i+1}| \leq 1 , \qquad (**)$$

where $\sigma_{2m+1} = \sigma_0$. If segment *i* is shifted by one position, it is possible that

- one X enters the segment, one X leaves, then $|\Delta\sigma_i| = 0$
- one X enters, none leave, then $|\Delta\sigma_i| = 1$
- none enter, none leave, then $|\Delta\sigma_i| = 0$
- none enter, one leaves, then $|\Delta\sigma_i| = 1$.

Two cases are possible:
a) $\exists i : \sigma_i > k$. Then for (*) to hold it follows that $\exists j : \sigma_j < k$. Without loss of generality, assume that $i < j$. From (**) it follows that $\exists l_1 : \sigma_{l_1} = k$ and $\exists l_2 : \sigma_{l_2} = k$ such that $i < l_1 < j < l_2$.
b) $\neg\exists i : \sigma_i > k \Rightarrow \neg\exists j : \sigma_j < k$. Consequently $\forall i : \sigma_i = k$.
Thus there always exist at least two locations that can provide positive utility. □

LEMMA A3. Any segment of the type $\{O\}_l \{X\}_{k+1} \{O\}_{k-l}$, where 2*k* is the size of a neighborhood, $0 < l < k$, and the number of each type on the circle is *m*>*k*, can be transformed into a seed $[\{O\}_k \{X\}_k]$ by means of positive probability moves.

PROOF. The proof is done by induction.
Inductive base: segment $\{O\}_{k-1} \{X\}_{k+1} O$ can be preceded by either O or X. The former case yields a seed $[\{O\}_k \{X\}_k]XO$.
Consider the latter case of $X\{O\}_{k-1} \{X\}_{k+1} O$. By Lemma A2 there exist at least two positions with positive utility. At the same time between any of the two Xs, which are inside the fragment $\{X\}_{k+1}$ and immediately to the left of $\{X\}_{k+1}$: $\sigma > k$. Consequently, $\sigma = k$ in at least two places outside this fragment.
a) If $\sigma = k$ immediately to the right of $\{X\}_{k+1}$, then we have a seed.

b) If $\sigma = k$ anywhere enclosed between horizontal bars $\mathrm{X}|\{\mathrm{O}\}_{k-1}|\{\mathrm{X}\}_{k+1}\mathrm{O}$ (bars included), then by PPM we insert there O and obtain a seed $\mathrm{X}[\{\mathrm{O}\}_k\{\mathrm{X}\}_k]\mathrm{XO}$.

c) If $\sigma = k$ is outside the segment $\mathrm{X}\{\mathrm{O}\}_{k-1}\{\mathrm{X}\}_{k+1}\mathrm{O}$, then the following action will be a positive probability move: $\mathrm{X}\{\mathrm{O}\}_{k-1}\mathrm{X}\{\mathrm{X}\}_k\mathrm{O} \rightarrow \mathrm{X}\{\mathrm{O}\}_{k-1}\{\mathrm{X}\}_k\mathrm{O}$. Since the destination is outside the segment, the moved X could not have constituted a part of its neighborhood. Finally, $\mathrm{X}\{\mathrm{O}\}_{k-1}\{\mathrm{X}\}_k\mathrm{O} \rightarrow \mathrm{X}\{\mathrm{O}\}_{k-1}\mathrm{XO}\{\mathrm{X}\}_{k-1}\mathrm{O} \rightarrow \mathrm{X}[\{\mathrm{O}\}_k\{\mathrm{X}\}_k]\mathrm{O}$.

Inductive step: assume that $\{\mathrm{O}\}_l\{\mathrm{X}\}_{k+1}\{\mathrm{O}\}_{k-l}$ (*) can be transformed into a seed. It needs to be shown that segment $\{\mathrm{O}\}_{l-1}\{\mathrm{X}\}_{k+1}\{\mathrm{O}\}_{k-l+1}$ (**) can also be transformed into a seed. The segment (**) can be preceded by either O or X. In the former case we get: $\{\mathrm{O}\}_l\{\mathrm{X}\}_{k+1}\{\mathrm{O}\}_{k-l}\mathrm{O}$. The obtained segment contains (*) and therefore can also be transformed into a seed by assumption.

Consider the latter case when X precedes (**): $\mathrm{X}\{\mathrm{O}\}_{l-1}\{\mathrm{X}\}_{k+1}\{\mathrm{O}\}_{k-l+1}$. As with the inductive step, we know that inside $\{\mathrm{X}\}_{k+1}$ and immediately to the left of $\{\mathrm{X}\}_{k+1}$ we have $\sigma > k$.

a) If $\sigma = k$ immediately to the right of $\{\mathrm{X}\}_{k+1}$, then we already have a seed.

b) If $\sigma = k$ anywhere enclosed between horizontal bars $\mathrm{X}|\{\mathrm{O}\}_{l-1}|\{\mathrm{X}\}_{k+1}\{\mathrm{O}\}_{k-l+1}$ (bars included), then by PPM move there O from the right hand side so that the resulting segment $\mathrm{X}\{\mathrm{O}\}_l\{\mathrm{X}\}_{k+1}\{\mathrm{O}\}_{k-l}$ contains (*).

c) If $\sigma = k$ anywhere within the rightmost braces $\{\mathrm{O}\}$, then insert there the free O. Three cases are possible:

i) A seed has formed: $\mathrm{X}\{\mathrm{O}\}_{l-1}\mathrm{X}[\{\mathrm{X}\}_k\{\mathrm{O}\}_k]$.

ii) If in the rightmost braces $\{\mathrm{O}\}$ there still is a position with $\sigma = k$, then repeat c), i.e., insert there the O, and see what happens.

iii) If there is no position in the rightmost $\{\mathrm{O}\}$ with $\sigma = k$, then the following is possible:

$$\mathrm{X}\{\mathrm{O}\}_{l-1}\{\mathrm{X}\}_{k+1}\{\mathrm{O}\}_{k-l+1} = \mathrm{X}\{\mathrm{O}\}_{l-1}\{\mathrm{X}\}_{k-l}\mathrm{X}\{\mathrm{X}\}_l\{\mathrm{O}\}_{k-l+1} \rightarrow \mathrm{X}\{\mathrm{O}\}_{l-1}\{\mathrm{X}\}_{k-l}\{\mathrm{X}\}_l\{\mathrm{O}\}_{k-l+1} \rightarrow \mathrm{X}\{\mathrm{O}\}_{l-1}\{\mathrm{X}\}_{k-l+1}\mathrm{O}\{\mathrm{X}\}_{l-1}\{\mathrm{O}\}_{k-l+1} \rightarrow \{\mathrm{O}\}_{l-1}\{\mathrm{X}\}_{k-l+1}\mathrm{O}\{\mathrm{X}\}_l\{\mathrm{O}\}_{k-l+1} \rightarrow \{\mathrm{O}\}_{l-1}\{\mathrm{X}\}_{k-l}\mathrm{O}\{\mathrm{X}\}_{l+1}\{\mathrm{O}\}_{k-l+1} \rightarrow \{\mathrm{O}\}_{l-1}\{\mathrm{X}\}_{k-l-1}\mathrm{O}\{\mathrm{X}\}_{l+2}\{\mathrm{O}\}_{k-l+1} \rightarrow \ldots \rightarrow \{\mathrm{O}\}_{l-1}\mathrm{XO}\{\mathrm{X}\}_k\{\mathrm{O}\}_{k-l+1} \rightarrow \{\mathrm{O}\}_l\{\mathrm{X}\}_{k+1}\{\mathrm{O}\}_{k-l}\mathrm{O}.$$

The obtained segment contains (*) and therefore can also be transformed into a seed. □

PROPOSITION A1. If a neighborhood on a circle of size $2m$ is defined as k neighbors to the left and k neighbors to the right, then, if m is the number of each type, $m > k$, and the utility function is spiked, then the process of best-responses has a unique recurrent class consisting of all completely segregated states.

PROOF. If an allocation contains a seed, then construction of complete segregation by means of PPMs is trivial. Otherwise construct a seed. If no agent enjoys positive utility, some agent will go to a location that ensures one (it exists by Lemma B1). Now at least one agent has positive utility. Take one such agent and consider all possible configurations of his neighborhood. Without loss of generality we can assume that X is such an agent. Since its utility is positive, its neighborhood is the segment $\{\mathrm{X},\mathrm{O}\}_{k-l,l}\mathrm{X}\{\mathrm{X},\mathrm{O}\}_{l,k-l}$. By Lemma A1 this can be transformed into $\{\mathrm{O}\}_l\{\mathrm{X}\}_{k+1}\{\mathrm{O}\}_{k-l}$ and then by Lemma A3 into $[\{\mathrm{O}\}_k\{\mathrm{X}\}_k]$ with positive probability. Thus a seed can always be constructed and complete segregation then built. On a completely segregated ring, there are two borders between ghettos. These borders are the only locations that offer positive utility. Hence, any agent can only move to such a border, which does not affect the integrity of the ghettos. □

Appendix B. On MNE on a ring

Assume a ring with an equal number of each type of agents. Now take any one agent out, say, without loss of generality, take agent X out. Then the following lemma applies.

LEMMA B1. On any circle of size $2m-1$, where m and $m-1$ is the number of each type, and $m > k$, where k is the size of a $k+k$ neighborhood, there exist at least two perfectly integrated positions between two neighboring agents.

PROOF. Borrow the definition of mapping σ_i and its properties from Lemma A2. Without loss of generality assume that there are m Os and $m-1$ Xs. The following equality will hold:

$$\sum_{i=1}^{2m-1} \sigma_i = 2k(m-1) \qquad (*)$$

The proof proceeds by contradiction. Assume there exist no two i, for which $\sigma_i = k$.

From (*) it follows that $\exists i : \sigma_i < k$ for otherwise

$$\sum_{i=1}^{2m-1} \sigma_i > k(2m-1) > 2k(m-1),$$

so that (*) would not hold. Since for some i $\sigma_i < k$ and for no two i $\sigma_i = k$, then $\neg\exists i : \sigma_i > k$, because σ_i can only change by increments of 1. Hence, for at most one i $\sigma_i = k$, while for the remaining ones $\sigma_i < k$. Consequently,

$$\sum_{i=1}^{2m-1} \sigma_i \le (2m-2)(k-1) + k .$$

The necessary condition for (*) to be satisfied is

$(2m-2)(k-1) + k \ge 2k(m-1)$ or $m \le (k+2)/2$.

This, however, contradicts the premise of the lemma requiring $m > k$. Therefore there exist at least two i, for which $\sigma_i = k$. □

PROPOSITION B1. Suppose, on a circle, there are m agents of each type, where $m > k$, and the utility function is maximized at a perfectly integrated neighborhood (e.g. the utility function may be spiked, peaked or flat). Then an allocation is an MNE, if and only if all agents enjoy the maximal possible utility.

PROOF. The proof proceeds by contradiction. Suppose an allocation is an MNE, and there is an agent who enjoys less than maximal possible utility. Remove this agent. Lemma B1 applies: there will always exist a perfectly integrated location, where the agent can go to obtain the maximal utility. Hence, an allocation cannot be an MNE. Thus, in an MNE, all agents enjoy the maximal utility. Further, whenever all agents enjoy the maximal utility, no agent will have an incentive to move elsewhere, so such an allocation is an MNE. □

PROPOSITION B2. Suppose, on a circle, there are m agents of each type, where $m > k$, and the utility function is maximized at a perfectly integrated neighborhood. Then there exist no strict MNE.

PROOF. The proof proceeds by contradiction. Assume a strict MNE exists, and pick any agent. Take it out. Lemma B1 applies: on the remainder of the ring, there will always be at least two perfectly integrated locations which ensure maximal possible utility. One of these may be the position the agent occupied, but there will be at least one more perfectly integrated position elsewhere. Hence, there is always a choice to move elsewhere to a place with maximal possible utility. This contradicts the definition of a strict MNE. Therefore, there are no strict MNE. □

PROPOSITION B3. Suppose, on a circle, there are *m* agents of each type, where $m > k$, and the utility function is maximized at a perfectly integrated neighborhood. Then, in an MNE, there can be no cluster larger than *k*+1, where *k* is the size of a neighborhood parameter.

PROOF. A cluster bigger than *k*+1 requires that all its agents in the cluster, except possibly those on the two edges of the cluster, have less than maximal possible utility. But, by Proposition B1, this contradicts the assumption that the cluster is a part of a MNE allocation. Hence, no cluster can be larger than *k*+1. □

PROPOSITION B4. The sets of MNE for all utility functions, which attain their maximum at and only at a perfectly integrated neighborhood (e.g. spiked and peaked utility functions), coincide.

PROOF. For a circle, Lemma B1 guarantees (provided $m > k$) an opportunity for any agent to migrate to a perfectly integrated location. Since the considered utility functions have peaks at perfect integration, it is impossible for any agent in an MNE not to enjoy a perfectly integrated neighborhood. By Proposition B1, this is a necessary and sufficient condition for an allocation to be an MNE. Hence, the sets of MNE are the same for all utility functions uniquely maximized at perfect integration. □

PROPOSITION B5. Suppose the utility function attains its maximum at and only at a perfectly integrated neighborhood, and let the neighborhood size parameter *k* be odd. Then an allocation is an MNE, if and only if agents *i* and *i*+*k*+1 for all *i* are of opposite types.

PROOF. The proof is constructive. Given an arbitrary segment of 2*k*+1 agents $a_0 a_1 ... a_k ... a_{2k}$, with the central agent a_k enjoying maximal utility, it is sometimes possible to continue the string adding agents a_{2k+1} onwards so that agents a_{k+1} onwards also enjoy maximal utility. Call this a complementary step procedure. The initial segments, for which it is possible to continue the process until the repetition starts, are compatible with maximal utility for all agents. Since maximal utility for every agent is a necessary and sufficient condition for an allocation to be a MNE (by Proposition B1), this procedure will allow to construct all patterns, which can serve to build a MNE.

The complementary step procedure will necessarily reproduce the initial segment. It is impossible to produce a string of an infinite length for which all segments consisting of 2*k*+1 agents would be unique, because the number of such segments is finite. Thus the string obtained by means of this procedure can be joined in a circle. This ensures that not only agents a_k onwards enjoy maximal utility, but also that agents from a_0 through a_{k-1} enjoy maximal utility by construction.[2]

Let $a_i = -1$ if location *i* is occupied by X and $a_i = 1$ otherwise. Then the recursive formula used in the complementary step procedure to add agent *i* given the preceding segment is

$$a_i = a_{i-2k-1} - a_{i-k-1} + a_{i-k} . \qquad (1)$$

[2] It is also crucial that a complementary step procedure is uniquely defined, so it is possible to 'scroll back' to the initial allocation. For assume the sequence does eventually repeat, but not from the initial segment. Then it would be possible from any place on the formed circle, which excludes the initial segment, to return both to the initial segment and never return to it. This is a contradiction. Hence repetition starts from the initial segment.

The formula ensures that if in segment $a_{i-2k-1}a_{i-2k}...a_{i-k-1}a_{i-k}...a_{i-1}a_i$ agent a_{i-k-1} enjoyed maximal utility, then a_{i-k} will also enjoy maximal utility. It is undefined for $a_{i-2k-1} = a_{i-k} = -a_{i-k-1}$.

To prove necessity, suppose the contrary: $\exists r : a_r = a_{r+k+1} = \alpha_0$. Without the loss of generality, set r=0. Applying (1) yields $a_{2k+1} = a_0 - a_k + a_{k+1} = 2\alpha_0 - a_k$. Hence, $a_{2k+1} = a_k = \alpha_0$. Applying (1) again gives $a_{3k+1} = a_k - a_{2k} + a_{2k+1}$, so that $a_{3k+1} = a_{2k} = \alpha_0$. Reiteratively using (1), the following obtains:

$$a_{zk} = a_{zk+1} = \alpha_0, \tag{2}$$

where $z \in Z$. Apply (1) once again: $a_{zk} = a_{(z-2)k-1} - a_{(z-1)k-1} + a_{(z-1)k}$. Now by (2) it follows that

$$a_{(z-2)k-1} = a_{(z-1)k-1} \text{ or, equivalently, } a_{zk-1} = \alpha_1. \tag{3}$$

Next, $a_{zk-1} = a_{(z-2)k-2} - a_{(z-1)k-2} + a_{(z-1)k-1}$, which gives $a_{zk-2} = \alpha_2$, and in general

$$a_{zk-l} = \alpha_l, \tag{4}$$

where $1 \le l \le k-2$. Combining (2), (3) and (4) ensures that a string consists of the following repeated segments:

$$\alpha_{k-2}\alpha_{k-3}...\alpha_2\alpha_1\alpha_0\alpha_0$$

Take a fragment of the string as given below

$$\alpha_0\alpha_0\alpha_{k-2}\alpha_{k-3}...\alpha_2\alpha_1[\alpha_0]\alpha_0\alpha_{k-2}\alpha_{k-3}...\alpha_2\alpha_1\alpha_0.$$

The condition for the bracketed agent to enjoy maximal utility, so that the fragment could be a part of an MNE, is (having divided through by 2)

$$2\alpha_0 + \alpha_1 + \alpha_2 + ... + \alpha_{k-3} + \alpha_{k-2} = 0. \tag{5}$$

Recall that k is odd and $|\alpha_i| = 1$. Hence condition (5) can never be satisfied and

$$\forall i : a_i = -a_{i+k+1}. \tag{6}$$

To prove sufficiency consider any segment containing a full neighborhood (the index of the first element is normalized to 0)

$$a_0 a_1 ... a_k ... a_{2k-1} a_{2k}$$

The condition for the middle agent to enjoy the maximal possible utility is

$$a_0 + a_1 + ... + a_{k-1} + a_{k+1} + ... + a_{2k-1} + a_{2k} = 0 \tag{7}$$

Substitution of (6) into (7) readily shows that (7) is an identity. Hence if (6) holds, then all agents have positive utility and an allocation is a MNE. □

PROPOSITION B6. Suppose the utility function attains its maximum at and only at a perfectly integrated neighborhood, and let the neighborhood size parameter k be even. Then an allocation is an MNE, if and only if either (i) agents i and i+k+1 are of opposite types or (ii) agents i and i+k are of the same type and $a_i + a_{i+1} + ... + a_{i+k-2} + a_{i+k-1} = 0$ for all i.

PROOF. The proof draws heavily on the proof of Proposition B5. In the necessary part of the proof of Proposition B5 it has been demonstrated that if an allocation is an MNE, then either agents i and i+k+1 are of opposite types or agents i and $i+k$ are of the same type. Moreover if agents i and $i+k$ are of the same type, then condition (5) holds, and in any pattern there is at least one pair of agents of the same type (of type α_0). The only pattern with all adjacent agents of different types is a string of alternating X and O. This MNE falls under type (i). Condition (5) is equivalent to one in (ii), where two agents of the same type are gathered in one term. Condition (5) can now be satisfied, because k is even.

To prove sufficiency, consider any segment containing a full neighborhood (the index of the first element is normalized to 0) $a_0 a_1 ... a_k ... a_{2k-1} a_{2k}$. The condition for the middle agent to enjoy the maximal possible utility is (7). If a pattern is of type (i), then substitution of (6) into (7) turns it into identity. Hence if (6) holds, then all agents have positive utility and an allocation is a MNE. If a pattern is of type (ii), then the fact that $a_i = a_{i+k}$ and the condition in (ii) ensures that (7) and (5) hold, so that the allocation is an MNE. □

LEMMA B2. In a MNE with the flat utility function adjacent agents of different type enjoy perfect integration.

PROOF. Consider two neighboring agents: X' and O', each of which prefers at least k neighbors of his own type. The neighborhood of X' is different from the neighborhood of O' in that it contains two more agents, one of which is O', and omits two agents, one of which is X'. Both X' and O' can be satisfied only if the neighborhood of O' contains exactly k Os, then it is possible to ensure k Xs for X' by stipulating that an extra agent is of X type and an omitted agent is of O type. Thus, adjacent agents of different type can only have perfectly integrated neighborhoods.

Let a_i = -1 if location i is occupied by X and a_i = 1 otherwise, then

$$\Omega_{X'} = \Omega_{O'} + 2 + a' - a'' \leq 0 \text{ and } \Omega_{O'} \geq 0, \qquad (*)$$

where $\Omega_{X'}$ and $\Omega_{O'}$ are the sum of agents in the neighborhood of X' and O' respectively, a' and a'' are additional and excluded agents. Both inequalities are satisfied only if $\Omega_{X'} = \Omega_{O'}$.

So far it has been implicitly assumed that we are either operating on a circle or considering agents on a line who are sufficiently remote from edges. The difference the possibility of edges introduces is that either a' or a'' can turn into zero. If X' is closer to the border than O' then a' might turn into zero, otherwise a'' might be zero. If this is the case, then it is never possible to satisfy (*): both X' and O' cannot be satisfied simultaneously. Hence, the allocation is not a MNE. □

PROPOSITION B7. The set of all MNE on a circle with a flat utility function and the neighborhood size k ($m > k$) contains the set of all MNE on a circle with a peaked utility function for corresponding k, plus all possible combinations of clusters, each populated by at least k+1 agents with at least one cluster exceeding k+1 agents. For $k \leq 4$ there are no other MNE than these.

PROOF. In any MNE with the spiked utility function all agents enjoy maximum possible utility. With the flat utility function, all agents would continue to enjoy the highest possible utility for each of these allocations. Hence all MNE with the spiked function are also MNE with the flat function. Further, if all clusters are k+1 or larger, than agents on edges enjoy perfectly integrated neighborhoods, while the rest are surrounded by a majority of their own type. Hence, no one has an incentive to move and such an allocation is a MNE. It remains to be shown that there are no other MNE.

In MNE, clusters of size k+1 cannot co-exist with smaller clusters. Assume the opposite. Consider any cluster of size at least k+1, let it be X-type, which neighbors a cluster of k agents or less (O-type):

$$...X\{O\}_{<k}\dot{O}\{X\}_{\geq k+1}...$$

Then the agent marked with a dot will be in a minority, and hence willing to go to an allocation with higher utility, which is known to exist. It needs to be demonstrated, that there is no MNE with all cluster sizes less or equal to *k* (for $k \leq 4$) where some agents would not enjoy a perfectly integrated neighborhood.

If *k*=1 then a single agent forming a cluster of size 1 is always dissatisfied. If *k*=2, then any agent from a cluster sized 1 or 2, by Lemma B2, in MNE would enjoy a perfectly integrated neighborhood, if such a MNE exists. If *k*=3, again, by Lemma B2, we should consider only clusters of size 3. Such a cluster needs to be enveloped by two clusters of the opposite type comprising at least 4 agents each for bordering agents of the opposite type to be satisfied. Hence, no agent in cluster of size 3 will be satisfied, and this allocation will not be a MNE.

If *k*=4, then, by Lemma B2, we can consider only central agents in clusters of size 3 and 4. For the cluster of size 3, it is impossible that the central agent (e.g., O) is in a majority while the bordering agents of the opposite type (Xs) are satisfied:

…XXXOXOOOXOOX…

The cluster of size 4, trivially, will be surrounded by clusters of the opposite type of 5 agents or more for the border agents of the opposite type to be satisfied. This renders central agents dissatisfied:

…XXXXXOOOOXXXXX…

Thus it is impossible to construct a MNE with cluster sizes not exceeding *k* for ($k \leq 4$), such that there exists an agent who is in a majority. □